5

Solitons and two-dimensional integrable models

5.1 Introduction

In the previous chapters we have addressed 2D field theories with no scale. As we discussed in Chapter 2, one cannot define an S-matrix for such theories. Generically physical systems are characterized by certain energy scales and the notion of S-matrix plays an important role. It is thus time to move forward and examine non-conformal field theories. Again we start our journey with the theory of a free scalar field, but now a massive one. We then move on and discuss interacting theories equipped with infinite numbers of conserved charges, the so-called integrable models, that resemble the free massive theory in a way that will be explained below.

5.2 From the theory of a massive free scalar field to integrable models

The classical action of a massive free scalar field is obviously the action of a massless scalar field with an additional mass term,

$$S = \int d^2x \left[\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \right], \tag{5.1}$$

where m is the mass scale which momentarily will be shown to be the mass of the particle associated with the field ϕ. Unlike the analysis of CFTs there is no advantage here to the use of complex coordinates, so we will use real ones.

The corresponding equation of motion,

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0, \tag{5.2}$$

is solved for the case of uncompactified space-time by the following Fourier transform,

$$\phi(x^0, x^1) = \int \frac{dk^1}{\sqrt{2\pi}\sqrt{k^0}} \left[a(k^1) e^{-ik\cdot x} + a^\dagger(k^1) e^{ik\cdot x} \right], \tag{5.3}$$

where $(k^0)^2 - (k^1)^2 = m^2$.

A dramatic difference between the massless field discussed in Chapter 2, and the massive one we discuss here, shows up when analyzing the symmetries of the system.

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The only transformations that leave the action invariant are the $ISO(1,1)$ Poincare transformations, namely, the space and time translations and a single Lorentz transformation. These are,

$$x^0 \to x^0 + a^0 \quad x^1 \to x^1 + a^1$$
$$x^0 \to x^0 + a^0_1 x^1 \quad x^1 \to x^1 + a^1_0 x^0, \tag{5.4}$$

where the transformation parameters are constants and $a^0_1 = a^1_0$. The fact that the parameters are constants, and not holomorphic and anti-holomorphic functions of the complex coordinates, has a tremendous impact, since it implies the absence of the powerful infinite-dimensional Virasoro algebra.

The corresponding Noether currents associated with the Poincare transformations are,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathcal{L},$$
$$J^\mu_{\text{Lor}} \equiv J^{\mu 01}_{\text{Lor}} = \epsilon_{\rho\nu} T^{\mu\rho} x^\nu. \tag{5.5}$$

However, since the space-time is two dimensional, there is an additional conserved current, the so-called *topological current*,

$$J^\mu_{\text{top}} = \epsilon^{\mu\nu}\partial_\nu \phi, \tag{5.6}$$

which is conserved regardless of the equations of motion, since obviously $\partial_\mu J^\mu_{\text{top}} = \epsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$. In fact this current is conserved for any interacting scalar field in 2d, and as we will see later on it plays an important role in the analysis of soliton solutions of integrable models.

The theory of a free massive scalar field, as well as other scalar theories that will be addressed in this chapter, are obviously invariant under the discrete symmetry of,

$$\phi \to -\phi. \tag{5.7}$$

The canonical quantization was described in Section 1.6. The normal ordered Hamiltonian and momentum expressed in terms of the creation and annihilation operators take the form,

$$H = \int \mathrm{d}k^1 \sqrt{(k^1)^2 + m^2}\, a^\dagger(k^1)a(k^1) \quad \mathbf{P} = \int \mathrm{d}k^1 k^1 a^\dagger(k^1)a(k^1). \tag{5.8}$$

The state $a^\dagger(k^1)|0>$ is characterized by,

$$\mathbf{P}a^\dagger(p^1)|0> = p^1 a^\dagger(p^1)|0> \quad Ha^\dagger(p)|0> = \sqrt{(p^1)^2 + m^2}\, a^\dagger(p^1)|0>, \tag{5.9}$$

and hence it is interpreted as a single free massive relativistic particle.

For the case of free single particles, the momentum and Hamiltonian can be generalized to an infinite set of conserved charges, like $Q_n \equiv P^n$ as,

$$Q_n \; a^\dagger(p)|0> = (p^1)^n a^\dagger(p)|0>, \tag{5.10}$$

and similarly for powers of p^0.

The conserved charge Q_n can also be represented as an integral in space. For odd $n = 2k + 1$,

$$Q_{2k+1} = \int \mathrm{d}x[\phi^{(2k+1)}(t,x)\dot{\phi}(t,x) + \text{hermitian conjugate}], \tag{5.11}$$

where $\phi^{(n)}$ is n derivatives of space on ϕ, t is x^0 and x is x^1. For even n it is a bit more complicated, but can be evaluated similarly. Note that the expression is local.

We elevate the field theory associated with a free massive scalar particle into a non-trivial interacting integrable model by replacing the mass term with a potential for the scalar field. It will be shown that identifying in such interacting field theories, an infinite set of conserved charges similar to the one of the free theory, will be a key ingredient in constructing integrable models. This will be discussed in Section 5.10.

A more general construction of integrable models is based on perturbing conformal field theories, which were discussed in Chapter 3, with relevant primary fields, namely, those that have conformal dimension $\Delta + \bar{\Delta} < 2$.[1] This class of models, which will include in particular the integrable minimal models, will be discussed in Section 5.9.

A very basic notion in scalar theories with interacting potential is the solitonic classical configurations, which will be the topic of the next section.

5.3 Classical solitons

We now let the massive particles interact with each other. The interaction is introduced in the form of a potential added to the Lagrangian of the free scalar field theory. Our first task in analyzing this type of field theories is to determine the solutions of the classical equations of motion. We start first with *solitons*, which are static solutions of finite energy, and then move on to time-dependent solutions.

The "classical" material about solitons is described in great detail in several books, in particular in [182], [66] and [183]. For "nontopological solitons" see [149].

Consider a two-dimensional scalar field described by the following action,

$$\begin{aligned} S &= \int \mathrm{d}^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \\ &= \int \mathrm{d}^2x \left[\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - V(\phi)\right], \end{aligned} \tag{5.12}$$

[1] So far in the context of conformal field theory we have denoted the conformal dimension by h. Here in the chapter on integrable models it will be denoted by Δ.

where $\dot{}$ and $'$ refer to time and space derivatives and $V(\phi)$ is a positive semi-definite function of ϕ. The corresponding equation of motion is given by,

$$\partial_\mu \partial^\mu \phi + \partial_\phi V(\phi) = \ddot{\phi} - \phi'' + \partial_\phi V(\phi) = 0. \tag{5.13}$$

The energy associated with a given configuration of ϕ is,

$$E = \int \mathrm{d}x \left[\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + V(\phi) \right]. \tag{5.14}$$

Let us assume that the potential has a set of N absolute minima at which it vanishes, namely $V(\phi^i) = 0$ for $i = 1, \ldots, N$. If ϕ^i are constants independent of space-time, then the corresponding energy vanishes, and in fact $E(\phi) = 0$ if and only if $\phi(x,t) = \phi^i$.

Static solutions of the equation of motion are determined by,

$$\phi'' - \partial_\phi V(\phi) = 0. \tag{5.15}$$

Solitons which have finite energy, must have $\phi^{'}$ and $V(\phi)$ vanish rapidly enough at $\pm\infty$, and thus must approach asymptotically one of the configurations ϕ^i that minimizes the potential, namely

$$\lim_{x \to \infty} \phi(x) \to \phi^i, \quad \lim_{x \to -\infty} \phi(x) \to \phi^j. \tag{5.16}$$

Solving (5.15) is equivalent to solving a mechanical system where x becomes the time, ϕ the coordinate of a point particle of a unit mass subjected to a potential $-V(\phi)$, and $E_{\mathrm{mech}} = \frac{1}{2}\phi'^2 - V(\phi)$ is the conserved energy of the system. The boundary conditions where at $x \to \pm\infty$ $V(\phi) \to 0$ and $\phi' \to 0$ implies that $E_{\mathrm{mech}} = 0$. The energy of the field theory (5.14) translates into the action of the mechanical system. The particle trajectory is therefore characterized by having finite action and vanishing mechanical energy. The virial theorem for the particle system has the form,

$$\frac{1}{2}(\phi')^2 = V(\phi), \tag{5.17}$$

which is also easily derivable in field theory language by multiplying (5.15) by ϕ', integrating over x and using the boundary conditions.

From the mechanical analog it is clear that:

(i) there is no non-trivial solution for a potential with a single minimum.
(ii) For a potential with n minima there are $2(n-1)$ solutions associated with trajectories starting at $x \to -\infty$ at ϕ^i and ending at $x \to \infty$ at ϕ^{i+1} and vice versa. Trajectories where instead the particle ends at $\phi^{j>i+1}$ or back to ϕ^i are impossible, since all the derivatives $\frac{d^n \phi}{dx^n}$ vanish at ϕ^{i+1} so the particle that gets to this point will not be able to leave it.

The equation of motion (5.15) has solutions of the form,

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{\mathrm{d}\tilde{\phi}}{\sqrt{2V(\tilde{\phi})}}, \tag{5.18}$$

where x_0 the integration constant is any arbitrary point where the field has the value of $\phi(x_0)$. The integral is non-singular apart from the end-points since everywhere else $V(\phi)$ is positive.

Classical solitons of $\lambda\phi^4$ theory

Let us now demonstrate the general features of solitons discussed in the previous section with the prototype model of the a potential with a quartic interaction. Consider the potential,

$$V(\phi) = \frac{1}{4}\lambda\left(\phi^2 - \frac{m^2}{\lambda}\right)^2, \tag{5.19}$$

which has two minima at $\phi = \pm\frac{m}{\sqrt{\lambda}}$ and is obviously invariant under $\phi \to -\phi$. Substituting this potential into (5.18) and inverting it one finds when setting $\phi(x_0) = 0$ the following two possible solutions,

$$\phi(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{5.20}$$

which corresponds to either starting at $\phi = -\frac{m}{\sqrt{\lambda}}$ and ending at $\phi = \frac{m}{\sqrt{\lambda}}$, or vice versa. The former will be called a "kink" and the latter an "anti-kink". The invariance under $\phi \to -\phi$ and parity transformation are easily realized in the kink anti-kink system, namely, $\phi_{\text{kink}}(x) = -\phi_{\text{anti-kink}}(x)$, and for $x_0 = 0$ also $= -\phi_{\text{kink}}(-x)$ (otherwise reflect around x_0).

The energy density of the kink solution is given by

$$\epsilon(x) = \frac{1}{2}(\phi')^2 + V(\phi) = \frac{m^4}{2\lambda}\mathrm{sech}^4\left[\frac{m}{\sqrt{2}}(x - x_0)\right]. \tag{5.21}$$

The total classical energy, which is refered to as the classical mass of the kink is (as our soliton is like a particle at rest),

$$M_{\text{cl}} = \int_{-\infty}^{\infty} \mathrm{d}x\epsilon(x) = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{5.22}$$

Classical solitons of sine-Gordon theory

The sine-Gordon model, which will serve as a prototype model throughout this chapter is defined by the action given in (5.12) with the potential,

$$V(\phi) = -\frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right]. \tag{5.23}$$

Later in Section 5.4 we will adopt a different convention where,

$$\frac{\sqrt{\lambda}}{m} = \beta \quad \mu^2 = \frac{m^3}{\sqrt{\lambda}}. \tag{5.24}$$

In terms of this parametrization the potential reads,

$$V(\phi) = -\frac{\mu^2}{\beta}[\cos(\beta\phi) - 1]. \tag{5.25}$$

The potential has an infinite set of discrete vacua at $\phi^k = 2\pi k \frac{m}{\sqrt{\lambda}}$ and again it is invariant under $\phi \to -\phi$. As for the ϕ^4 case, here as well the integral in (5.18) can be solved analytically. For the soliton that goes from 0 to $\frac{m}{\sqrt{\lambda}} 2\pi$ and vice versa for the anti-soliton, and choosing $\phi(x_0) = \frac{m}{\sqrt{\lambda}}\pi$, we get,

$$\phi(x) = 4\frac{m}{\sqrt{\lambda}} \tan^{-1}[\mathrm{e}^{\pm m(x-x_0)}]. \tag{5.26}$$

Adding $\frac{m}{\sqrt{\lambda}} 2n\pi$ to this, gives a soliton that goes from $\frac{m}{\sqrt{\lambda}} 2n\pi$ to $\frac{m}{\sqrt{\lambda}} 2(n+1)\pi$, and vice versa for the anti-soliton. The soliton has a topological charge associated with (5.6) of $Q = 1$, the anti-soliton $Q = -1$.

Substituting the explicit expression of the soliton profile (5.26) into the expression for the energy one finds that the mass of the SG soliton is

$$M_{\mathrm{SGsol}} = \frac{8m^3}{\lambda} = \frac{8m}{\beta^2}. \tag{5.27}$$

Classical stability of the solitons

So far we have shown that scalar field theories with degenerate vacua admit soliton solutions. Let us now address the question of whether these solutions are stable against small time-dependent perturbation. Consider the field configuration,

$$\phi(x,t) = \phi_{\mathrm{sol}}(x) + \delta\phi(x,t), \tag{5.28}$$

where $\phi_{\mathrm{sol}}(x)$ is a time-independent soliton solution and $\delta(x,t)$ is a small perturbation. Substituting this configuration into the equation of motion and retaining only linear terms in the perturbation we get,

$$\partial^\mu \partial_\mu \delta\phi + V''(\phi_{\mathrm{sol}})\delta\phi = 0. \tag{5.29}$$

Since the equation is invariant under time translation, we express the perturbation as a superposition of normal modes in the following form,

$$\delta\phi(x,t) = \sum_n Re[a_n \mathrm{e}^{iw_n t}\delta_n(x)]. \tag{5.30}$$

The normal modes obey the equation,

$$-\frac{\mathrm{d}^2\delta_n}{\mathrm{d}x^2} + V''(\phi_{\mathrm{sol}})\delta_n = w_n^2\delta_n, \tag{5.31}$$

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which is in fact a one-dimensional Schrodinger equation with $V''(\phi_{\rm sol})$ as a potential. If this equation has eigenmodes with negative eigenvalues, the soliton is unstable.

It is easy to construct one eigenmode. Since the soliton is invariant under space translation $\phi_{\rm sol}(x) \to \phi_{\rm sol}(x+a)$, $\delta_0 = \frac{{\rm d}\phi_{\rm sol}(x)}{{\rm d}x}$ is an eigenmode with a vanishing eigenvalue. Now since the soliton is a monotonic function of x, δ_o does not have nodes. A theorem about a one-dimensional Schrodinger equation tells us that the eigenmode with no nodes has the lowest eigenvalue and hence there are no negative modes and the soliton for any $V(\phi)$ is indeed stable.

The topological charge

Any two-dimensional scalar field theory in two dimensions admits the topological current (5.6), $J^\mu_{\rm top} = {\rm e}^{\mu\nu}\partial_\nu\phi$. Thus, the following difference is a conserved charge,

$$Q_{\rm top} = \int {\rm d}x\phi' = [\phi(t,+\infty) - \phi(t,-\infty)] \equiv \phi_+ - \phi_-. \tag{5.32}$$

Often one refers to $\phi_\pm$ as the topological indices. In fact for theories with a potential that has a discrete number (finite or infinite) of vacua, non-singular field configurations of finite energy have both ϕ_+ and ϕ_- separately conserved. This results from the following argument. Finite energy implies that both ϕ_+ and ϕ_- are at absolute minima of the potential. Now since the non-singular configurations are continuous in time, and the potential has a set of discrete (finite or infinite) vacua, $\phi(t,\infty)$ must be stationary at ϕ_+, or $\partial_0\phi(t,\pm\infty) = \partial_0\phi_\pm = 0$, namely the indices are conserved.

In fact this conservation can be used to show the existence of non-dissipative solutions. For instance in the ϕ^4 theory we can show that a configuration with $\phi_+ = -\phi_-$ is non dissipative. By continuity in x there must be, for any t, some x for which $\phi = 0$. At this point $T_{00} \geq V(0)$ and since the definition of a dissipative solution is that the $\lim_{t\to\infty}\max_x T_{00} = 0$ it is clear that it is non-dissipative. Similar arguments hold for other cases of solitons.

Thus, one can divide the space of finite-energy non-singular solutions into topologically disconnected sub-spaces that are characterized by the two indices $\phi_\pm$. Such a sub-space cannot be continuously deformed into another one unless the finite energy condition is violated. For instance, in the ϕ^4 theory, the potential has two minima so that $\phi_+ = \frac{m}{\lambda}$ and $\phi_- = -\frac{m}{\lambda}$. Hence, there are four subspaces $(-,+), (+,-), (-,-), (+,+)$ associated with the soliton, the anti-soliton and the two trivial constant vacuum solutions. For the sine-Gordon the solitons belong to the subspaces characterized by $\phi_- = 2\pi n\frac{m}{\sqrt{\lambda}}$ and $\phi_+ = 2\pi(n+1)\frac{m}{\sqrt{\lambda}}$.

Obviously, non-trivial topological charges require multiple vacua. The latter situation occurs if and only if there is a spontaneous breaking of a symmetry. For instance in ϕ^4 and sine-Gordon it is the discrete $\phi \to -\phi$ symmetry which is broken.

Derrick's theorem

Consider a scalar field theory in $D+1$ space-time dimensions described by the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V(\phi), \tag{5.33}$$

where the potential $V(\phi)$ is non-negative and vanishes at its minima. The theorem states that *for $D \geq 2$ the only non-singular time-independent solutions of finite energy are the vacua.*

Let us denote by $\phi(\mathbf{x})$ a time-independent solution of the equation of motion. We now introduce a one-parameter family of field configurations defined as,

$$\phi(\lambda, \mathbf{x}) = \phi(\lambda \mathbf{x}), \tag{5.34}$$

where λ is a positive real number. The energy associated with the configuration $\phi(\lambda, \mathbf{x})$ is,

$$E(\lambda) = \lambda^{-D} \int \mathrm{d}^D\mathbf{x} \left[\frac{1}{2}\lambda^2 (\nabla\phi)^2 + V(\phi)\right]. \tag{5.35}$$

By Hamilton's principle the energy as a function of λ is stationary at $\lambda = 1$ so that,

$$\int \mathrm{d}^D\mathbf{x} \left[\frac{1}{2}(D-2)(\nabla\phi)^2 + DV(\phi)\right] = 0. \tag{5.36}$$

For $D > 2$ the two terms in the integral have to vanish separately, which occurs only for the vacua. For $D = 2$, the potential term has to vanish, which again occurs only for the vacua. This proves the theorem.

The following remarks are in order:

(i) Derrick's theorem applies only to time-independent configurations.
(ii) It applies to field theories with only scalar fields. Once one introduces additional fields like gauge fields or fermions the theorem is not valid (see Section 20.3).

5.4 Breathers or "doublets"

So far we have discussed only time-independent solutions of the equations of motion. A natural question to ask is whether the equations also admit exact time-dependent solutions. Another question, seemingly unrelated, is that of the interactions between solitons and between solitons and anti-solitons. We will see shortly that these two puzzles are in fact related. We now proceed to examine these questions in the laboratory of the sine-Gordon model.

The following periodic configuration,

$$\phi(x,t) = \frac{4}{\beta}\tan^{-1}\left[\frac{\eta \sin(wt)}{\cosh(\eta w x)}\right], \tag{5.37}$$

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where $\eta = \frac{\sqrt{(m^2 - w^2)}}{w}$, with $w \leq m$, is a solution of the equation of motion (5.13). We will show now that this solution is related to a bound state of a soliton and anti-soliton.

Consider first the simple case of small w, namely $w \ll m$. For positive $\sin(wt)$ and finite x, the argument of $\tan^{-1}$ is very large, and thus $\phi \sim \frac{2\pi}{\beta}$. When x approaches $-\infty$ we can approximate $\phi(x,t)$ as,

$$\phi(x,t) \sim \frac{4}{\beta} \tan^{-1}\left[\exp\left(mx + \ln\left[\frac{2m}{w}\sin(wt)\right]\right)\right], \tag{5.38}$$

which looks like a soliton to the left. Similarly, it looks like an anti-soliton to the right. The soliton and anti-soliton move further apart as $\sin(wt)$ increases to one, and then when $\sin(wt)$ decreases they approach each other. As $\sin(wt) \to 0$ the approximation that lead to (5.38) is no longer valid, in accordance with the fact that in this region the soliton and anti-soliton are on top of each other. A similar discussion applies also for negative $\sin(wt)$. It is thus clear that the solution (5.37) describes an oscillation of a soliton anti-soliton pair around their common center of mass.

Revealing a bound state solution implies that the system must be attractive at least in a certain region of the "coupling constant". Indeed if one uses the coupling constant,

$$\xi = \frac{\pi\beta^2}{8\pi - \beta^2}, \tag{5.39}$$

then

$$\begin{aligned} \infty > \xi > \pi \quad & \textit{repulsive interaction} \\ \xi = \pi \quad & \textit{free particle} \\ \pi > \xi > 0 \quad & \textit{attractive interaction} \end{aligned} \tag{5.40}$$

As will be clarified in the next chapter, the case of $\xi = \pi$ corresponds to a free massive Dirac fermion. This will be further discussed in Section 6.1 as a bosonization of the free massive Dirac fermion. Also, the attractive region corresponds to positive coupling of a four-fermion interaction, namely attraction, while the repulsive region above corresponds to a negative coupling four-fermion. The case of negative ξ leads to no ground state.

If indeed the breather describes a bound state, it has to have a mass which is smaller than twice the mass of the soliton. It is easy to compute the classical energy associated with (5.37) at $t = 0$, since both the potential and the ϕ' vanish. Thus,

$$E^{(\text{clas})}_{\text{breather}} = \int \mathrm{d}x \left[\frac{1}{2}(\dot{\phi})^2\right] = \frac{8(\eta w)^2}{\beta^2} \int \mathrm{d}x \frac{1}{\cosh^2(\eta w x)} = 2M_{\text{sol}}\sqrt{1 - \left(\frac{w}{m}\right)^2}, \tag{5.41}$$

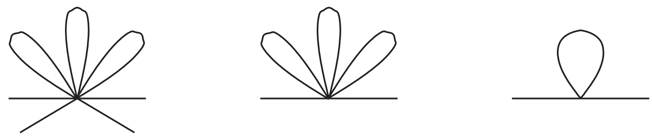

Fig. 5.1. Normal-order correction of the coupling (a), a correction to m^2 (b) and the lowest-order contribution to δm^2 (c).

where $M_{\rm sol}$ is the mass of the soliton. This verifies the existence of a binding energy since the mass of the breather is less than twice the mass of the soliton. In Section 5.5.1 the quantum description of the bound states will be addressed, and their scattering processes in Section 5.6.

5.5 Quantum solitons

The quantization of the soliton and breather was worked out in [74].

The classical mass of the soliton is $M = \frac{8m}{\beta^2}$, as in (5.27). Quantum mechanically this mass is corrected due to quantum fluctuations. We parametrize these space-time-dependent fluctuations as,

$$\phi_s \to \phi_s + \sum_n Re[\mathrm{e}^{iw_n t}\delta_n(x)]. \tag{5.42}$$

The normal modes $\delta_n(x)$ obey the equation,

$$[-\partial_x^2 + m^2 \cos(\beta\phi_s)]\delta_n(x) = w_n^2 \delta_n(x). \tag{5.43}$$

The leading order of the quantum mass is then given by

$$M_{\rm quantum} = M_s + \frac{1}{2}\sum_n w_n + V_{\rm ct}(\phi_s), \tag{5.44}$$

where $V_{\rm ct}(\phi_s)$ is a counter term that one has to add to the Lagrangian.

In two dimensions the only source of UV divergences in any order of perturbation theory are diagrams that contain a loop consisting of a single internal line. Stated differently, UV divergences are due to the fact that the action is not normal ordered. The corresponding diagrams are depicted in Figure 5.1. In fact the corrections (a) and (b) cancel and the only corrections follow from (c).

Let us first recall the normal ordering of $\phi^2(x)$, namely,

$$\phi^2(x) =: \phi^2(x) :_m + \frac{1}{4\pi}\int \frac{\mathrm{d}k}{\sqrt{k^2+m^2}} \tag{5.45}$$

where $: :_m$ indicates that the normal ordering is performed for a scalar of mass m. The last integral in obviously divergent so one introduces a cutoff $\Lambda \gg m$

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such that

$$\frac{1}{4\pi}\int_{-\Lambda}^{\Lambda}\frac{\mathrm{d}k}{\sqrt{k^2+m^2}} = \frac{1}{4\pi}\ln\frac{4\Lambda^2}{m^2} + O\left(\frac{m^2}{\Lambda^2}\right). \tag{5.46}$$

For a general potential $V(\phi)$, Wick's theorem tells us that,

$$V(\phi) =: \left[\mathrm{e}^{\frac{1}{8\pi}\ln\frac{4\Lambda^2}{m^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}}V(\phi)\right] :_m . \tag{5.47}$$

If one expands the exponent one finds a term with no contraction, next, with one contraction etc. We can pass from normal ordering at mass m to normal ordering at $\tilde{m}$. This transformation is independent of the cutoff, since

$$:\phi^2 :_m =: \phi^2 :_{\tilde{m}} + \frac{1}{4\pi}\ln\frac{m^2}{\tilde{m}^2}, \tag{5.48}$$

and hence

$$: V(\phi) :_m =: \left[\mathrm{e}^{\frac{1}{8\pi}\ln\frac{m^2}{\tilde{m}^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}}V(\phi)\right] :_{\tilde{m}} . \tag{5.49}$$

When applied to the sine-Gordon case, the normal-ordered potential takes the form,

$$\frac{m^2}{\beta^2} : [\cos(\beta\phi) - 1] :_m \frac{(m^2 - \delta m^2)}{\beta^2}[\cos(\beta\phi) - 1], \tag{5.50}$$

and where to the lowest order in β,

$$\delta m^2 = -\frac{m^2\beta^2}{4\pi}\int^{\Lambda}\frac{\mathrm{d}k}{\sqrt{k^2+m^2}}. \tag{5.51}$$

Thus the counterterm potential reads,

$$V_{\mathrm{counter}}(\phi) = -\frac{\delta m^2}{\beta^2}\int_{-\infty}^{\infty}\mathrm{d}x[(1-\cos(\beta\phi_s)] - E_{\mathrm{vacuum}}, \tag{5.52}$$

where we further subtracted the energy of the vacuum. Finally the quantum mass takes the form

$$M_{\mathrm{quantum}} = M_s + \frac{1}{2}\sum_n w_n - \frac{\delta m^2}{\beta^2}\int_{-\infty}^{\infty}\mathrm{d}x[(1-\cos(\beta\phi_s)] - \frac{1}{2}\sum_n\sqrt{k_n^2+m^2}, \tag{5.53}$$

where $k_n = \frac{2\pi}{L}$, with L the size of the quantization length, to be sent to ∞ at the end of the calculation.

When substituting the set of all the frequencies w_n associated with solutions of (5.43) one finds that the quantum mass is finite and reads,

$$M_{\mathrm{quantum}} = \frac{8m}{\beta^2} - \frac{m}{\pi} = \frac{m}{\xi} \tag{5.54}$$

up to corrections of order $m\beta^2$.

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5.5.1 Quantization of the breather

Next we discuss the quantization of the classical time-dependent breather solution. First, as a warm up exercise, compute the spectrum approximately, using the Bohr–Sommerfeld "old quantization procedure". Adapting this recipe to field theory states that a one parameter family of periodic fields characterized by the period $\tau = \frac{2\pi}{w}$ has an energy eigenstate whenever

$$\int_0^\tau \mathrm{d}t \int \mathrm{d}x \pi(x,t)\partial_0\phi(x,t) = 2\pi N, \tag{5.55}$$

where N is an integer.

Using the relation between the Hamiltonian and Lagrangian densities, $\mathcal{H} = \pi\partial_0\phi - \mathcal{L}$, we find after integrating over one period that,

$$E\tau = 2\pi N - \int_0^\tau \mathrm{d}t \int \mathrm{d}x\mathcal{L}. \tag{5.56}$$

By differentiating with respect to τ (with N varying as a function of it, by analytic continuation), and using the equations of motion we find,

$$\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{1}{w} = \frac{1}{m}\frac{1}{\sqrt{1-\frac{E^2}{4M^2}}}, \tag{5.57}$$

where the expression for w in terms of E and M follows from the calculation of the energy which can be performed most conveniently at $t = 0$, as was done in (5.41).

Integrating this equation and using a natural boundary condition that $N = 0$ for $E = 0$ the Bohr–Sommerfeld procedure predicts the following spectrum,

$$M_N = 2M_{\mathrm{sol}} \sin\left(\frac{N\beta^2}{16}\right) \quad N = 1, 2, \ldots, < \frac{8\pi}{\beta^2}. \tag{5.58}$$

Next we would like to describe the quantization procedure of Dashes, Hasslacher and Neveu (DHN). The classical action of the breather solution per period $\tau = \frac{2\pi}{w}$ is determined by substituting the breather solution (5.37) into the action and integrating,

$$S_{\mathrm{cl}}(\phi_b) = \frac{32\pi}{\beta^2}\left[\cos^{-1}\left(\frac{w}{m}\right) - \eta\right]. \tag{5.59}$$

The stability of the breather solution is determined by the requirement that there are no negative eigenmodes to the stability equation,

$$[\partial^\mu\partial_\mu + m^2\cos(\beta\phi_b)]\delta_n(x,t) = 0, \tag{5.60}$$

where $\delta_n(x,t)$, which obeys $\delta_n(x, t+\tau) = \mathrm{e}^{i\nu_n}\delta_n(x,t)$, is the fluctuation of the breather solution. The set of all the solutions of this equation was written down by DHN [74].

The corresponding spectrum of ν_n reads,

$$\nu_0 = 0 \quad \nu_1 = 0 \quad \nu_n = \frac{2\pi}{w}\sqrt{m^2 + q_n^2}, \tag{5.61}$$

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where q_n obeys the equations

$$Lq_n + f(q_n) = 2\pi n \quad f(q_n) = 4\tan^{-1}\left(\frac{\eta w}{q_n}\right), \tag{5.62}$$

and where L is the size of the space direction. The two vanishing frequencies are associated with the invariance under space and time translations.

The WKB semi-classical quantization determines the energy level of the breather solution via the conditions,

$$\begin{aligned} E_{\text{cl}}(\phi_b) + E_{\text{ct}}(\phi_b) - 1/2\sum_0^\infty \frac{w^2}{2\pi}\partial_w \nu_n &= E \\ S_{\text{cl}}(\phi_b) + S_{\text{ct}}(\phi_b) + \frac{2\pi E}{w(\phi_b)} - 1/2\sum_0^\infty \nu_n &= 2\pi N, \end{aligned} \tag{5.63}$$

where E_{ct} and S_{ct} are the energy and action associated with the counterterm. In the limit of $L \to \infty$ the sum over q_n turns into an integral. The integral has a quadratic as well as logarithmic divergences. These divergences will be cancelled out by the contribution of the counterterm such that $S_{\text{cl}}(\phi_b) + S_{\text{ct}}(\phi_b) - 1/2\sum_0^\infty \nu_n$ is the same as the $S_{\text{cl}}(\phi_b)$ given above with the renormalization of the coupling constant $\beta^2 \to \xi = \frac{\pi\beta^2}{8\pi - \beta^2}$. Using this result it is easy to determine the energy,

$$E = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(S_{\text{cl}}(\phi_b) + S_{\text{ct}}(\phi_b) - 1/2\sum_0^\infty \nu_n\right)\frac{2w\eta}{\xi} \tag{5.64}$$

Substituting this into the second equation of (5.63) one finds that the energy levels take the form,

$$M_N = \frac{2m}{\xi}\sin\left(\frac{N\xi}{2}\right) \quad N = 1, 2, \ldots, < \frac{\pi}{\xi}. \tag{5.65}$$

Note that in spite of the fact that the quantization condition permits any N, only if it is smaller than $\frac{\pi}{\xi}$ the classical breather solution exists. Thus the interpretation of this result is that there is a finite number of quantum bound states corresponding to the classical breather solution. Even though the derivation of the mass spectrum of the bound states was based on a Wentzel, Krames and Brillonin (WKB) approximation, the final result turns out to be **exact**. This statement follows from the analysis of the physical poles of soliton anti-soliton scattering, and already indicated in perturbation theory from two loops. The latter, though, works for mass ratios only, in view of scale dependence for the normal ordering of each individual mass.

The spectrum (5.65) can be re-written in terms of the mass of the quantum soliton. Using (5.54) this takes the form,

$$M_N = 2M_{\text{sol}}\sin\left(\frac{N\xi}{2}\right) \quad N = 1, 2, \ldots, < \frac{\pi}{\xi}, \tag{5.66}$$

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which indicates that the quantum breather states are indeed bound states of a quantum soliton anti-soliton pair.

At weak coupling $N\beta^2 \ll 1$, the mass spectrum reads,

$$M_N = Nm\left[1 - \frac{1}{6}\left(\frac{N\beta^2}{16}\right)^2 + O(N^2\beta^6)\right]. \tag{5.67}$$

Thus at weak coupling the lowest bound state has a mass of,

$$M_1 = m\left[1 - \frac{1}{6}\left(\frac{\beta^2}{16}\right)^2 + O(\beta^6)\right], \tag{5.68}$$

showing that first bound state is in fact the "elementary" boson of the theory. Moreover the higher bound states have a mass which is $NM_1 + O[(\beta^2)^2 N(1-N^2)]$, namely bound states of N elementary bosons. These bound states are loosely bound with a binding energy of $\frac{m}{6}(\frac{\beta^2}{16})^2 N(N^2-1)$. Using perturbation theory one can show that each of these states is stable against decay to states with lower N. In fact the stability turns out to be an exact statement. The source of the stability of these states is the set of infinitely conserved charges as will be discussed in the following section.

5.6 Integrability and factorized S-matrix

One of the first papers that discusses integrability of the S-matrix is the seminal paper [235]. We follow this paper in describing the basic notions of integrability. The Yang–Baxter relations were derived in [230] and [29] and S-matrix results for solitons and breathers of the sine-Gordon model are analyzed in [198].

Consider an integrable theory with ∞ of conserved charges Q^n diagonalized in the single particle base such that

$$Q_n|p^{(a)}>= w_n^{(a)}(p)|p^{(a)}>, \tag{5.69}$$

where p is the momentum of the particle and (a) denotes its type. For the sine-Gordon case the eigenvalues $w_n^{(a)}(p)$ are given by,

$$w_{2n+1}^{(a)}(p) = p^{2n+1}, \quad w_{2n}^{(a)}(p) = p^{2n}\sqrt{p^2+m_a^2}. \tag{5.70}$$

In general one assumes that the $w_n^{(a)}(p)$ form a set of independent functions. A multiple particle in or out state obeys

$$Q_n|p_1^{(a_1)}\dots p_k^{(a_k)}, \text{in}> = \sum_{i=1}^{k} w_n^{(a_i)}(p_i)|p_1^{(a_1)}\dots p_k^{(a_k)}, \text{in}>, \tag{5.71}$$

and since the charges Q_n are conserved one finds that,

$$\sum_{i\in\text{in}} w_n^{(a_i)}(p_i) = \sum_{i\in\text{out}} w_n^{(a_i)}(p_i). \tag{5.72}$$

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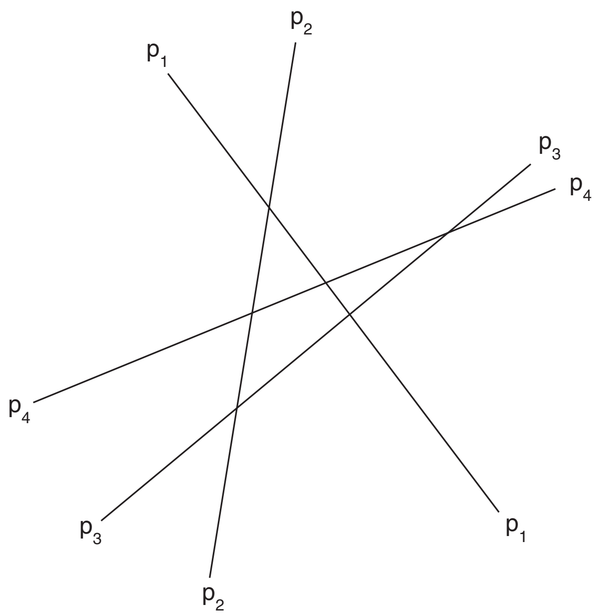

Fig. 5.2. Space-time picture of the multi-particle factorized scattering.

From this conservation one can deduce that,

- For any given mass m_a the number of initial and final particles of this mass is the same.
- The final set of momenta is the same as the initial one.

These two rules, that should apply also to intermediate states where particles are far enough from each other, together with the special kinematics of two dimensions, are behind the assertion that the multi-particle S-matrix of theories equipped with infinitely many conserved charges, can be expressed in terms of two-particle ones.

The factorized S-matrix corresponds to the following scattering process:

- In the infinite past a set of N particles with momenta $p_1 > p_2 > \ldots > p_N$ are spatially arranged in the opposite order, namely, $x_1 < x_2 < \ldots < x_N$.
- In the interaction region the particles collide in pairs. In each collision the momenta are conserved and in between collisions the particles move as free particles.
- The final state of the outgoing particles, achieved after $\frac{N(N-1)}{2}$ pair collisions, is built from the N particles arranged along the x coordinate in the order of increasing momenta.

The factorized scattering of the N particles is represented, for $N = 4$, by the space-time diagram Fig. 5.2, in which time is flowing up.

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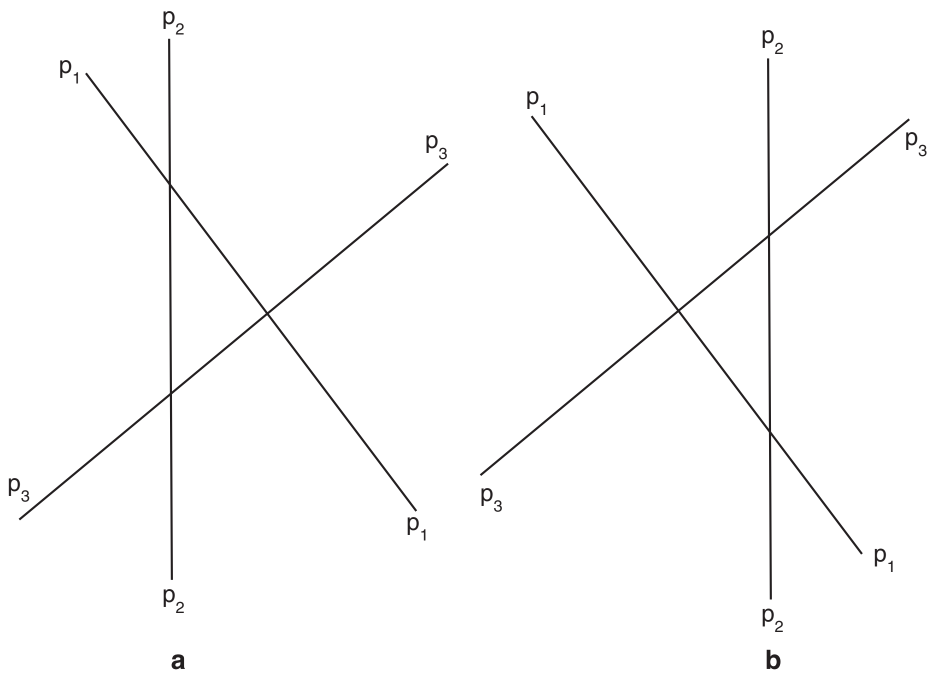

Fig. 5.3. Two possible ways of three-particle scattering.

- Each line corresponds to a given value of the momentum associated with the slope of the line.
- Each vertex corresponds to a two-particle collision. The two-particle amplitude $S_{ij}(p_i, p_j)$ has to be attached to each vertex.
- The total S-matrix element of the process is the product of all the $\frac{N(N-1)}{2}$ two-particle amplitudes $\prod_{ij} S_{ij}$, and then a sum over the different kinds of particles in the internal lines.

Take for example the case of $N = 3$. The same scattering can be represented in two ways, as is shown in Fig. 5.3. These two differ only by a parallel translation of a line, and thus represent the same process.

5.7 Yang–Baxter equations

The amplitudes and phases of the two diagrams should be the same. The requirement that they are indeed the same imposes cubic equations on the two-particle matrix elements which are called factorization equations or Yang–Baxter equations. We now proceed to analyze these conditions of factorization.

It is more convenient to discuss the S-matrix in terms of the rapidities of the massive particles. The rapidity β of a particle of mass m is defined via,[2]

$$p_{\pm} = m \exp(\pm\beta). \tag{5.73}$$

The scattering amplitude of a system of two particles $S(p_1, p_2)$ is a function of the rapidity difference $\beta = \beta_1 - \beta_2$ as can be seen from the fact that the s-channel invariant

$$s = (p_1 + p_2)^2, \tag{5.74}$$

is given by

$$s = m_1^2 + m_2^2 + 2m_1 m_2 \cosh(\beta_1 - \beta_2). \tag{5.75}$$

We now analytically continue s and define the amplitude $S(s)$ as an analytic function in the complex s-plane. This function has two cuts along the real axis for $s \geq (m_a + m_b)^2$ and $s \leq (m_a - m_b)^2$. The points $s = (m_a - m_b)^2$ and $s = (m_a + m_b)^2$ are square root branching points of $S(s)$. Using (5.75), $S(s)$ is mapped to $S(\beta)$, where $\beta = \beta_1 - \beta_2$. The physical sheet is mapped into the strip of $0 < \text{Im}\beta < \pi$, the branch cuts to the lines $\text{Im}\beta = 0$ and $\text{Im}\beta = \pi$, and the crossing transformation of $s \to 2m_1^2 + 2m_2^2 - s$ to $\beta \to i\pi - \beta$. If one assumes that there are no other cuts, then $S(\beta)$ is a meromorphic function in the above strip. In the non-relativistic limit, for $m_i \gg p_i^1$, the rapidity goes into the velocity $\beta_i \to v_i = \frac{p_i}{m_i}$.

5.8 The general solution of the S-matrix

Consider the two-particle S-matrix,

$$\begin{aligned} {}_{ik}S_{jl} &= \langle A_j(\beta_1')A_l(\beta_2'), out | A_i(\beta_1)A_k(\beta_2), in\rangle \\ &= \delta(\beta_1' - \beta_1)\delta(\beta_2' - \beta_2)[\delta_{ik}\delta_{jl}S_1(s) + \delta_{ij}\delta_{kl}S_2(s) + \delta_{il}\delta_{kj}S_3(s)] \\ &\quad \pm [i \leftrightarrow k, \beta_1 \leftrightarrow \beta_2], \end{aligned} \tag{5.76}$$

where $i, j, k, l = 1, 2$ so that the particles are in doublets of $O(2)$, and the $\pm$ refers to bosons $(+)$ and fermions $(-)$. This can be generalized to $O(N)$ in a straightforward way. Here we analyze only the case of the doublet. The amplitudes S_2, S_3 are the transition and reflection amplitudes, respectively, and S_1 corresponds to the process $A_i + A_i \to A_j + A_j$ for $(i \neq j)$. The $A_i(\beta)$, non-commutative

[2] Please note, that the β we had before, in the term $cos(\beta\phi)$ in the action, is not to be confused with the present one, which denotes the rapidity.

variables representing the particles, obey the relation,

$$A_i(\beta_1)A_j(\beta_2) = \delta_{ij}S_1(\beta)\sum_n A_n(\beta_2)A_n(\beta_1) + S_2(\beta)A_j(\beta_2)A_i(\beta_1) + S_3(\beta)A_i(\beta_2)A_j(\beta_1). \tag{5.77}$$

Incoming states are represented by products arranged by order of decreasing rapidities, while outgoing by increasing rapidities. The crossing symmetry relations are,

$$S_2(\beta) = S_2(i\pi - \beta) \quad S_1(\beta) = S_3(i\pi - \beta). \tag{5.78}$$

The unitarity conditions for the two-particle S-matrix are,

$$\begin{aligned} S_2(\beta)S_2(-\beta) \ &+ \ S_3(\beta)S_3(-\beta) = 1 \\ S_2(\beta)S_3(-\beta) \ &+ \ S_2(-\beta)S_3(\beta) = 0 \\ 2S_1(\beta)S_1(-\beta) \ &+ \ S_1(\beta)S_2(-\beta) + S_1(\beta)S_3(-\beta) + S_2(\beta)S_1(-\beta) \\ &+ \ S_3(\beta)S_1(-\beta) = 0. \end{aligned} \tag{5.79}$$

Unitarity (5.79) and crossing symmetry (5.78) do not fix the S-matrix. The additional conditions one has to impose are those of the factorization or Yang–Baxter equations. The latter are obtained by considering all possible three-particle in-products $A_i(\beta_1)A_j(\beta_2)A_k(\beta_3)$, reordering them to get out-products using (5.77), and requiring that the results be independent of the order of successions of the two particle commutations, one then finds,

$$\begin{aligned} S_2S_1S_3 + S_2S_3S_3 + S_3S_3S_2 &= S_3S_2S_3 + S_1S_2S_3 + S_1S_1S_2 \\ S_3S_1S_3 + S_3S_2S_3 &= S_3S_3S_1 + S_3S_3S_2 + S_2S_3S_1 \\ +S_2S_3S_3 + 2S_1S_3S_1 + S_1S_3S_2 &+ S_1S_3S_3 + S_1S_2S_1 + S_1S_1S_1, \end{aligned} \tag{5.80}$$

where, for each of the terms, the arguments for the three S factors are $\beta, \beta + \beta', \beta'$, respectively.

The general solution of the factorization equations is expressed in terms of one function which we take to be $S_2(\beta)$. The solution reads,

$$\begin{aligned} S_3(\beta) &= ictg\left(\frac{4\pi\delta}{\gamma}\right) cth\left(\frac{4\pi\beta}{\gamma}\right) S_2(\beta) \\ S_1(\beta) &= ictg\left(\frac{4\pi\delta}{\gamma}\right) cth\left(\frac{4\pi(i\delta - \beta)}{\gamma}\right) S_2(\beta) \end{aligned} \tag{5.81}$$

with γ and δ real, but so far arbitrary. This solution as well as the restriction from unitarity (5.79) is valid also for a non-relativistic system. Crossing symmetry is a restriction that shows up only in the relativistic case. Imposing the latter (5.78) on the general solution fixes $\delta = \pi$. A "minimum" solution for $S_2(\beta)$ then takes the form,

$$S_2(\beta) = \frac{2}{\pi}\sin\left(\frac{4\pi^2}{\gamma}\right) \text{sh}\left(\frac{4\pi\beta}{\gamma}\right) \sin\left(\frac{4\pi(i\pi - \beta)}{\gamma}\right) U(\beta), \tag{5.82}$$

where,

$$U(\beta) = \Gamma\left(\frac{8\pi}{\gamma}\right)\Gamma\left(1+i\frac{8\beta}{\gamma}\right)\Gamma\left(1-\frac{8\pi}{\gamma}-i\frac{8\beta}{\gamma}\right)\prod_{n=1}^{\infty}\frac{R_n(\beta)R_n(i\pi-\beta)}{R_n(0)R_n(i\pi)},$$
$$R_n(\beta) = \frac{\Gamma\left(2n\frac{8\pi}{\gamma}+i\frac{8\beta}{\gamma}\right)\Gamma\left(1+2n\frac{8\pi}{\gamma}+i\frac{8\beta}{\gamma})\right)}{\Gamma\left((2n+1)\frac{8\pi}{\gamma}+i\frac{8\beta}{\gamma}\right)\Gamma\left(1+(2n-1)\frac{8\pi}{\gamma}+i\frac{8\beta}{\gamma}\right)}. \tag{5.83}$$

It is a "minimal" in the number of singularities along the imaginary β axis, and more general solutions can be obtained from it by multiplying with a meromorphic function of the form $f(\beta) = \prod_{k=1}^{L}\frac{\mathrm{sh}\beta+i\sin\alpha_k}{\mathrm{sh}\beta-i\sin\alpha_k}$ for arbitrary real numbers α_k.

5.8.1 The S-matrix of the sine-Gordon model

The sine-Gordon model has a hidden $O(2)$ invariance, which is simplest to see via the soliton solutions, where the soliton and anti-soliton are incorporated in an $O(2)$ doublet. In terms of the $A_1(\beta)$ and $A_2(\beta)$ the soliton and anti-soliton amplitudes are,

$$A(\beta) = A_1(\beta) + iA_2(\beta) \quad \bar{A}(\beta) = A_1(\beta) - iA_2(\beta). \tag{5.84}$$

In terms of A and $\bar{A}$, (5.77) takes the form,

$$\begin{aligned} A(\beta_1)\bar{A}(\beta_2) &= S_T(\beta)\bar{A}(\beta_2)A(\beta_1) + S_R(\beta)A(\beta_2)\bar{A}(\beta_1) \\ A(\beta_1)A(\beta_2) &= S(\beta)A(\beta_2)A(\beta_1) \\ \bar{A}(\beta_1)\bar{A}(\beta_2) &= S(\beta)\bar{A}(\beta_2)\bar{A}(\beta_1), \end{aligned} \tag{5.85}$$

where $S(\beta), S_T(\beta)$ and $S_R(\beta)$ are the scattering amplitude of identical solitons, transition and reflection amplitude for soliton anti-soliton, which are related to $S_1(\beta), S_2(\beta)$ and $S_3(\beta)$ as,

$$\begin{aligned} S(\beta) &= S_3(\beta) + S_2(\beta) \\ S_T(\beta) &= S_1(\beta) + S_2(\beta) \\ S_R(\beta) &= S_1(\beta) + S_3(\beta). \end{aligned} \tag{5.86}$$

It follows from crossing symmetry (5.78) that,

$$S(\beta) = S_T(i\pi-\beta) \quad S_R(\beta) = S_R(i\pi-\beta). \tag{5.87}$$

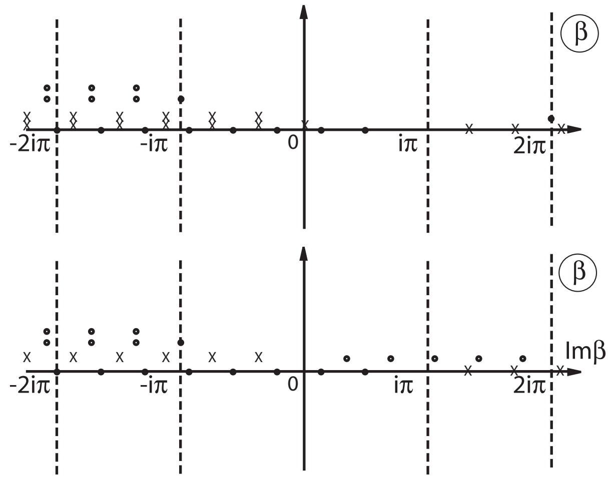

Fig. 5.4. The zeros (crosses) and poles (dots) of $S_T(\beta)$ (upper) and $S_R(\beta)$ (lower). All the singularities have pure imaginary values even though some of them are displaced from the imaginary axis for clarity [235].

Substituting these relations in the general solution derived in the previous section one finds for the SG model,

$$S_T(\beta) = -i\frac{\text{sh}\left(\frac{8\pi\beta}{\gamma}\right)}{\sin\left(\frac{8\pi^2}{\gamma}\right)}S_R(\beta)$$

$$S(\beta) = -i\frac{\text{sh}\left(\frac{8\pi(i\pi-\beta)}{\gamma}\right)}{\sin\left(\frac{8\pi^2}{\gamma}\right)}S_R(\beta), \tag{5.88}$$

where,

$$S_R(\beta) = \frac{1}{\pi}\sin\left(\frac{8\pi^2}{\gamma}\right)U(\beta). \tag{5.89}$$

The zeros and the poles of $S_T(\beta)$ and $S_R(\beta)$ are shown in Fig. 5.4.

The solution (5.88) is in fact the exact solution of the S-matrix of the SG model. This assertion is supported by the following properties:

- The poles of $S_T(\beta)$ are located at equidistance, and their values are in accordance with the semi-classical mass spectrum if one equates $\gamma = 8\xi$.
- Note that for the value of $\xi = \pi$ where the coupling of the associated Thirring model vanishes, and the SG model is a bosonized version of a free Dirac fermion,

$$S_T(\beta) \equiv S(\beta) = 1 \quad S_R(\beta) = 0. \tag{5.90}$$

- At $\xi \geq \pi$ all bound states, including the "elementary particle" associated with the field of the SG model, become unstable and the spectrum includes only soliton and anti-soliton. This situation follows from the fact that at this region

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the Thirring coupling is negative and there is a repulsion between the solition and anti-soliton.

- At $\xi = \frac{\pi}{n}$ the reflection amplitude vanishes identically.
- Expanding (5.88) in powers of $[(\frac{8\pi}{\gamma}) - 1]$, which means small coupling of the massive Thirring model, matches the perturbative expansion of the latter model.
- The limit $\beta^2 \to 0$ of the exact result (5.88) is equal to the semi-classical expression of the two-particle S-matrix.

The explicit expression for the two-particle S-matrix (5.88) enables one to also write down the S-matrix for any number of solitons and anti-solitons and the scattering of any number of bound states. This general approach to solving the S-matrix can be applied to the various integrable models. Here we demonstrate it on the sine-Gordon model. For soliton and anti-solitons we find the following S-matrix elements:

$$\begin{aligned}
&S^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\beta) = S^{-\frac{1}{2},-\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}}(\beta) = S(\beta)\\
&S^{\frac{1}{2},-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}(\beta) = S^{-\frac{1}{2},\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\beta) = -\frac{\mathrm{sh}\left(\frac{\pi}{\xi}\beta\right)}{\mathrm{sh}\left(\frac{\pi}{\xi}(\beta - \pi i)\right)} S(\beta)\\
&S^{\frac{1}{2},-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\beta) = S^{-\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}(\beta) = \frac{\mathrm{sh}\left(\frac{\pi^2 i}{\xi}\right)}{\mathrm{sh}\left(\frac{\pi}{\xi}(\beta - \pi i)\right)} S(\beta)\\
&S^{\epsilon_1',\epsilon_2'}_{\epsilon_1,\epsilon_2}(\beta) = 0 \quad \epsilon_1 + \epsilon_2 \neq \epsilon_1' + \epsilon_2'.
\end{aligned} \tag{5.91}$$

S can also be expressed as an exponential of an integral,

$$S(\beta) = -\exp\left[-i\int_0^\infty \frac{\sin(\kappa\beta)\,\mathrm{sh}\left(\frac{\pi-\xi}{2}\kappa\right)}{\kappa\mathrm{ch}\left(\frac{\pi\kappa}{2}\right)\mathrm{sh}\left(\frac{\xi\kappa}{2}\right)}\mathrm{d}\kappa\right]. \tag{5.92}$$

5.9 From conformal field theories to integrable models

So far we have analyzed integrable theories based on a scalar field theory with an integrable interacting potential. As was mentioned in Section 1 there is a more general scheme of constructing integrable models. This scheme is based on perturbing conformal field theories with relevant primary fields so that the action takes the form,

$$S = S_{\mathrm{CFT}} + \sum_i \lambda_i \int \mathrm{d}^2 z \Phi_i(z,\bar{z}). \tag{5.93}$$

Note that just as for the conformal field theories we use here complex two-dimensional coordinates. The $\Phi_i(z,\bar{z})$ are primary fields of conformal dimension $\Delta_i + \bar{\Delta}_i < 2$, namely relevant operators. Since these operators are

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super-renormalizable, they do not affect the short distance behavior but do affect the structure of the IR domain. In the analogy with statistical mechanical systems, where the CFT describes the behavior of the system at its fixed point, the perturbation with the relevant primary fields describes the scaling region around the fixed point.

A system described by an action of the form (5.93), is integrable provided that one can identify a set of infinitely many conserved charges just as for the systems described previously. An important class of such theories are the integrable minimal models, for example the tricritical Ising model $\mathcal{M}_{4,5}$ perturbed by $\Phi_{\frac{3}{5},\frac{3}{5}}$ and the tricritical Potts model $\mathcal{M}_{6,7}$ perturbed by $\Phi_{\frac{1}{7},\frac{1}{7}}$.

The renormalization group (RG) flow of the integrable systems follows a trajectory that starts at a fixed point and may end on another one in the IR, or on a point that corresponds to a massive QFT. An important property of these flows is the **c-theorem**[3] which states the following:

Quantum field theories which possess rotational invariance, reflection positivity, and conservation of the energy momentum tensor admit a function $c(\lambda_i)$ of the coupling constants λ_i which is non-increasing along the RG trajectories and is stationary only at fixed points.

The proof of the theorem is as follows. Consider the correlators of $T \equiv T_{zz}$ and $T_{z\bar{z}}$,

$$<T(z,\bar{z})T(0,0)> = \frac{F(z\bar{z})}{z^4}, \quad <T(z,\bar{z})T_{z\bar{z}}(0,0)> = \frac{G(z\bar{z})}{z^3\,\bar{z}},$$
$$<T_{z\bar{z}}(z,\bar{z})T_{z\bar{z}}(0,0)> = \frac{H(z\bar{z})}{z^2\,\bar{z}^2}. \tag{5.94}$$

We now use the conservation law,

$$\bar{\partial}T + \partial T_{z\bar{z}} = 0, \quad \partial\bar{T} + \bar{\partial}T_{z\bar{z}} = 0, \tag{5.95}$$

to deduce the following differential equations for F, G and H,

$$\dot{F} + (\dot{G} - 3G) = 0 \quad \dot{G} - G + \dot{H} - 2H = 0, \tag{5.96}$$

where $\dot{A} = \frac{\mathrm{d}A(x)}{\mathrm{d}\log x}$. Since the positivity condition implies that $H \geq 0$, the following c function is non-increasing,

$$c = 2F - 4G - 6H, \quad \dot{c} = -12H. \tag{5.97}$$

At the fixed points $T_{z\bar{z}} = 0$, hence $G = H = 0$ and $c = 2F$, as indeed it should be. Recall that the OPE in CFT is of the form $T(z\bar{z})T(0,0) = \frac{(\frac{c}{2})}{z^4} + \ldots$

One can further write down an expression for the integral of $\dot{c}$, namely for the difference of c in the UV and IR regions. For the case of a perturbation with a single operator Φ, the trace of the energy-momentum tensor is

[3] Zamolodchikov c-theorem was derived in [232].

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$T_{z\bar{z}} = \pi\lambda(1-\Delta)\Phi$, where the total conformal dimension is 2Δ. Using (5.97) it was shown that,

$$c_{\rm UV} - c_{\rm IR} = 12\pi\lambda^2(1-\Delta)^2 \int \mathrm{d}^2x|x|^2 <\Phi(x)\Phi(0)> . \tag{5.98}$$

This result has been applied to integrable minimal models yielding the correct difference in the Virasoro anomalies. Another example where this relation between the conformal data and the properties of the non-conformal theory can be tested is the sine-Gordon model. The model can be thought of as a perturbation on a free massless scalar field which has $c = 1$, and the massive model has $c = 0$. Let us check for this case the outcome of the relation (5.97). The perturbation now is,

$$\lambda\Phi = \frac{m^2}{\beta^2} : (\cos\beta\phi - 1) : \quad 2\Delta = \frac{\beta^2}{4\pi}. \tag{5.99}$$

If we expand $\lambda\Phi$ in β, then the leading order $\lambda\Phi$ is $\frac{1}{2}m^2\phi^2$ for which $\Delta = 0$ and,

$$<\lambda\Phi(x)\lambda\Phi(0)>= \frac{1}{4}m^2 <\phi(x)\phi(0)>^2= \frac{m^4}{8\pi^2}K_0^2(m|x|), \tag{5.100}$$

where K_0 is a Bessel function. Inserting this into (5.97) we get,

$$c_{\rm UV} - c_{\rm IR} = 3\pi\frac{m^4}{2\pi^2}\int \mathrm{d}^2x|x|^2K_0^2(m|x|) = 3\int_0^\infty \mathrm{d}r r^3K_0^2(r) = 1, \tag{5.101}$$

which verifies our data about the Virasoro anomalies at the two ends of the trajectory. One can further show that higher-order terms in β are in accordance with the fact that $c_{\rm UV} - c_{\rm IR}$ is β independent.

5.10 Conserved charges and classical integrability

We will now show that the classical sine-Gordon theory incorporates an infinite set of conserved charges. This will imply an exact determination of any S-matrix element of the theory. This property of having a set of an infinite number of conserved currents and charges is referred to as classical integrability. In the next section we will discuss the fate of the integrability in the quantum domain. We choose here to describe the sine-Gordon model, however this structure applies to a class of models.

There are several methods to determine these classically conserved charges. Here we will follow two of them: The Lax pair approach and a method based on a generating function.

The infinitely many charges of the integrable models were analyzed using various different techniques. The Lax pair method applied to the sine-Gordon is described in [61]. The method of multilocal charges from an integral equation was presented in [153]. The inductive method was introduced in [45].

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5.10.1 The Lax pair method

In the Lax pair approach the idea is to rewrite the sine-Gordon equation of motion in terms of a commutator relation between two operators. Let us first rewrite the equation of motion of the sine-Gordon system in the light-cone coordinates,

$$\tilde{\partial}_+ \tilde{\partial}_- \tilde{\phi} = -\sin(\tilde{\phi}), \tag{5.102}$$

where $\tilde{\phi} = \beta\phi$ and $\tilde{\partial}_\pm = m\partial_\pm$.

Next we define the following pair of 2×2 matrix Lax operators,

$$\begin{aligned} L &= 2\sigma_3 \tilde{\partial}_- + \sigma_2(\tilde{\partial}_- \tilde{\phi}) \\ B &= \frac{1}{2}[\sigma_3 \cos(\tilde{\phi}) + \sigma_2 \sin(\tilde{\phi})]L^{-1}. \end{aligned} \tag{5.103}$$

In terms of these operators the sine Gordon equation takes the form,

$$\tilde{\partial}_+ L = [L, B]. \tag{5.104}$$

The reason for rewriting the equation of motion in this form is the fact that the spectrum of L is conserved. To realize this property of the Lax pair, notice first that the solution of (5.104) can be parameterized as,

$$L(x^+) = S(x^+)L(0)S^{-1}(x^+) \quad \partial_+ S = -BS. \tag{5.105}$$

Now consider the eigenvalue problem,

$$L(x^+)v(x^+) = \lambda v(x^+). \tag{5.106}$$

It is easy to check that if $v(0)$ is an eigenfunction of $L(0)$ with eigenvalue λ, then $v(x^+)$ is an eigenfunction of $L(x^+)$ with the same eigenvalue λ, where,

$$v(x^+) = S(x^+)v(0). \tag{5.107}$$

Since x^+ is the light-cone time direction, this implies the conservation of the eigenvalues. This conservation of the spectrum is the origin of the infinite set of conserved charges.

For the case that L can be represented by a finite matrix, it is obvious from (5.105), using the cyclicity of the trace, that $Q^n = Tr[L^n]$ are conserved charges. In general, and in particular in our case, L does not act only in a space of finite-dimensional matrices but also in the continuous space whose base vectors are $|x_-\rangle$. Thus the trace takes the form of the integral $\int_{-\infty}^{\infty} \mathrm{d}x_-$. Due to the unitarity of $S(x^+)$ the cyclicity property of the trace is maintained and hence $Q^n = Tr[L^n]$ are indeed conserved charges.

It turns out that one can map the Lax pair of the sine-Gordon system to that of the Korteweg-deVries (KdV) equation $\partial_0 u(x,t) = 6u\partial_1 u - \partial_1^3 u$. This map is useful since it is more convenient to express the conserved charges of the sine-Gordon system in terms of the L operator of the KdV equation. In this format

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the first four charges of the set of infinite charges which are classically conserved are,

$$Q_1 = -\left(\frac{1}{4}\right)\int_{-\infty}^{\infty}(\partial_-\tilde{\phi})^2\mathrm{d}x_-$$
$$Q_2 = +\left(\frac{1}{4}\right)^2\int_{-\infty}^{\infty}[(\partial_-\tilde{\phi})^4 - 4(\partial_-^2\tilde{\phi})^2]\mathrm{d}x_-$$
$$Q_3 = -\left(\frac{1}{4}\right)^3\int_{-\infty}^{\infty}[(\partial_-\tilde{\phi})^6 - 20(\partial_-\tilde{\phi})^2(\partial_-^2\tilde{\phi})^2 + 8(\partial_-^3\tilde{\phi})^2]\mathrm{d}x_-$$
$$Q_4 = +\left(\frac{1}{4}\right)^4\int_{-\infty}^{\infty}[(\partial_-\tilde{\phi})^8 - \frac{112}{5}(\partial_-^2\tilde{\phi})^4 - 56(\partial_-\tilde{\phi})^4(\partial_-^2\tilde{\phi})^2 + \frac{224}{5}(\partial_-\tilde{\phi})^2(\partial_-^3\tilde{\phi})^2 - \frac{64}{5}(\partial_-^4\tilde{\phi})^2]\mathrm{d}x_-. \quad (5.108)$$

5.10.2 The generating function method

A second method to determine the set of infinite conserved charges is based on a generating function. Define the generating function,

$$\psi = \phi + \frac{1}{\beta}\sin^{-1}(\epsilon\beta\partial_-\psi). \quad (5.109)$$

When ϕ obeys the sine-Gordon equation, ψ obeys the following equation,

$$\partial_+\left(\frac{1-\sqrt{1-\beta^2\epsilon^2(\partial_-\psi)^2}}{\epsilon^2}\right) - m^2\partial_-\left(\cos(\beta\psi)-1\right). \quad (5.110)$$

Equation (5.109) determines ψ as a power series in ϕ and ϵ. Upon substituting this into (5.110), we get an infinite set of conserved charges, that are the coefficients of the even powers of ϵ. A dual sequence of charges can be obtained by interchanging ∂_+ and ∂_- in the equations above. The set of conserved currents is labeled as,

$$\partial_+ J^+_{2n} + \partial_- J^-_{2n} = 0, \quad (5.111)$$

where $2n$ relates to the power of ϵ. The lowest-order current is the energy-momentum tensor,

$$J_0^+ \equiv T_{--} = \frac{1}{2}(\partial_-\phi)^2, \quad J_0^- \equiv T_{+-} = -\frac{m^2}{\beta^2}(\cos(\beta\phi)-1). \quad (5.112)$$

The second-order currents are,

$$J_2^+ = \frac{1}{2}(\partial_-^2\phi)^2 - \frac{\beta^2}{8}(\partial_-\phi)^2, \quad J_2^- = \frac{m^2}{2}(\phi_-)^2\cos(\beta\phi), \quad (5.113)$$

and similarly one can write the expressions for the higher order currents J_{2n}.

One can map the charges derived by the Lax pair procedure to those derived by the generating function method.

5.11 Multilocal conserved charges

In the previous section we analyzed the set of infinitely many conserved charges associated with local currents. It will be shown later that these conservation laws are responsible for the fact that the system is integrable, namely there is no particle production and the S-matrix is factorizable. We would like to show now that this type of structure may also follow from conservation laws associated with multilocal currents. The construction of the multi-local currents will be presented in two ways: (i) via an integral equation, (ii) by an inductive procedure.

We then show that the two types of charges are in fact equivalent.

5.11.1 Multilocal charges from integral equation

Consider first the $O(N)$ non-linear sigma model defined by the Lagrangian density,

$$S_{O(N)} = \frac{1}{2g_0^2}\int \mathrm{d}^2x \left[(\partial_+\vec{n})\cdot(\partial_-\vec{n}) - u(\vec{n}^2-1)\right], \tag{5.114}$$

where $\vec{n}$ is an N-dimensional real vector and g_0 is the coupling constant. The fact that the field $\vec{n}$ is constrained $\vec{n}^2 = n_i n^i = 1$ $i = 1,\ldots,N$ is incorporated via the Lagrange multiplier u. The equations of motion that follow from this action are,

$$\partial_-\partial_+\vec{n} + u\vec{n} = 0, \quad \vec{n}^2 = 1. \tag{5.115}$$

So u is actually the action density $u = (\partial_+\vec{n})\cdot(\partial_-\vec{n})$. Instead we can write the equation of motion as,

$$\partial_\mu\partial^\mu\vec{n} + \vec{n}(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}) = 0. \tag{5.116}$$

The action is classically invariant under $O(N)$ global symmetry, by construction. For the special case of $n = 3$, namely $O(3)$, the current takes the form,

$$j_\mu^k = \epsilon^{ijk}(n^i\partial_\mu n^j - \partial_\mu n^i n^j). \tag{5.117}$$

It is easy to check, using the equations of motion, that this current is indeed conserved. In addition the energy-momentum is conserved,

$$\partial_+T_{--} = \partial_+\left[\frac{1}{2}(\partial_-\vec{n})^2\right] = \partial_-\vec{n}\partial_-\partial_+\vec{n} = -u\partial_-\left[\frac{1}{2}(\vec{n})^2\right] = 0. \tag{5.118}$$

Thus classically the trace of the energy-momentum tensor vanishes $T_{+-} = 0$.

In addition there is an infinite set of currents, the simplest of which takes the form,

$$J_- = \frac{1}{2|\partial_-\vec{n}|}\left[\partial_-\left(\frac{\partial_-\vec{n}}{|\partial_-\vec{n}|}\right)\right]^2, \quad J_+ = -\frac{u}{|\partial_-\vec{n}|}. \tag{5.119}$$

An alternative way to write down a set of non-local classically conserved charges is the following. We define at any given t the (2×2) $U(t,x)$ operator via the equation,

$$\partial_x U(t,x) = i\frac{w}{1-w^2}\left[j_1^i - wj_0^i\right]\sigma^i U(t,x), \tag{5.120}$$

and the boundary condition $U(t,-\infty)=1$, for any Cauchy data $n^i(t,x), \partial_t n^i(t,x)$, where w is the "spectral parameter" which is a complex parameter with $w \neq \pm 1$. Following this definition and using the equations of motion one can show that,

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(w) \equiv \frac{\mathrm{d}}{\mathrm{d}t}U(t,\infty) = 0. \tag{5.121}$$

If one expands Q in terms of the spectral parameter one finds a set of infinitely many conserved charges,

$$Q(w) = \sum_{n=0}^{\infty} Q_n w^n \qquad \frac{\mathrm{d}}{\mathrm{d}t}Q_n = 0. \tag{5.122}$$

We can now rewrite the differential equation (5.120) in terms of the integral equation,

$$U(t,x) = 1 + \frac{w}{1-w^2}\int_{-\infty}^{x}\mathrm{d}y\left[j_0^i - wj_1^i\right](t,y)\sigma^i U(t,y). \tag{5.123}$$

Inserting the expansion,

$$U(t,x) = \sum_{n=0}^{\infty} U_n w^n, \tag{5.124}$$

into the integral equation for $U(t,x)$, we find the following recurrence relation,

$$\begin{aligned} U_n(t,x) = i\int_{-\infty}^{x}\mathrm{d}y\Bigg[& j_0^i(t,y)\sigma^i \sum_{0\le k\le \frac{(n-1)}{2}} U_{n-2k-1}(t,y) \\ & - j_1^i(t,y)\sigma^i \sum_{1\le l\le \frac{(n)}{2}} U_{n-2l}(t,y)\Bigg], \end{aligned} \tag{5.125}$$

with $U_0(t,x)=1$. Thus we can calculate Q_n recursively deriving explicit non-local expressions for the set of infinitely many conserved charges. The "lowest" charges are given by,

$$\begin{aligned} Q_1^0 &= 0 \qquad Q_1^i = \int \mathrm{d}y j_0^i(t,y) \\ Q_2^0 &= -\frac{1}{2}Q_1^i Q_1^i \\ Q_2^i &= \int \mathrm{d}x \int \mathrm{d}y\mathrm{d}y' \epsilon_{ijk} j_0^j(t,y) j_0^k(t,y')\theta(y-y') - \int \mathrm{d}y j_1^i(t,y). \end{aligned} \tag{5.126}$$

The algebraic structure associated with these charges is the Yangian symmetry, the description of which is beyond the scope of this book. In the reference list we mention several that deal with these algebras. In Section 5.12 it will be further shown that both the charges of the form (5.119) as well as (5.126) are quantum mechanically conserved.

5.11.2 Charges by inductive procedure

The second method of obtaining non-local currents is as follows. Assume that the system admits a (non-abelian) conserved current which is also a pure gauge, namely,

$$J_\mu = g^{-1}\partial_\mu g, \quad \partial^\mu J_\mu = 0, \tag{5.127}$$

where g is a non-singular matrix (for instance $U(N)$ or $O(N)$ matrix). It follows that the current J is a flat gauge connection, since for $D_\mu = \partial_\mu + J_\mu$ we find that $[D_\mu, D_\nu] = 0$ and also $\partial_\mu D_\mu = D_\mu \partial_\mu$. Using the terminology of differential forms these properties can be rewritten as,

$$DJ \equiv \mathrm{d}J + J \wedge J = 0 \quad d * J = 0, \tag{5.128}$$

together with $[D, D] = 0$ and $D * d = d * D$. Now let us assume that there is an n-th conserved current $J_\mu^{(n)}$, then there is a function $\chi^{(n)}$ such that,

$$J_\mu^{(n)} = \epsilon_{\mu\nu}\partial^\nu \chi^{(n)}. \tag{5.129}$$

Then there is an (n+1)-th conserved current,

$$J_\mu^{(n+1)} = D_\mu \chi^{(n)}, \quad \partial^\mu J_\mu^{(n+1)} = 0. \tag{5.130}$$

The conservation follows easily,

$$d * J^{(n+1)} = d * D\chi^{(n)} = D * d\chi^{(n)} = DJ^{(n)} = DD\chi^{(n-1)} = 0, \tag{5.131}$$

or directly,

$$\partial^\mu J_\mu^{(n+1)} = D^\mu \partial_\mu \chi^{(n)} = -\epsilon^{\mu\nu} D_\mu J_\nu^{(n)} = -\epsilon^{\mu\nu} D_\mu D_\nu \chi^{(n-1)} = 0. \tag{5.132}$$

The sequence of $\chi^{(n)}$ starts with $\chi^{(0)} = 1$ and $J_\mu^{(1)} = J_\mu$.

Associated with the set of infinitely many conserved currents, there is obviously also a set of infinite number of conserved charges,

$$Q^{(n)} = \int \mathrm{d}x J_0^{(n)}(t, x). \tag{5.133}$$

Consider the special case $n = 2$,

$$\begin{aligned} Q^{(2)}(t) &= \int \mathrm{d}x J_0^{(2)} = \int \mathrm{d}x(\partial_0 + j_0^{(1)})\chi^{(1)} \\ &= -\int \mathrm{d}x j_1^{(1)}(t, x) + \int \mathrm{d}x j_0^{(1)}(t, x)\chi^{(1)}(t, x). \end{aligned} \tag{5.134}$$

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We can now re-express $\chi^{(1)}$ as $\chi^{(1)}(t,x) = \int dx' j_0^{(1)}(t,x')$. When substituting this to the previous equation we discover the structure of multi-local charges of (5.126).

5.12 Quantum integrable charges in the $O(N)$ model

In Section 5.11 it was demonstrated that certain two-dimensional interacting models have an infinite set of conserved classical charges. This property constitutes the classical integrability of a given system. A natural question to ask is whether this integrability persists also in the quantum regime. We will analyze this question in the context of two models, the $O(N)$ sigma model in this section and the sine-Gordon model in the following one. In fact we have already seen in Section 5.8 how the quantum integrability of the sine-Gordon model fully determines the S-matrix of the theory.[4]

In Section 5.11.1 it was shown that classically the $O(N)$ model is scale invariant as the trace of the energy-momentum vanishes. Following our discussion of the Virasoro anomaly in Chapter 2, it is clear that quantum mechanically the classical conformal invariance of the $O(N)$ model is broken by an anomaly. The right-hand side of (5.118) should not vanish any more but rather be equal to some local terms. These terms can be determined by Lorentz $x^+ \to ax^+,\ x^- \to a^{-1}x^-$ and scale invariance $x^+ \to ax^+,\ x^- \to ax^-$. It turns out that up to a constant the quantum relation is,

$$\partial_+ \left[\frac{1}{2}(\partial_-\vec{n})^2\right] = \hat{\beta}\partial_- u = \hat{\beta}\partial_-(\partial_-\vec{n}\partial_+\vec{n}). \tag{5.135}$$

In fact one can show that the constant $\hat{\beta}$ is the one-loop beta function.

Consider now the next conservation law. Classically it reads,

$$\partial_- \left[\frac{1}{2}u(\partial_-\vec{n})^2\right] + \partial_+ \left[\frac{1}{2}(\partial_-^2\vec{n})^2\right] = \frac{3}{2}(\partial_-\vec{n})^2(\partial_- u). \tag{5.136}$$

One can show that this classical conservation is directly related to the conservation of the current in (5.119). For this make use of the classical scale invariance $x_- \to f(x_-)$ and the classical conservation $\partial_+(\partial_-\vec{n})^2 = 0$, to choose a gauge where $(\partial_-\vec{n})^2 = 1$. Now the classical conservation law takes the form,

$$\partial_+ \left[\frac{1}{2}(\partial_-^2\vec{n})^2\right] = (\partial_- u). \tag{5.137}$$

This is exactly the conservation of the current (5.119), when inserting the gauge $(\partial_-\vec{n})^2 = 1$. To get to the form in general coordinates, namely the form as in

[4] Quantum integrability was discussed in [176].

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(5.119), make the substitution,

$$\partial_- \to \frac{1}{|\partial_- \vec{n}|}\partial_-, \tag{5.138}$$

noting that it also implies $u \to \frac{u}{|\partial_- \vec{n}|}$.

Quantum mechanically again the right-hand side of equation (5.136) can be corrected by several terms. To eliminate possible terms that are total derivatives it is convenient to analyze the integrated form of this conservation law,

$$\partial_+ \int \mathrm{d}x^- \left[\frac{1}{2}(\partial_-^2 \vec{n})^2\right] = (3+\gamma)\int \mathrm{d}x^- \frac{1}{2}(\partial_- \vec{n})^2(\partial_- u), \tag{5.139}$$

and finally using (5.135) to eliminate $\partial_- u$ we get the quantum conservation law,

$$I = \int \mathrm{d}x^- \left[\frac{1}{2}(\partial_-^2 \vec{n})^2 - \frac{(3+\gamma)}{4\hat{\beta}}(\partial_- \vec{n})^4\right], \quad \partial_+ I = 0. \tag{5.140}$$

One can now show that this conservation law implies the conservation of $\sum\limits_i P_-{}_i^3$. Since I commutes with the S matrix $[I, S] = 0$ one has,

$$<b \text{ out}|I|b \text{ out}> \; <b \text{ out}|a \text{ in}> \; = \; <b \text{ out}|a \text{ in}> \; <a \text{ in}|I|a \text{ in}> . \tag{5.141}$$

For asymptotic states with N particles and momenta $P_1, \ldots, P_N$ we have

$$<N|I|N> = \text{Constant} \sum_i P_-{}_i^3. \tag{5.142}$$

The reason that the conserved charge on an asymptotic state has to be proportional to $\sum_i P_-{}_i^3$ is that its tensorial structure is of the form $_{---}$, the only conserved quantity with $_-$ Lorentz index is P_- and there are no higher tensorial charges that are not products of P_-.

In a similar manner one has a similar conservation law for P_+ so that,

$$\sum_{\text{in}} P_-^3 = \sum_{\text{out}} P_-^3 \quad \sum_{\text{in}} P_+^3 = \sum_{\text{out}} P_+^3. \tag{5.143}$$

If we now write these conservation laws for a $2 \to N$ process combined with the ordinary conservation of momenta we get four equations for two quantities which, combined with the analyticity of the S-matrix, implies that there cannot be any multiple production, and the only allowed process is $2 \to 2$.

It can also be shown that the classically conserved charges constructed by an integral equation (5.126) are also quantum mechanically conserved.

5.13 Non-local charges and quantum groups

The discussion of quantum groups and non-local charges follows the paper [39]. For a review see, for instance, [114].

In the previous sections we have derived in various forms sets of infinitely many conserved charges and argued that they constitute the integrability of the corresponding models. In particular we described in Section 5.11 non-local charges. Here we will establish the algebraic structure of these charges. It will be shown that they involve non-trivial braiding and that they obey the algebra associated with "quantum groups".

Rather than discussing the generalities of this algebraic structure we analyze it in the context of our laboratory model, the sine-Gordon model. We now rewrite the Lagrangian density of the model as,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z\partial\phi\bar{\partial}\phi + \frac{\hat{\lambda}}{\pi}\int \mathrm{d}^2 z : \cos(\hat{\beta}\phi) : . \tag{5.144}$$

It is straightforward to relate $\hat{\lambda}$ of this formulation to m, β of (5.23), and $\hat{\beta}$ here equals β there.

Recall (Section 5.9) that this action can be considered as a conformal field theory plus a relevant perturbation of the form (5.93). In such a case one can identify a conserved current that obeys the relation,

$$\bar{\partial}J(z,\bar{z}) = \partial H(z,\bar{z}) \quad \partial\bar{J}(z,\bar{z}) = \bar{\partial}\bar{H}(z,\bar{z}), \tag{5.145}$$

where for the conformal limit one has $\bar{\partial}J = \partial\bar{J} = 0$, and $H, \bar{H}$ are defined via,

$$\mathrm{Res}_{z=w}(\phi_{\mathrm{pert}}(w)J^a(z)) = \partial h^a(z) \quad H^a(z,\bar{z}) = 2\hat{\lambda}h^a(z)\bar{\phi}_{\mathrm{pert}}(\bar{z}), \tag{5.146}$$

where the perturbation term is written, in the conformal limit, as $\phi_{\mathrm{pert}}(z)\bar{\phi}_{\mathrm{pert}}(\bar{z})$, and all this under the condition that the Res above are indeed total derivatives. A similar construction applies also for the anti-holomorphic current.

We can now identify a pair of non-local fields

$$\tilde{\phi}_{\pm}(t,x) = \frac{1}{2}\left[\phi(t,x) \pm \int_{-\infty}^{x} \mathrm{d}y\partial_0\phi(t,y)\right], \tag{5.147}$$

with which we can write a pair of conserved currents $J_{\pm}$ of the form (5.145) as,

$$\begin{aligned} J_{\pm} &= \mathrm{e}^{\pm\frac{2i}{\hat{\beta}}\tilde{\phi}_{+}(t,x)} \\ H_{\pm} &= \lambda\frac{\hat{\beta}^2}{\hat{\beta}^2-2}\mathrm{e}^{\pm i\left(\frac{2}{\hat{\beta}}-\hat{\beta}\right)\tilde{\phi}_{+}(t,x)\mp\frac{i}{\hat{\beta}}\tilde{\phi}_{-}(t,x)}. \end{aligned} \tag{5.148}$$

The conserved charges associated with the pair of currents are

$$Q_{\pm} = \frac{1}{2\pi i}\left(\int \mathrm{d}zJ_{\pm} + \int \mathrm{d}\bar{z}H_{\pm}\right) \quad \bar{Q}_{\pm} = \frac{1}{2\pi i}\left(\int \mathrm{d}\bar{z}\bar{J}_{\pm} + \int \mathrm{d}z\bar{H}_{\pm}\right). \tag{5.149}$$

The charges $Q_{\pm}$ are non-local, as a consequence of being built from the non-local field $\tilde{\phi}$.

Using the basic canonical commutation relation $[\phi(t,x), \partial_0\phi(t,y)] = 4\pi i\delta(x-y)$ and $e^A e^B = e^{[A,B]}e^B e^A$ (when [A,B] commutes with A and B) one finds the following braiding relations,

$$\begin{aligned} J_{\pm}(t,x)\bar{J}_{\mp}(t,y) &= \frac{1}{q^2}\bar{J}_{\mp}(t,y)J_{\pm}(t,x) \\ J_{\pm}(t,x)\bar{J}_{\pm}(t,y) &= q^2\bar{J}_{\pm}(t,y)J_{\pm}(t,x), \end{aligned} \tag{5.150}$$

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where,

$$q = \mathrm{e}^{-\frac{2\pi i}{\hat{\beta}^2}}. \tag{5.151}$$

These non-trivial braiding relations of the currents imply similar relations for the conserved charges,

$$\begin{aligned}
Q_+\bar{Q}_+ - q^2\bar{Q}_+Q_+ &= 0\\
Q_-\bar{Q}_- - q^2\bar{Q}_-Q_- &= 0\\
Q_+\bar{Q}_- - \frac{1}{q^2}\bar{Q}_-Q_+ &= a(1-q^{2Q_{\text{top}}})\\
Q_-\bar{Q}_+ - \frac{1}{q^2}\bar{Q}_+Q_- &= a(1-q^{-2Q_{\text{top}}})\\
[Q_{\text{top}}, Q_\pm] = \pm 2Q_\pm \quad [Q_{\text{top}}, \bar{Q}_\pm] &= \pm 2\bar{Q}_\pm,
\end{aligned} \tag{5.152}$$

where $a = \frac{\lambda}{2\pi i}\gamma^2$, $\gamma^{-1} = \Delta = -\bar{\Delta}(Q_\pm)$ and the topological charge $Q_{\text{top}} = \frac{\hat{\beta}}{2\pi}(\phi(x=\infty) - \phi(x=-\infty))$ (compare with (5.6)).

This algebra of the charges is referred to as "q-deformation" $\hat{S}L_q(2)$ of the $SL(2)$ affine Lie algebra with zero center. Recall the basic $SL(2)$ algebra in the Chevaley basis (3.5),

$$[H, E_\pm] = \pm 2E_\pm \quad [E_+, E_-] = H. \tag{5.153}$$

Introducing the spectral parameter w the infinitely many generators of the $SL(2)$ affine Lie algebra are defined via $J^a = \sum_n J^a_n w^n$ with $J^a = H, E_\pm$. We then define the Chevaley basis of the affine algebra $S\hat{L}(2)$ as,

$$\begin{aligned}
E_{+1} &= wE_+ \quad & E_{-1} &= w^{-1}E_-\\
E_{+0} &= wE_- \quad & E_{-0} &= w^{-1}E_+\\
H_1 &= H & H_0 &= -H.
\end{aligned} \tag{5.154}$$

In terms of these generators the $\hat{S}L_q(2)$ algebra reads,

$$\begin{aligned}
[H_i, E_{+j}] &= a_{ij}E_{+j}\\
[H_i, E_{-j}] &= -a_{ij}E_{-j}\\
[E_{+i}, E_{-j}] &= \delta_{ij}E_{-j}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}},
\end{aligned} \tag{5.155}$$

and a_{ij} is the Cartan matrix of $SL(2)$.

The relations between the non-local charges $Q_\pm$ and $\bar{Q}_\pm$ and the generators of the $\hat{S}L_q(2)$ algebra are,

$$\begin{aligned}
Q_+ &= cE_{+1}q^{\frac{H_1}{2}} \quad & Q_- &= cE_{+0}q^{\frac{H_0}{2}}\\
\bar{Q}_- &= cE_{-1}q^{\frac{H_1}{2}} \quad & \bar{Q}_+ &= cE_{-0}q^{\frac{H_0}{2}}\\
Q_{\text{top}} &= H_1 = -H_0
\end{aligned} \tag{5.156}$$

where $c^2 = \frac{\lambda}{2\pi i}\gamma^2(q^{-2} - 1)$.

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5.14 Integrable spin chain models and the algebraic Bethe ansatz

The discussion of the algebraic Bethe ansatz follows closely the pedagogical paper of Faddeev [88] and also Beisert [31]. The use of the ansatz in a continuous system that we present follows that of Zamolodchikov [234].

A very useful class of two-dimensional integrable models are the spin chain models. In these models the space is divided into a discrete number of sites where spin variables are placed. So far, and in fact also in the rest of this book, we do not discuss discretized field theories. In this chapter we do since the spin chain models will be shown, in Section 18, to be intimately related to integrable sectors of gauge theories in four dimensions. We will demonstrate the techniques used to solve the spin chain models by applying them to a prototype model, the $XXX_{1/2}$ model. We will describe the model, write down the Bethe ansatz equations associated with it, solve them and extract the spectrum of the model. We then apply the technique to the discretized sine-Gordon model.

A discrete circle with N ordered points is taken to be the space direction. The "space" is periodic so that each site is identified with $i \equiv i + N$. The formal continuum limit can be taken by introducing a lattice spacing Δ such that $\Delta \to 0,\ N \to \infty$ while $x = N\Delta$ is kept finite.

At each site there is a dynamical variable X_i^α where i denotes the site and α is a set of finite number of values. One defines a quantum algebra of observables $\mathcal{A}$ by fixing a set of commutation relations between the X_i^α. When $[X_i^\alpha, X_j^\beta] = 0$ for any $i \neq j$ the algebra is called ultra local. Examples are canonical variables, and spin variables which will be used in the $XXX_{1/2}$ model we are about to describe.

The Hilbert space of the representations of the ultra local algebra has a natural tensor product,

$$\mathcal{H} = \prod_{i=1}^{N} \bigotimes h_i = h_1 \bigotimes h_2 \ldots \bigotimes h_i \ldots \bigotimes h_N , \tag{5.157}$$

and the variables X_i^α act nontrivially only on h_i.

5.14.1 The $\mathbf{XXX_{1/2}}$ model

The $XXX_{1/2}$ describes a spin chain model with N sites. At each site there is a spin variable $S_i^\alpha = \frac{\hbar}{2}\sigma^\alpha$ where σ^α are the Pauli matrices. The Hilbert space at each site is $\mathcal{C}^2$, the two-dimensional complex numbers. The Hamiltonian that defines the model is based on a nearest neighbor interaction of the form,

$$H = \sum_{i,\alpha} \left(S_i^\alpha S_{i+1}^\alpha - \frac{1}{4} \right) . \tag{5.158}$$

The spin of the system which is given by,

$$S^\alpha = \sum_i S_i^\alpha , \tag{5.159}$$

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and is conserved,

$$[H, S^\alpha] = 0. \tag{5.160}$$

The notion XXX associates with the fact that the coefficient of $(S_i^\alpha S_{i+1}^\alpha - \frac{1}{4})$ is a constant independent on i and α. In the case where the coefficient is α dependent, namely $J^\alpha(S_i^\alpha S_{i+1}^\alpha - \frac{1}{4})$, the model is referred as the XYZ model.

To extract the spectrum of the model we make use of the *Lax operator* defined by,

$$L_{k,a}(\lambda) = \lambda I_k \bigotimes I_a + i \sum_\alpha S_k^\alpha \bigotimes \sigma^\alpha, \tag{5.161}$$

where λ is a complex parameter referred to as the *spectral parameter*, I_k, S_k^α are the identity and spin operators acting on h_k, and I_a, σ^α act on an auxiliary space V which is also $\mathcal{L}^2$.

The Lax operator can also be written in the form,

$$L_{k,a}(\lambda) = \left(\lambda - \frac{i}{2}\right) I_{k,a} + iP_{k,a}, \tag{5.162}$$

where $P_{i,a}$ is the permutation operator in $\mathcal{L}^2 \bigotimes \mathcal{L}^2$, namely,

$$Pa \bigotimes b = b \bigotimes a, \tag{5.163}$$

and is given by,

$$P = \frac{1}{2}(I \bigotimes I + \sum_\alpha \sigma^\alpha \bigotimes \sigma^\alpha). \tag{5.164}$$

Equation (5.163) implies,

$$\begin{aligned} P_{a_1,a_2} &= P_{a_2,a_1} \\ P_{n,a_1} P_{n,a_2} &= P_{a_1,a_2} P_{n,a_1} = P_{n,a_2} P_{a_2,a_1}. \end{aligned} \tag{5.165}$$

The Hamiltonian can also be expressed in terms of the permutation operator,

$$H = \frac{1}{2} \sum_i P_{i,i+1} - \frac{N}{2}. \tag{5.166}$$

The idea now is to relate the Hamiltonian and other conserved charges to the monodromy of a string of Lax operators along the whole chain. For that we have to analyze the commuting structure of the Lax operators. This structure is controlled by the *fundamental commutation relations* (FCR) which will be shown later to be part of the Yang–Baxter relations and read,

$$R_{a_1,a_2}(\lambda - \mu) L_{i,a_1}(\lambda) L_{i,a_2}(\mu) = L_{i,a_2}(\mu) L_{i,a_1}(\lambda) R_{a_1,a_2}(\lambda - \mu), \tag{5.167}$$

where,

$$R_{a_1,a_2}(\lambda) = \lambda I_{a_1,a_2} + iP_{a_1,a_2}. \tag{5.168}$$

To prove this relation one makes use of the relations of the permutation operator (5.165). The Lax operator can be interpreted as a connection along the chain in the sense,

$$\psi_{i+1} = L_i \psi_i. \tag{5.169}$$

This can be generalized to an ordered product which transports from i_1 to i_2,

$$T^{i_2}_{i_1,a}(\lambda) = L_{i_2-1,a}(\lambda) \dots L_{i_1,a}(\lambda), \tag{5.170}$$

and to the full monodromy along the spin chain,

$$T_{N,a}(\lambda) = L_{N,a}(\lambda) \dots L_{1,a}(\lambda). \tag{5.171}$$

We parameterize this monodromy operator in terms of a 2×2 matrix in the auxiliary space as,

$$T_{N,a}(\lambda) = \begin{pmatrix} A_N(\lambda) & a^\dagger_N(\lambda) \\ \tilde{a}_N(\lambda) & D_N(\lambda) \end{pmatrix}, \tag{5.172}$$

with entries in the full Hilbert space $\mathcal{H}$. In analogy to the FCR of the basic Lax operator it is straightforward to realize that there is a similar relation for the monodromy operator,

$$R_{a_1,a_2}(\lambda-\mu)T_{a_1}(\lambda))T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu). \tag{5.173}$$

From this relation it follows that the trace of the monodromy operator,

$$F(\lambda) \equiv Tr[T(\lambda)] = A(\lambda) + D(\lambda), \tag{5.174}$$

is commuting, namely $[F(\lambda), F(\mu)] = 0$. We can now expand both T_N and $F(\lambda)$ as a polynomial of order N in λ as,

$$\begin{aligned} T_{a,N}(\lambda) &= \lambda^N + i\lambda^{N-1}\sum_\alpha S^\alpha \bigotimes \sigma^\alpha + \dots \\ F(\lambda) &= 2\lambda^N + \sum_{l=0}^{N-2} Q_l \lambda^l. \end{aligned} \tag{5.175}$$

We will see shortly that the set of $N-1$ operators Q_l are commuting and constitute the set of conserved charges, including the Hamiltonian. Next we expand the monodromy at $\lambda = \frac{i}{2}$,

$$T_{a,N}\left(\frac{i}{2}\right) = i^N P_{N,a}P_{N-1,a}\dots P_{1,a} = i^N P_{1,2}P_{2,3}\dots P_{N-1,N}P_{N,a}. \tag{5.176}$$

This follows from $L_{i,a}(\frac{i}{2}) = iP_{i,a}$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}L_{i,a} = I_{i,a}$ and then taking the permutations one after the other of the term in the middle. The trace over the auxiliary space is $Tr[P_{N,a}] = I_N$ so that we can now define a shift operator in $\mathcal{H}$,

$$U = e^{i\mathcal{P}} \equiv i^{-N} Tr_a\left[T_N\left(\frac{i}{2}\right)\right] = P_{1,2}P_{2,3}\dots P_{N-1,N}. \tag{5.177}$$

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Using the properties of the permutation operator that $P^* = P$ and $P^2 = 1$, it follows that U is a unitary operator. Moreover, one can show that indeed it is a shift operator, namely,

$$U^{-1} X_i U = X_{i-1}. \tag{5.178}$$

To expand $F(\lambda)$ in the vicinity of $\lambda = \frac{i}{2}$ we first observe that,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} T_a(\lambda)|_{\lambda=1/2} = i^{N-1} \sum_i P_{N,a} \dots \hat{P}_{i,a} \dots P_{1,a}, \tag{5.179}$$

where ˆ means that the corresponding factor is absent. Using the same procedure as above we find that,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F_a(\lambda)|_{\lambda=i/2} = i^{N-1} \sum_i P_{1,2} P_{2,3} \dots P_{N-1,N}. \tag{5.180}$$

Most of the permutations can be cancelled by multiplying with U^{-1} so that

$$\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} F_a(\lambda)\right] F_a(\lambda)^{-1}|_{\lambda=i/2} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln(F_a(\lambda))|_{\lambda=i/2} = \frac{1}{i} \sum_i P_{i,i+1}. \tag{5.181}$$

Recalling the expression we found earlier for the Hamiltonian (5.166) we can now see that,

$$H = \frac{i}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln(F_a(\lambda))|_{\lambda=i/2} - \frac{N}{2}. \tag{5.182}$$

We have just shown that the Hamiltonian is part of a set of $N-1$ commuting operators generated by $F(\lambda)$, the trace of the monodromy. In fact there are N such conserved charges if we add also one component, say S^3, of the spin. The model is characterized by its N degrees of freedom and is equipped with N conserved charges and hence is (at least classically) *integrable*.

5.14.2 Bethe ansatz equations

To diagonalize the family of operators $F(\lambda)$ one can generalize the procedure used in the quantum harmonic oscillator. In that case we have a non-trivial commutation relation $[a, a^\dagger] = 1$, a Hamiltonian which is $H = a^\dagger a + 1$ and a ground state which is annihilated by a, namely $a|0> = 0$. Let us start first with the commutation relations. These are determined from the FCR as,

$$\begin{aligned}
&[\tilde{a}(\lambda), \tilde{a}(\mu)] = 0 \\
&A(\lambda)\tilde{a}(\mu) = \frac{\lambda-\mu-i}{\lambda-\mu} \tilde{a}(\mu) A(\lambda) + \frac{i}{\lambda-\mu} \tilde{a}(\lambda) A(\mu) \\
&D(\lambda)\tilde{a}(\mu) = \frac{\lambda-\mu+i}{\lambda-\mu} \tilde{a}(\mu) D(\lambda) - \frac{i}{\lambda-\mu} \tilde{a}(\lambda) D(\mu).
\end{aligned} \tag{5.183}$$

The last two relations generalize the relations,

$$aH = (H+1)a \quad a^\dagger H = (H-1)a^\dagger, \tag{5.184}$$

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of the harmonic oscillator. To derive the above one uses an explicit 4×4 matrix formulation for the operators in $V \bigotimes V$. A natural basis for these matrices is

$$e_1 = e_+ \bigotimes e_+, \quad e_2 = e_+ \bigotimes e_-, \quad e_3 = e_- \bigotimes e_+, \quad e_4 = e_- \bigotimes e_-, \qquad (5.185)$$

where,

$$e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (5.186)$$

In this basis the permutation operator and the R matrix read,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.187)$$

$$R(\lambda) = \begin{pmatrix} \lambda + i & 0 & 0 & 0 \\ 0 & \lambda & i & 0 \\ 0 & i & \lambda & 0 \\ 0 & 0 & 0 & \lambda + i \end{pmatrix}. \qquad (5.188)$$

The matrices $T_{a_1}(\lambda)$ and $T_{a_2}(\mu)$ read,

$$T_{a_1}(\lambda) = \begin{pmatrix} A(\lambda) & & a(\lambda) & \\ & A(\lambda) & & a(\lambda) \\ a^\dagger(\lambda) & & D(\lambda) & \\ & a^\dagger(\lambda) & & D(\lambda) \end{pmatrix} \qquad (5.189)$$

$$T_{a_2}(\lambda) = \begin{pmatrix} A(\mu) & a(\mu) & & \\ a^\dagger(\mu) & D(\mu) & & \\ & & A(\mu) & a(\mu) \\ & & a^\dagger(\mu) & D(\mu) \end{pmatrix}. \qquad (5.190)$$

Explicit multiplication of these matrices yields (5.183).

Similarly to the case of the harmonic oscillator we now define the *ground state*,

$$a(\lambda)|0\rangle = a(\lambda) \prod_i \bigotimes |0\rangle_i = 0. \qquad (5.191)$$

We choose $|0\rangle_i = e_+$ so that,

$$S^3|0\rangle = \frac{N}{2}|0\rangle \quad S^+|0\rangle = 0. \qquad (5.192)$$

Thus this state is the “highest weight state”. It also follows that (the operator * is left unspecified),

$$L_n(\lambda)|0\rangle_i = \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} |0\rangle_i, \qquad (5.193)$$

and hence,

$$T(\lambda)|0\rangle = \begin{pmatrix} (\lambda + \frac{i}{2})^N & * \\ 0 & (\lambda - \frac{i}{2})^N \end{pmatrix} |0\rangle, \qquad (5.194)$$

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which means that $|0>$ is an eigenstate of both $A(\lambda)$ and $D(\lambda)$ and thus also of $F(\lambda)$. Higher excited states are created from the ground state $|0>$ by a successive action with creation operators,

$$\Phi(\{\lambda\}) = a^\dagger(\lambda_1)\dots a^\dagger(\lambda_l)|0> . \tag{5.195}$$

Requiring that the state $\Phi(\{\lambda\})$ is an eigenstate of $F(\lambda)$ imposes a set of relations on the $\lambda_1, \dots, \lambda_l$. In particular using the FCR relation (5.183) we find,

$$\begin{aligned} &A(\lambda)a^\dagger(\lambda_1)\dots a^\dagger(\lambda_l)|0> \\ &= \prod_{k=1}^{l} \frac{\lambda-\lambda_k-i}{\lambda-\lambda_k}\left(\lambda+\frac{i}{2}\right)^N (\lambda)a^\dagger(\lambda_1)\dots a^\dagger(\lambda_l)|0> \\ &\quad + \sum_{k=1}^{l} M_k(\lambda,\{\lambda\})a^\dagger(\lambda_1)\dots \hat{a}^\dagger(\lambda_k)\dots a^\dagger(\lambda_l)|0> . \end{aligned} \tag{5.196}$$

The first term on the right-hand side of the equation has the form of an eigenstate equations but the rest of the terms do not. The idea is to choose the set $\{\lambda\}$ such that these terms will cancel out against similar terms in $D(\lambda)a^\dagger(\lambda_1)\dots a^\dagger(\lambda_l)|0>$. To get the value of the coefficient $M_1(\lambda,\{\lambda\})$ we use the second term on the right-hand side of the second equation in (5.183) when interchanging $A(\lambda)$ and $a^\dagger(\lambda_1)$ and in all other exchanges we use the first term. In this way we find that,

$$M_1(\lambda,\{\lambda\}) = \frac{i}{\lambda-\lambda_1}\prod_{k=2}^{l}\frac{\lambda_1-\lambda_k-i}{\lambda_1-\lambda_k}\left(\lambda_1+\frac{i}{2}\right)^N . \tag{5.197}$$

Interchanging now $\lambda_1 \to \lambda_j$ we get similarly the expressions for all $M_j(\lambda,\{\lambda\})$. The same type of manipulations yield,

$$\begin{aligned} &D(\lambda)a^\dagger(\lambda_1)\dots a^\dagger(\lambda_l)|0> \\ &= \prod_{k=1}^{l} \frac{\lambda-\lambda_k-i}{\lambda-\lambda_k}\left(\lambda-\frac{i}{2}\right)^N (\lambda)a^\dagger(\lambda_1)\dots a^\dagger(\lambda_l)|0> \\ &\quad + \sum_{k=1}^{l} N_k(\lambda,\{\lambda\})a^\dagger(\lambda_1)\dots \hat{a}^\dagger(\lambda_k)\dots a^\dagger(\lambda_l)|0>, \end{aligned} \tag{5.198}$$

with,

$$N_j(\lambda,\{\lambda\}) = \frac{i}{\lambda-\lambda_1}\prod_{k=2}^{l}\frac{\lambda_1-\lambda_k-i}{\lambda_1-\lambda_k}\left(\lambda_1-\frac{i}{2}\right)^N . \tag{5.199}$$

We now observe that if the set of $\{\lambda\}$ obey the condition,

$$\prod_{k\neq j}^{l}\frac{\lambda_1-\lambda_k-i}{\lambda_1-\lambda_k}\left(\lambda_1+\frac{i}{2}\right)^N = \prod_{k\neq j}^{l}\frac{\lambda_1-\lambda_k-i}{\lambda_1-\lambda_k}\left(\lambda_1-\frac{i}{2}\right)^N , \tag{5.200}$$

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then the undesirable terms in (5.196) cancel out and we end with an eigenstate of $F(\lambda)$ and hence of the Hamiltonian. For the former it takes the form,

$$F(\lambda)\Phi(\{\lambda\}) = \left(\lambda + \frac{i}{2}\right)^N \prod_{j=1}^{l} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + \left(\lambda - \frac{i}{2}\right)^N \prod_{k\neq j}^{l} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j}\Phi(\{\lambda\}). \tag{5.201}$$

These conditions can be rewritten as,

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^N = \prod_{k\neq j}^{l} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}. \tag{5.202}$$

These conditions on the eigenvectors were derived originally by Bethe, though in a completely different way, and hence the names, the "Bethe ansatz equation" (BAE) for (5.202) and the "Bethe vector" for $\Phi(\{\lambda\})$. It is straightforward to observe that the poles in the eigenvalue of $F(\lambda)$ cancel out so that it is a polynomial in λ of degree N. One can further show that the full spectrum can be recast just with nonequal λ_j.

We now want to determine the eigenvalues of spin, momentum and Hamiltonian operators of the eigenstates just found.

Using the FCR relation (5.173) in the limit of $\mu \to \infty$ the $SL(2)$ invariance of the monodromy in $\mathcal{H} \bigotimes V$ is determined via,

$$\left[T_a(\lambda), \frac{1}{2}\sigma^\alpha + S^\alpha\right] = 0, \tag{5.203}$$

which means in particular that,

$$[S^3, a^\dagger] = -a^\dagger \quad [S^+, a^\dagger] = A - D. \tag{5.204}$$

The spin of the state is therefore given by,

$$S^3\Phi(\{\lambda\}) = \left(\frac{N}{2} - l\right)\Phi(\{\lambda\}). \tag{5.205}$$

Furthermore it can be shown that the states $\Phi(\{\lambda\})$ are all highest weight states provided that the BAE is obeyed, namely,

$$S^+\Phi(\{\lambda\}) = 0. \tag{5.206}$$

Since the S^3 eigenvalue of the highest weight states is non-negative it is obvious that $l \leq \frac{N}{2}$. When N is odd the spin of the state is half-integer, whereas when it is even the spin is even and in particular for $l = \frac{N}{2}$ there is an $SL(2)$ invariant state with vanishing spin.

Let us determine now the eigenvalue of the momentum operator. From the definition of the shift operator (5.177) it follows that,

$$U\Phi(\{\lambda\}) = i^N F\left(\frac{i}{2}\right)\Phi(\{\lambda\}) = \prod \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\Phi(\{\lambda\}). \tag{5.207}$$

The eigenvalue of the momentum operator is therefore given by,

$$P\Phi(\{\lambda\}) = \sum_j p(\lambda_j)\Phi(\{\lambda\}) \quad p(\lambda) = \frac{1}{i}\log\left[\frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}}\right]. \tag{5.208}$$

The energy eigenvalue is determined from (5.182),

$$H\Phi(\{\lambda\}) = \sum_j e(\lambda_j)\Phi(\{\lambda\}) \quad e(\lambda) = -\frac{1}{2}\frac{1}{\lambda^2 + \frac{1}{4}}. \tag{5.209}$$

The last expressions calls for a *"quasi particle"* interpretation. The operator $a^\dagger(\lambda)$ creates a quasi particle which reduces the spin S^3 by one unit and adds to the momentum and energy $p(\lambda)$ and $e(\lambda)$, respectively. We further observe that,

$$e(\lambda) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}p(\lambda). \tag{5.210}$$

It is also evident that we can eliminate the dependence on the rapidity of the energy and momentum and read directly the dispersion relation,

$$e(p) = \cos(p) - 1. \tag{5.211}$$

Since this energy is always non-positive, the highest weight state $|0>$ can be considered a ground state only if we take $-H$ as the Hamiltonian rather than H. In fact it will be shown shortly that the latter corresponds to a system of an antiferromagnet whereas the former corresponds to that of a ferromagnet.

5.14.3 The thermodynamic Bethe ansatz

The thermodynamic limit of the spin chain models is the limit of $N \to \infty$. Recall that the continuum limit of the model when $N \to \infty$ and the spacing $\Delta \to 0$. In the BAE N appears only in the left-hand side. If we take the log of the ansatz we find for real $\{\lambda\}$,

$$Np(\lambda_j) = 2\pi Q_j + \sum_{k=1}^{l}\varphi(\lambda_j - l_k), \tag{5.212}$$

where Q_i are integers $0 \le Q_j \le N-1$ that define the branch of the log and $\varphi(\lambda)$ is a fixed branch of $log(\frac{\lambda+i}{\lambda-i})$. For large N and Q_j and fixed l one finds the usual expression for the momentum of a free particle on the chain,

$$p_j = 2\pi\frac{Q_j}{N}, \tag{5.213}$$

since the $\varphi(\lambda)$ is negligible.

The second term in (5.212) associates with the scattering of these particles. In fact by comparison with the quantum mechanics of a particle in a box we see that $\varphi(\lambda_i - \lambda_j)$ stands for the corresponding phase shift of the particles with rapidity λ_i and λ_j. Using this analogy we can now identify the S-matrix element with,

$$S(\lambda-\mu)=\frac{\lambda-\mu+i}{\lambda-\mu-i}. \tag{5.214}$$

The S-matrix also enters the large N commutation relations of the normalized creation operators $\tilde{a}^\dagger(\lambda)=a^\dagger(\lambda)A^{-1}(\lambda)$,

$$\tilde{a}^\dagger(\lambda)\tilde{a}^\dagger(\mu)=S(\lambda-\mu)\tilde{a}^\dagger(\mu)\tilde{a}^\dagger(\lambda). \tag{5.215}$$

In addition to the quasi-particle states in the Hilbert space, there are also bound states of the quasi-particles. These states correspond to complex solutions of the BAE. The simplest case is with two quasi-particles $l=2$. In this case the two BAE read,

$$\left(\frac{\lambda_1+i/2}{\lambda_1-i/2}\right)^N=\frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}\quad\left(\frac{\lambda_2+i/2}{\lambda_2-i/2}\right)^N\frac{\lambda_2-\lambda_1+i}{\lambda_2-\lambda_1-i}. \tag{5.216}$$

Using (5.208) it follows that $p(\lambda_1)+p(\lambda_2)$ is real. Furthermore, for $N\to\infty$ to compensate the exponential increase (decrease) of the left-hand side of the last equations, it is clear that the right-hand side must have $\mathrm{Im}(\lambda_1-\lambda_2)=i$ $(or -i)$ and thus the final form of λ_1 and λ_2 are,

$$\lambda_1=\lambda_{1/2}+\frac{i}{2}\quad\lambda_2=\lambda_{1/2}-\frac{i}{2}, \tag{5.217}$$

where $\lambda_{1/2}$ is real. The momentum and energy eigenvalues of the corresponding Bethe vector are,

$$p_{1/2}=\frac{1}{i}\ln\frac{\lambda+i}{\lambda-i}\quad e_{1/2}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}[p_{1/2}(\lambda)]=\frac{1}{\lambda^2+1}. \tag{5.218}$$

The state is considered as a bound state since its energy is less than the sum of the energies of the two constituents,

$$e_{1/2}<[e_0(p-p_1)+e_0(p_1)], \tag{5.219}$$

for any $0\leq p,p_1\leq 2\pi$.

The bound state of two quasi-particles $l=2$ can be generalized for $l>2$. The roots λ_l are combined into complexes M, where M takes half integer values $M=0,1/2,1,\dots$ with $l=\sum_M\nu_M(2M+1)$ where ν_M gives the number of complexes of type M. Each complex has roots of the type,

$$\lambda_{M,m}=\lambda_M+\mathrm{im}\qquad -M\leq m\leq M, \tag{5.220}$$

where λ_M is real and m are integers and half integers. The corresponding momentum and energy are given by,

$$p_M(\lambda)=\frac{1}{i}\ln\frac{\lambda+i(M+1/2)}{\lambda-i(M+1/2)}\quad e_M(\lambda)=\frac{1}{2}\frac{2M+1}{\lambda^2+(M+1/2)^2}. \tag{5.221}$$

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The S-matrix for scattering of complexes M and N are

$$S_{M,N} = \prod_{L=|M-N|}^{M+N} S_{0,L}(\lambda), \tag{5.222}$$

where,

$$S_{0,M}(\lambda) = \frac{\lambda + iM}{\lambda - iM}\frac{\lambda + i(M+1)}{\lambda - i(M+1)}. \tag{5.223}$$

To summarize, the ferromagnetic system with Hamiltonian $-H$ in the thermodynamics limit has a Hilbert space $\mathcal{H}_F$ with a ground state $|0\rangle = \prod(e_+)_i$. The excitations are quasi-particles characterized by M, $M = 0, 1/2, \ldots$ and the rapidity λ. The dispersion relation is given by $e_M = \frac{1}{2M+1}(1 - \cos(p_M))$ and the S-matrix is (5.222). Out of S^α, only S^3 makes sense as an operator on $\mathcal{H}_F$. The operators $S^\pm$ change the ground state at each site. This may be viewed as a symmetry-breaking phenomenon.

So far we have described the basic notions of the physics of the spin chain using the example of the $XXX_{1/2}$. One can further generalize these considerations in many directions such as the anti-ferromagnetic system, general spin states in the XXX model, namely, the $XXX_{s/2}$ model, the XXZ and many others.

The thermodynamic limit for the $XXX_{s/2}$ model

So far we have described the basic notions of the physics of the spin chain using the example of the $XXX_{1/2}$. One can further generalize these considerations also to the $XXX_{s/2}$ model. We state here the results without derivation. An eigenstate characterized by the set $\lambda_1, \ldots, \lambda_l$ which are determined by a straightforward generalization of the BAE (5.202),

$$\left(\frac{\lambda_j + (i/2)s}{\lambda_j - (i/2)s}\right)^N = \prod_{k\neq j}^{l} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}. \tag{5.224}$$

The eigenvalues of the Hamiltonian and momentum are given by,

$$E = \sum_{k=1}^{l} \frac{s}{\lambda_k^2 + \frac{1}{4}s^2} \quad P = \frac{1}{i}\sum_{k=1}^{l} \ln\left[\frac{\lambda_k + \frac{i}{2}s}{\lambda_k - \frac{i}{2}s}\right], \tag{5.225}$$

and the higher conserved charges that render the model integrable are,

$$Q_r = \frac{i}{r-1}\sum_{k=1}^{l}\left(\frac{1}{(\lambda_k + \frac{i}{2}s)^{r-1}} - \frac{1}{(\lambda_k - \frac{i}{2}s)^{r-1}}\right), \tag{5.226}$$

where $r \leq N$ and in particular $r = 2$ is the Hamiltonian.

In the thermodynamic limit, $N \to \infty$ states with low energy and zero momentum can be dealt with by introducing the scaling,

$$\tilde{E} = NE = \frac{1}{N}\sum_{k=1}^{l}\frac{s}{\tilde{\lambda}_k^2} \quad 2\pi n = \frac{1}{N}\sum_{k=1}^{l}\frac{s}{\tilde{\lambda}_k} \quad 2\pi n_k - \frac{s}{\tilde{\lambda}_k} = \frac{1}{N}\sum_{j=1,j\neq k}^{l}\frac{2}{\tilde{\lambda}-\tilde{\lambda}_k}, \tag{5.227}$$

where $\lambda_k = N\tilde{\lambda}_k$ and where the second and third expressions were derived by taking the log of the zero momentum condition $U = 1$ and of the BAE, respectively. Using the same scaling one finds for the higher charges and the transfer matrix the results,

$$\tilde{Q}_r = \frac{Q_r}{N^{r-1}} = \frac{1}{N}\sum_{k=1}^{l}\frac{s}{\tilde{\lambda}_k^r} \quad -i\log\tilde{T}(\tilde{\lambda}) = -i\log T(N\tilde{\lambda}) = \frac{1}{N}\sum_{j=1,j\neq k}^{l}\frac{s}{\tilde{\lambda}_j-\tilde{\lambda}_k}. \tag{5.228}$$

For $N \to \infty$ it is plausible to assume that the Bethe roots accumulate on smooth contours $(\mathcal{C}_1,\ldots\mathcal{C}_A) \equiv \mathcal{C}$ which are referred to as *"Bethe strings"*. Thus we replace the discrete $\tilde{\lambda}_k$ locations of the roots by a continuum variable $\tilde{\lambda}$ described by a density $\rho(\tilde{\lambda})$ so that the sum of the root translates into the integral,

$$\frac{1}{N}\sum_{k=1}^{l} \to \int_{\mathcal{C}} \mathrm{d}\tilde{\lambda}\rho(\tilde{\lambda}), \tag{5.229}$$

with the normalization that $\int_{\mathcal{C}} \mathrm{d}\tilde{\lambda}\rho(\tilde{\lambda}) = \frac{1}{N}$. Using the continuum formulation we can now rewrite the expressions for the Bethe ansatz, the energy and the higher charges as,

$$\begin{aligned} 2\pi n = s\int_{\mathcal{C}}\frac{\mathrm{d}\tilde{\lambda}\rho(\tilde{\lambda})}{\tilde{\lambda}} \quad & 2\pi n_{\tilde{\lambda}} - \frac{s}{\tilde{u}} = 2\int_{\mathcal{C}}\frac{\mathrm{d}\tilde{\lambda}'\rho(\tilde{\lambda}')}{\tilde{\lambda}'-\lambda}, \\ \tilde{E} = s\int_{\mathcal{C}}\frac{\mathrm{d}\tilde{\lambda}\rho(\tilde{\lambda})}{\tilde{\lambda}^2} \quad & \tilde{Q}_r = s\int_{\mathcal{C}}\frac{\mathrm{d}\tilde{\lambda}\rho(\tilde{\lambda})}{\tilde{\lambda}^r}, \end{aligned} \tag{5.230}$$

where $n_{\tilde{\lambda}}$ is the mode number n_k at the point $\tilde{\lambda} = \tilde{\lambda}_k$. It is expected to be a constant along each contour $\mathcal{C}_a$. In the second integral of the first line, a principal part prescription is implemented. An important result is that one can determine the set of conserved charges $\tilde{Q}_r$ using a resolvent, as,

$$G(\tilde{\lambda}) = |s|\int_{\mathcal{C}}\frac{\mathrm{d}\tilde{\lambda}'\rho(\tilde{\lambda}')}{\tilde{\lambda}'-\lambda} \quad G(\tilde{\lambda}) = \sum_{r=1}^{\infty}\tilde{\lambda}^{r-1}\tilde{Q}_r. \tag{5.231}$$

The resolvent is related to the transfer matrix,

$$\tilde{T}(\lambda) = \mathrm{e}^{iG(\tilde{\lambda})} + \mathrm{e}^{-iG(\tilde{\lambda}-\frac{i}{\tilde{\lambda}})} = \mathrm{e}^{-\frac{i}{2\tilde{u}}}2\cos\left(G(\tilde{\lambda}) + \frac{1}{2\tilde{\lambda}}\right). \tag{5.232}$$

We wish to conclude this chapter with a model that will enable us to connect the discussion of the spin chain models to continuum integrable models.

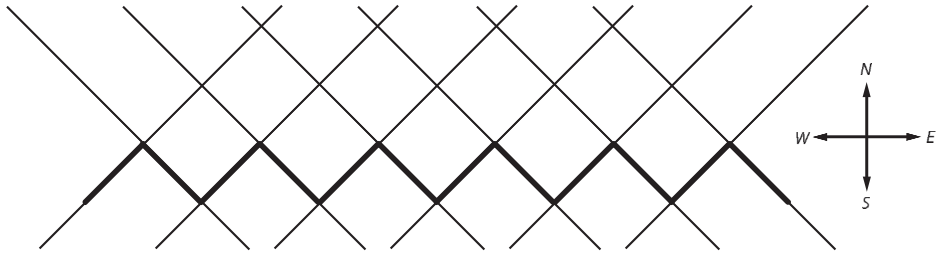

Fig. 5.5. Saw path on the two dimensional lattice.

5.14.4 Spin chain model in discrete time

So far we have discussed the two-dimensional integrable model with a discretized space dimension and a continuous time dimension. It turns out that to connect the spin chain models to continuum two-dimensional field theory it is useful to further also discretize the time direction. The shift operator (13.77) discussed above was determined from a trace of the monodromy at a particular value of the spectral parameter. To describe the system with both space and time discretized, one needs to distinguish values of the spectral parameter $\lambda \pm w$ for some fixed w. We define now an inhomogeneous monodromy built from the Lax operator $L_{i,f}$ acting on the quantum Hilbert space h_i and the auxiliary space V_f,

$$T_f(\lambda, w) = L_{2N,f}(\lambda + w)L_{2N-1,f}(\lambda - w)\ldots L_{2,f}(\lambda + w)L_{1,f}(\lambda - w). \quad (5.233)$$

The light-like shift operators U_+ and U_- are given by,

$$U_+ = tr_f[T_f(w, w)] \quad U_- = tr_f[T_f(-w, w)]. \quad (5.234)$$

The monodromy is along a saw path on the two-dimensional lattice as can be seen in Fig. 5.5.

$L_{2n,f}(\lambda + w)$ is a transport along the NW direction and $L_{2N-1,f}(\lambda - w)$ along the SW direction.

In analogy to the definition of the monodromy as a trace of $T(\lambda)$ (13.77), we now define the monodromy as,

$$F(\lambda, w|a, i, \mu) = \mathrm{tr}_f[T_f(\lambda, w|a, i, \mu)], \quad (5.235)$$

where

$$\begin{aligned} T_f(\lambda, w|a, i, \mu) &= L_{2N,f}(\lambda + w)L_{2N-1,f}(\lambda - w)\ldots \\ &\ldots L_{2i,f}(\lambda + w)L^{-1}_{f,a}(\mu - \lambda)L_{2i-1,f}(\lambda - w)\ldots L_{2,f}(\lambda + w)L_{1,f}(\lambda - w). \end{aligned} \quad (5.236)$$

It can be shown that $T_f(\lambda, w|a, i, \mu)$ is subjected to the FCR relations in a similar manner to $T(\lambda)$ (13.77). Due to the commutativity of F we get a zero curvature condition on the transport around an elementary plaquette of the space-time

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lattice,

$$L_{2i,a}(\lambda+w)L_{2i-1,a}(\lambda-w)=U_-L_{2i-1,a}(\lambda-w)U_-^{-1}U_+^{-1}L_{2i,a}(\lambda+w)U_+. \tag{5.237}$$

The light-like shifts $U_\pm$ are related to the shift in space and time in the usual form, namely,

$$U_+=\mathrm{e}^{-i(H-P)/2} \quad U_-=\mathrm{e}^{-i(H+P)/2}. \tag{5.238}$$

As for the case of only space dimension discretized, here too one finds that the condition of having an eigenvector of the energy and momentum is the BAE which takes the form,

$$\left(\frac{\alpha(\lambda_j+w)\alpha(\lambda_j-w)}{\delta(\lambda_j+w)\delta(\lambda_j-w)}\right)^N=\prod_{j\neq k}S(\lambda_j-\lambda_k), \tag{5.239}$$

where $\alpha(\lambda),\delta(\lambda)$ are local eigenvalues and $S(\lambda)$ is the quasi particle phase factor.

5.14.5 The discretized version of the sine-Gordon model

As a particular example of an integrable lattice model, we consider a model with Weyl variables rather than spin variables on the lattice sites. These variables obey at each site the relations,

$$u_iv_i=qv_iu_i, \quad q=\mathrm{e}^{i\gamma}. \tag{5.240}$$

The corresponding Lax operator is

$$L_{i,a}(x)=\begin{pmatrix} u_i & xv_i \\ -xv_i^{-1} & u_i^{-1}\end{pmatrix}. \tag{5.241}$$

To reduce the number of degrees of freedom per site from two to one we impose the constraint,

$$u_{2i}u_{2i-1}v_{2i}v_{2i-1}^{-1}=1. \tag{5.242}$$

The BAE for this case take the form,

$$\left(\frac{\sinh(\lambda_j+w+i\gamma/2)\sinh(\lambda_j-w+i\gamma/2)}{\sinh(\lambda_j+w-i\gamma/2)\sinh(\lambda_j-w-i\gamma/2)}\right)^{N/2}=\prod_{k\neq j}^{l}\frac{\sinh(\lambda_j-\lambda_k+i\gamma)}{\sinh(\lambda_j-\lambda_k-i\gamma)}, \tag{5.243}$$

where we have substituted $x=\mathrm{e}^{\lambda}$ and $\kappa=\mathrm{e}^{w}$.

The shift operator in the time direction related to the Hamiltonian can be determined in the following manner. Using the explicit form of the Lax operator we have,

$$L_{i_2,a}\left(\frac{1}{x}\right)L_{i_1,a}\left(\frac{1}{y}\right)r\left(\frac{x}{y}\right)=r\left(\frac{x}{y}\right)L_{i_1,a}\left(\frac{1}{y}\right)L_{i_2,a}\left(\frac{1}{x}\right). \tag{5.244}$$

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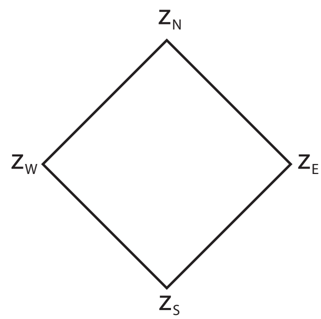

Fig. 5.6. Elementary plaquette.

It can be shown that this condition reduces to the functional equation,

$$\frac{r(x,qw)}{r(x,q^{-1}w)} = \frac{xw+1}{x+w}. \tag{5.245}$$

Denoting

$$w_{2i} = u_{2i}u_{2i-1}v_{2i}^{-1}v_{2i-1} \quad w_{2i+1} = u_{2i+1}u_{2i}v_{2i+1}^{-1}v_{2i}, \tag{5.246}$$

we find that the time shift operator is given by,

$$\mathrm{e}^{-iH} = \prod r(\kappa^2, w_{2i}) \prod r(\kappa^2 w_{2i+1}). \tag{5.247}$$

With this operator at hand we can now determine the equation of motion of the model and show that it corresponds in the continuuum limit to the sine-Gordon equation. To accomplish this we define now z_i such that,

$$w_i = \frac{z_{i+1}}{z_{i-1}}. \tag{5.248}$$

It is easy to see that z_i does not commute only with one w, namely w_i for which we have,

$$z_i w_i = q^2 w_i z_i. \tag{5.249}$$

Now we apply the time evolution operator on z_i. It reads,

$$\hat{z}_i = \mathrm{e}^{iH} z_i \mathrm{e}^{-iH}. \tag{5.250}$$

When we substitute (5.247) we get the equation of motion,

$$\hat{z}_{2i+1} = z_{2i+1}\frac{\kappa^2 q^{-1} z_{2i+2} + z_{2i}}{q^{-1}z_{2i+2} + \kappa^2 z_{2i}}, \tag{5.251}$$

and similarly for z_{2i}. This equation connects z along an elementary plaquette (see Fig. 5.6). The equation can also be rewritten as,

$$(q^{-1}z_N z_W - z_S z_E) = \kappa^2(q^{-1}z_S z_W - z_N z_E). \tag{5.252}$$

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If we now define the variable,

$$\chi = \mathrm{e}^{i\varphi} = \begin{pmatrix} z \\ z^{-1} \end{pmatrix}, \tag{5.253}$$

alternatively on each second SE characteristic line on our lattice, then the equations of motion take the form,

$$\chi_N = \chi - S^{-1} \frac{\kappa^2 \chi_W \chi_E + 1}{q^{-1} \chi_W \chi_E + \kappa^2}, \tag{5.254}$$

so that for large κ^2 and classical limit $q = 1$ we get,

$$\frac{\chi_N \chi_S}{\chi_E \chi_W} = 1 + \frac{1}{\kappa^2} \left(\frac{1}{\chi_E \chi_W} - \chi_E \chi_W \right) + \dots, \tag{5.255}$$

and in terms of φ,

$$\frac{\chi_N \chi_S}{\chi_E \chi_W} = \mathrm{e}^{i \frac{\Delta^2}{2} (\partial_t^2 \varphi - \partial_x^2 \varphi) + \dots}, \tag{5.256}$$

and with the scaling $\frac{1}{\kappa^2} = m^2 \Delta^2$ we finally discover the sine-Gordon equation,

$$(\partial_t^2 \varphi - \partial_x^2 \varphi) + 2m^2 \sin(2\varphi) = 0. \tag{5.257}$$

In the quantum version one modifies the scaling to take into account the mass renormalization.

5.15 The continuum thermodynamic Bethe ansatz

Here we describe again the thermodynamic Bethe ansatz but now in the context of continuous models. This will not be a straightforward transition from a discretized to a continuous model via a certain limit, but rather a completely different derivation. Obviously, here as well the Bethe wave-function, the thermodynamic limit and the interplay between the spectrum and S-matrix elements will enter as essential ingredients.

Consider an integrable Euclidean field theory defined on a two-dimensional torus. We denote the two cycles of the torus as *cycle* a and *cycle* b, with corresponding circumferences of R and L and coordinates x and y, as shown in Fig. 5.7. Obviously, one can define in a twofold manner the states of the system and its Hamiltonian. We can consider the space of states on a denoted by $\mathcal{A}$ with the time direction along y, with the Hamiltonian,

$$H_a = \frac{1}{2\pi} \int_a T_{yy} \mathrm{d}x, \tag{5.258}$$

and with the momentum,

$$P_a = \frac{1}{2\pi} \int_a T_{xy} \mathrm{d}x, \tag{5.259}$$

which has quantized eigenvalues $\frac{2\pi n}{R}$.

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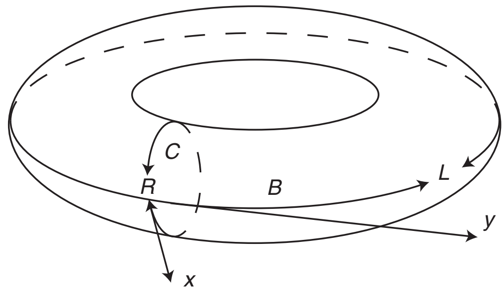

Fig. 5.7. Flat torus generated by two orthogonal geodesics C and B of circumference R and L, respectively.

Alternatively, one can consider the space of states $\mathcal{B}$ along the contour b with the time direction along $-x$, with the Hamiltonian,

$$H_b = \frac{1}{2\pi} \int_b T_{xx} \mathrm{d}y, \tag{5.260}$$

and with the momentum,

$$P_b = -\frac{1}{2\pi} \int_b T_{yx} \mathrm{d}y, \tag{5.261}$$

where now the quantization condition is that the eigenvalues of P_b are quantized in units of $\frac{2\pi n}{L}$.

Let us consider the cylindrical geometry via the limit $L \to \infty$ ($L \gg R$). For this case the partition function $Z(L, R)$ is dominated by the ground state of H_a with the ground state energy $E_0(R)$,

$$Z(R, L) \sim \mathrm{e}^{-LE_0(R)}. \tag{5.262}$$

On the other hand,

$$Z(R, L) = Tr_{\mathcal{B}}[\mathrm{e}^{-RH_b}]. \tag{5.263}$$

In $\mathcal{B}$, the thermodynamic limit, namely infinite space $L \to \infty$ which is the analog of the large N limit in Section 5.14.3, gives the free energy $f(R)$ at temperature $1/R$, via $\log Z(R, L) \sim -L\beta f(\beta)$, where $\beta = R$ is the inverse of the temperature. Hence,

$$E_0(R) = Rf(R). \tag{5.264}$$

The ground state energy $E_0(R)$, which can be referred to also as the Casimir energy, can be related in the limit of conformal field theory, to the Virasoro anomaly. Define the scaling factor $\tilde{c}(r)$ via,

$$E_0(R) = -\frac{\pi \tilde{c}(r)}{6R}, \tag{5.265}$$

where the dimensionless quantity $r = m_1 R$, with m_1 the lowest mass in the theory. The scaling factor will be determined by the TBA. On the other hand, in the case of conformal invariance, when $R \to 0$, using the relation between the Hamiltonian and L_0 and $\bar{L}_0$ we have,

$$E_0(R) = \frac{2\pi}{R}\left(\Delta_{\min} + \bar{\Delta}_{\min} - \frac{1}{12}c\right), \tag{5.266}$$

where $\Delta_{\min}$ denotes the conformal dimension of the lowest state. This means that the scaling factor should reduce to the effective Virasoro anomaly,

$$\lim_{r\to 0} \tilde{c}(r) = c - 24\Delta_{\min}. \tag{5.267}$$

Here we took the cases of $\Delta_{\min} = \bar{\Delta}_{\min}$.

The TBA method enables one to compute the spectrum of energies and momenta by combining the thermodynamic limit with the factorizable scattering amplitudes. We consider first for simplicity the case with only one type of particle of mass m with a pair scattering amplitude $S(\beta_1 - \beta_2)$ (see Section 5.7). Recall (5.73) that the energy and momentum are related to the rapidities β_i as

$$e_i(\beta) = m\cosh\beta_i \quad p_i(\beta) = m\sinh\beta_i, \tag{5.268}$$

and that the amplitudes obey the unitarity and crossing symmetry (5.78),

$$S(\beta)S(-\beta) = 1 \quad S(\beta) = S(i\pi - \beta). \tag{5.269}$$

For regions of configuration space where the particles are highly separated, namely where $|x_i - x_j| \gg R_c, \;\; i,j = 1,\ldots,N$ with R_c denoting the scale of the interaction, the particles can be treated as approximately free. In these regions it makes sense to introduce the wavefunction of the system $\Phi(x_1,\ldots,x_N)$ (in regions where the particles are not well separated, particle creation and annihilation prevent the use of a single wavefunction).

For integrable systems at any free region the number of particles will be the same, namely N, as well as the set of momenta p_i. The set of particles in a free region will be denoted by $(i_1, i_2, \ldots, i_N)$ where $x_{i_1} < x_{i_2} < \ldots < x_{i_N}$. A scattering process that yields a transition between $(i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_N)$ and $(i_1, i_2, \ldots, i_{k+1}, i_k, \ldots, i_N)$ affects the wavefunctions, such that the latter wavefunction is given by the former multiplied by the scattering amplitude $S(\beta_k - \beta_{k+1})$. These matching conditions on the wavefunctions combined with the quantization of the momenta due to the fact that the "space" coordinate is compact, lead to the relation,

$$\mathrm{e}^{ip_i L}\prod_{j\neq i} S(\beta_i - \beta_j) = 1, \quad Or \quad mL\sinh\beta_i + \sum_{j\neq i}\delta(\beta_i - \beta_j) = 2\pi n_i, \tag{5.270}$$

where for real β we rewrote the scattering amplitude in terms of a unimodular function $S(\beta)e^{i\delta(\beta)}$, with $\delta(\beta)$ a real phase.

This set of transcendental equations selects the admissible sets of rapidities and hence energies and momenta. The energy and momentum of the state

$(\beta_1, \ldots, \beta_N)$ are given by,

$$H_b = \sum_{i=1}^{N} m \cosh\beta_i, \quad P_b = \sum_{i=1}^{N} m \sinh\beta_i. \tag{5.271}$$

In the thermodynamic limit of $L \to \infty$ the set of equations (5.270) can be simplified. In this limit the number of particles becomes large since it grows $\sim L$ and the distance between adjacent levels behaves as $(\beta_i - \beta_{i+1}) \sim \frac{1}{mL}$. In that case one naturally defines the notion of a continuous rapidity density of particles $\rho_1(\beta)$. If there are n particles in a $\Delta\beta$ we take $\rho_1(\beta) = \frac{n}{\Delta\beta}$. We can now exchange the sum over β_i with an integral over β of the particle density, provided that the latter is independent on $\Delta\beta$ for $\frac{1}{mL} \ll \Delta\beta \ll 1$, so that (5.270) takes the form,

$$mL \sinh\beta_i + \int \delta(\beta_i - \beta')\rho_1(\beta')\mathrm{d}\beta' = 2\pi n_i. \tag{5.272}$$

If one further defines the level density $\rho(\beta)$ such that $n = \int \rho(\beta)\mathrm{d}\beta$ we get,

$$mL \cosh\beta + \int \varphi(\beta - \beta')\rho_1(\beta')\mathrm{d}\beta' = 2\pi\rho(\beta), \tag{5.273}$$

where $\varphi(\beta) = \frac{\partial\delta(\beta)}{\partial\beta}$.

In terms of the particle density the energy of the system is,

$$H_b = \int m \cosh\beta \rho_1(\beta)\mathrm{d}\beta. \tag{5.274}$$

At this point one has to distinguish between bosons and fermions. From the unitarity condition one can have either $S(0) = 1$ or $S(0) = -1$. In the former case bosons occupy each rapidity value in any number whereas for fermions the occupation number is at most one. In the latter case the situation is in some sense the opposite. $S(0) = -1$ implies that for two particles with the same rapidity the wavefunction is antisymmetric in their coordinates, and since this is incompatible with bose statistics, this state should be excluded. Hence the bosonic particles for $S(0) = -1$ behave like fermions, and indeed we will refer to this case as "fermionic". On the other hand identical particles that are fermion states of identical rapidity are allowed, and will be referred to as part of a "bosonic" system.

Now we would like to address the issue of the entropy for both the bosonic and fermionic cases. For small intervals of the rapidity $\Delta\beta_\alpha \ll 1$ but such that $\frac{1}{mL} \ll \Delta\beta_\alpha$ there is a large number of levels $N_\alpha \sim \rho(\beta_\alpha)\Delta\beta_\alpha$ and large number of particles $n_\alpha \sim \rho_1(\beta_\alpha)\Delta\beta_\alpha$ that are distributed among these levels. The number of different distributions for the two cases are,

$$\text{“fermionic”} = \frac{(N_\alpha)!}{(n_\alpha)!(N_\alpha - n_\alpha)!}, \quad \text{“bosonic”} = \frac{(N_\alpha + n_\alpha - 1)!}{(n_\alpha)!(N_\alpha - 1)!}. \tag{5.275}$$

The entropy $\mathcal{S}(\rho, \rho_1) = \log \mathcal{N}(\rho, \rho_1)$, where $\mathcal{N}(\rho, \rho_1)$ is the total number of states, is given by,

$$\mathcal{S}_{\text{Fermi}} = \int \mathrm{d}\beta[\rho \log \rho - \rho_1 \log \rho_1 - (\rho - \rho_1) \log(\rho - \rho_1)],$$
$$\mathcal{S}_{\text{Bose}} = \int \mathrm{d}\beta[-\rho \log \rho - \rho_1 \log \rho_1 + (\rho + \rho_1) \log(\rho + \rho_1)]. \tag{5.276}$$

Computing the partition function by performing the trace over all the states of the system translates to the minimization of the free energy,

$$-RLf(\rho, \rho_1) = -RH_b(\rho_1) + \mathcal{S}(\rho, \rho_1), \tag{5.277}$$

with respect to the densities ρ and ρ_1, subjected to the constraints (5.273). Using (5.273) the extremum equations take the form,

$$-Rm\cosh\beta + \epsilon(\beta) \pm \int \varphi(\beta - \beta') \log(1 \pm \mathrm{e}^{-\epsilon(\beta')}) \frac{\mathrm{d}\beta'}{2\pi} = 0, \tag{5.278}$$

where the upper sign is for the fermionic case, the lower for the bosonic, and the "pseudoenergies" $\epsilon(\beta)$ are defined via,

$$\frac{\rho_1}{\rho} = \frac{\mathrm{e}^{-\epsilon(\beta)}}{1 \pm \mathrm{e}^{-\epsilon(\beta)}}. \tag{5.279}$$

The extremal free energy is,

$$Rf(R) = \mp m \int (\cosh\beta) \log(1 \pm \mathrm{e}^{-\epsilon(\beta)}) \frac{\mathrm{d}\beta}{2\pi}. \tag{5.280}$$

Comparing (5.273) with (5.280) determines a useful relation,

$$\rho(\beta) = \frac{L}{2\pi} \frac{\partial \epsilon(\beta)}{\partial R}. \tag{5.281}$$

Finally we can also write down the TBA expressions for the expectation values of $T_{\mu,\nu}$. Using the last relation and (5.279) we get,

$$< T_{xx} > = 2\pi \frac{\mathrm{d}E(R)}{\mathrm{d}R} = \frac{2\pi}{L} m \int \rho_1(\beta) \cosh\beta \mathrm{d}\beta,$$
$$< T^{\mu}_{\mu} > = \frac{2\pi}{R} \frac{\mathrm{d}[RE(R)]}{\mathrm{d}R}$$
$$= m \int \frac{2\pi}{L}[\rho_1(\beta) \cosh\beta - \frac{1}{R} \frac{\rho_1}{\rho} \frac{\partial \epsilon(\beta)}{\partial \beta} \sinh\beta] \mathrm{d}\beta. \tag{5.282}$$

These relations generalize in a straightforward manner to the more general case of N types of particles with masses m_a, $a = 1, \ldots, \hat{N}$ and scattering amplitudes S_{ab} which are now $\hat{N} \times \hat{N}$ matrices.

For integrable models where one can take the limit of $rRm_1 \to 0$, with m_1 being the lowest mass of the system, one can determine the scaling function $\tilde{c}(r)$. In this limit $\epsilon(\beta)$ become constant in the regime where $-\ln(2/r) \ll \beta \ll \ln(2/r)$. Their constant value is determined by the limit of equation (5.273) which now for the case of N particles reads,

$$\epsilon_a = -\sum_{b=1}^{\hat{N}} \int \mathrm{d}\beta \phi_{ab}(\beta) \ln[1 + \mathrm{e}^{-\epsilon_b}] = -\frac{1}{2\pi} \sum_{b=1}^{\hat{N}} [\delta_{ab}(+\infty) - \delta_{ab}(-\infty)] \ln[1 + \mathrm{e}^{-\epsilon_b}], \tag{5.283}$$

recall that $\phi_{ab}(\beta) = \partial_\beta \delta_{ab}(\beta)$.

Interpolating between these values of $\epsilon(\beta)$ for $\beta \ll \ln(2/r)$ and the values for $\beta \to \infty$ where $\epsilon(\beta)$ go exponentially to infinity, and taking finally the $r \to 0$ limit one finds the effective Virasoro anomaly using the relation between the free energy and the scaling factor (5.265),

$$\tilde{c}(0) = \sum_{a=1}^{N} \tilde{c}_a(\epsilon_a) = \frac{6}{\pi^2} \sum_{a=1}^{N} L\left(\frac{1}{1 + \mathrm{e}^{\epsilon_a}}\right), \tag{5.284}$$

where $\tilde{c}_a$ is the scaling factor associated with the particle of type a and $L(x)$ is the dilogarithm function,

$$L(x) = -\frac{1}{2} \int_0^x \left[\frac{\ln t}{1-t} + \frac{\ln(1-t)}{t}\right] \mathrm{d}t. \tag{5.285}$$

Thus the determination of the Virasoro anomaly of the underlying CFT of our massive integrable theory with a purely elastic S-matrix follows from the solution for the pseudo-energies (5.283) and plugging it into the expression of the scaling factor. One can proceed with the continuous TBA and determine not only the ground state energy but also the full spectrum of energies as well as the spectrum of the eigenvalues of all the conserved charges, as was described for the discretized case discussed in Section 5.14. This is beyond the scope of this book. We refer the reader to the relevant papers in the reference list.