BULL. AUSTRAL. MATH. SOC. 54E35, 54G05
VOL. 68 (2003) [191–203]

HYPERCONVEX SPACES REVISITED

MARCIN BORKOWSKI, DARIUSZ BUGAJEWSKI AND HUBERT PRZYBYCIEŃ

In this paper we describe a construction of a large class of hyperconvex metric spaces. In particular, this construction contains well-known examples of hyperconvex spaces such as $\mathbb{R}^2$ with the "river" metric or with the radial one.

Further, we investigate linear hyperconvex spaces with extremal points of their unit balls. We prove that only in the case of a plane (and obviously a line) is there a strict connection between the number of extremal points of the unit ball and the hyperconvexity of the space.

Some additional properties concerning the notion of hyperconvexity are also investigated.

1. INTRODUCTION

The notion of hyperconvexity of metric spaces was introduced by Aronszajn and Panitchpakdi [**2**] when studying extensions of uniformly continuous transformations between metric spaces. Their main result in this area states that a necessary and sufficient condition which guarantees the existence of an extension of any transformation T into a metric space X with conservation of a subadditive modulus of continuity is that X be hyperconvex. Recall that the linear version of this result was obtained by Aronszajn in 1929 ([**1**]) but it was never published. The recent increasing interest of fixed point theory in hyperconvex spaces goes back to Sine [**25**] and Soardi [**28**] (see also [**3**] for complete generality) who independently proved that the fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then many results have been shown to hold in hyperconvex spaces (see [**3, 5, 6, 9, 10, 11, 12, 14, 15, 16, 18, 21, 24, 26, 27, 29**]).

From the topological point of view a hyperconvex space is an absolute retract by a nonexpansive retraction (see [**2**, Theorem 8, Corollary 4, 5, p. 422]). Hyperconvex real Banach spaces can be treated, in view of the theorem of Nachbin and Kelley ([**23, 13**]), as Stonian spaces $C(K)$ of all real continuous functions on extremally disconnected compact Hausdorff spaces K. Hence infinite-dimensional hyperconvex real Banach spaces are not reflexive and both l^∞ and L^∞ are good examples of hyperconvex spaces

Received 19th November,2003
We would like to thank Professor Ryszard Urbański for his suggestions concerning Theorem 3.1

(see [20]). Another type of example of these spaces are $\mathbb{R}^2$ with the "river" metric or with the radial one (see [6, Lemmas 2 and 3]).

The principal topic of Section 3 is to construct a large class of hyperconvex metric spaces. This construction uses Chebyshev subsets of a normed space endowed with a hyperconvex metric. $\mathbb{R}^2$ with the "river" metric or with the radial one are special cases of this construction. In the former case the corresponding Chebyshev subset is $\{(x,y) \in \mathbb{R}^2 : y = 0\}$ and in the latter case it is $\{(0,0)\}$.

The described construction overlaps with the notion of $\mathbb{R}$-trees. There exist $\mathbb{R}$-trees which cannot be obtained using the presented construction; on the other hand, not every hyperconvex space constructed in the described way is an $\mathbb{R}$-tree. Nevertheless, both notions are connected: if the Chebyshev subset in question is an $\mathbb{R}$-tree, so is the resulting space.

Recall also that the unit sphere in a hyperconvex real Banach space always has an extremal point (Nachbin's conjecture; see also [13, 2]). In connection with these classical results there arises a question, whether it is possible to state that a given Banach space (recall here that hyperconvex spaces are always complete, see [2, Theorem 1, p. 417]) is hyperconvex by knowing how many extremal points its closed unit ball possesses. It appears that only in the case of a line or a plane is the answer positive. Note, however, that the case of a line is trivial while the case of a plane is much more complicated. Section 4 of this paper is devoted to these considerations.

In the last section we investigate the notion of hyperconvexity of a given space in connection with the diameter of each bounded subset included in it. More precisely, it is easy to verify that hyperconvexity of a given metric space implies that each of its bounded subsets is included in a ball of radius equal to half its diameter. In the case of Banach spaces the converse implication holds, too (see [7]). In Section 5 we show that this converse implication cannot be extended to metric spaces.

In the next section we include only the basic definitions and theorems concerning hyperconvexity which will be useful in the sequel.

2. Preliminaries

In this section we briefly recall the basic definitions and facts which we shall use in the sequel.

A metric space (X,d) is called hyperconvex (or $\mathfrak{m}$-hyperconvex) if for any indexed class of closed balls in this space, $\overline{B}(x_i, r_i)$ for $i \in I$ (or for any such class with $\operatorname{card}(I) < \mathfrak{m}$), satisfying the condition that $d(x_i, x_j) \leqslant r_i + r_j$ for any $i, j \in I$, the intersection $\bigcap_{i \in I} \overline{B}(x_i, r_i)$ is nonempty. Further, a metric space (X,d) is called metrically convex if for any two distinct points $x, y \in X$ and for some positive real numbers a, b such that $d(x,y) = a + b$ there exists a point $z \in X$ such that $d(x,z) = a$ and

$d(z,y)=b$. If such a point exists for any decomposition $d(x,y)=a+b,\ \ a,b>0$, then X is called totally convex. Thus, we can say that hyperconvex spaces are "the most" metrically convex spaces.

The following property (P) (or $(P_{\mathfrak{m}})$) is weaker than hyperconvexity (or $\mathfrak{m}$-hyperconvexity). A metric space (X,d) is said to have the property (P) (or $(P_{\mathfrak{m}})$, where $\mathfrak{m}\geqslant 3$) if for any class of closed balls (or for any such class of cardinality less than $\mathfrak{m}$) such that every pair of these balls intersects, all these balls intersect.

The following characterisation was given by Aronszajn and Panitchpakdi in [**2**].

THEOREM. *Hyperconvexity (or $\mathfrak{m}$-hyperconvexity) is equivalent to the property (P) (or $(P_{\mathfrak{m}})$) and total convexity.*

In the case of the property (P) (or $(P_{\mathfrak{m}})$ for $\mathfrak{m}>\aleph_0$) we can replace total convexity by metrical convexity.

An $\mathbb{R}$-tree (see [**17**]) is a nonempty metric space M, satisfying the following:

(a) any two points $p,q\in M$ are joined by a unique metric segment $[p,q]$ (that is, a set isometric to some interval $[a,b]\subset\mathbb{R}$ such that this isometry maps a onto p and b onto q);

(b) if $p,q,r\in M$ then $[p,q]\cap[p,r]=[p,w]$ for some $w\in M$;

(c) if $p,q,r\in M$ and $[p,q]\cap[q,r]=\{q\}$ then $[p,q]\cup[q,r]=[p,r]$.

Now let K be a nonempty subset of a normed space X. For each $x\in X$ we say that $y\in K$ is a best approximation to x from K, if

$$\|x-y\|=\inf\bigl\{\|x-z\|:z\in K\bigr\}.$$

The set K is said to have the property U_x if the best approximation to x from K is unique. Finally, K is called Chebyshev if it has the property U_x for all $x\in X$.

Let X be a real normed space and let $C\subset X$ be a Chebyshev subset. For any $x\in X$, we shall denote the best approximation to x from C (the "projection" of x onto C) by x^p.

Note that $x=x^p$ if and only if $x\in C$. Moreover, any Chebyshev set $C\subset X$ has to be closed. Let us also recall that any nonempty, closed and convex subset of a Hilbert space and any nonempty, closed and strictly convex subset of a finite-dimensional normed space is a Chebyshev subset.

For completeness we recall the definitions of two metrics in $\mathbb{R}^2$. The following metric:

$$d(v_1,v_2)=\begin{cases}|y_1-y_2|, & \text{if } x_1=x_2,\\ |y_1|+|y_2|+|x_1-x_2|, & \text{if } x_1\neq x_2,\end{cases}$$

where $v_i=(x_i,y_i)\in\mathbb{R}^2$ for $i=1,2$ is called the "river" metric in $\mathbb{R}^2$.

Next, the radial metric in $\mathbb{R}^2$ is defined as follows:

$$d(v_1, v_2) = \begin{cases} \rho(v_1, v_2), & \text{if } 0 = (0,0),\ v_1,\ v_2 \text{ are colinear,} \\ \rho(v_1, 0) + \rho(v_2, 0), & \text{otherwise,} \end{cases}$$

where ρ denotes the usual Euclidean metric in $\mathbb{R}^2$.

In the rest of the paper we shall denote the closed segment with the ends x, y, that is, the set $\{(1-\alpha)x + \alpha y : 0 \leqslant \alpha \leqslant 1\}$ by xy; the symbol $xy^{\rightarrow}$ will mean the halfline beginning at x and containing y: $xy^{\rightarrow} = \{(1-\alpha)x + \alpha y : 0 \leqslant \alpha\}$.

3. A construction of hyperconvex metric spaces

Let X be a normed space and $C \subset X$ be a Chebyshev subset. In this section the following property of pairs of points $x, y \in X$ will be frequently used:

$$x^p = y^p \text{ and } x,\ x^p,\ y \text{ are colinear.} \tag{3.1}$$

Let us introduce the following

DEFINITION 3.1: Let $C \subset X$ be a Chebyshev set in a normed space X and let d_C be any metric defined on C. Define $d: X \times X \to [0, +\infty)$ by the formula:

$$d(x,y) = \begin{cases} \|x - y\|, & \text{if } x \text{ and } y \text{ satisfy the condition (3.1)} \\ \|x - x^p\| + d_C(x^p, y^p) + \|y^p - y\|, & \text{otherwise.} \end{cases} \tag{3.2}$$

LEMMA 3.1. *A function d defined by (3.2) is a metric on X.*

PROOF: The proof of this lemma is straightforward, although it is necessary to verify several cases. Therefore we omit it. ∎

For convenience, we shall describe some properties of closed balls in the metric space (X, d) described in Definition 3.1. If $x \notin C$ and $r < \operatorname{dist}(x, C)$, then $\overline{B}(x, r)$ is a segment contained in the halfline beginning at x^p and containing x. Further, if any closed ball $\overline{B}(x, r)$ has a nonempty intersection with C, then $\overline{B}(x, r) \cap C$ is a closed ball in (C, d_C). In what follows we also apply the observation that if $x \notin C$ and $y \in x^p x$, then $y^p = x^p$.

To prove that the metric space (X, d) is hyperconvex, we shall need the following three lemmas.

LEMMA 3.2. *Let $x \notin C$. Then the set $S_x = \{y \in x^p x^{\rightarrow} : y^p = x^p\}$ is either the halfline $x^p x^{\rightarrow}$ or a closed segment $x^p z$, for some $z \in x^p x^{\rightarrow}$.*

PROOF: We can obviously identify points of the halfline $x^p x^{\rightarrow}$ with nonnegative numbers. Hence let $z = \sup\{y \in x^p x^{\rightarrow} : y^p = x^p\}$. If $z = +\infty$, then $S_x = x^p x^{\rightarrow}$.

If $z < +\infty$, then note that $z^p = x^p$. Indeed, let us consider an increasing sequence (z_n) such that $z_n \in x^p x^{\to}$ for $n \in \mathbb{N}$ and $z_n \to z$ as $n \to \infty$. For every $n \in \mathbb{N}$ we have $z_n^p = x^p$, so $\operatorname{dist}(z_n, C) = \|z_n - x^p\|$. Further, $\operatorname{dist}(z_n, C) \to \operatorname{dist}(z, C)$ and $\|z_n - x^p\| \to \|z - x^p\|$ as $n \to \infty$, so $\operatorname{dist}(z, C) = \|z - x^p\|$. As C is Chebyshev, we have $z^p = x^p$. Hence it is enough to put $S_x = z^p z$. $\square$

LEMMA 3.3. *If $x \notin C$ belongs to a certain closed ball $\overline{B}$ in (X, d) such that $\overline{B} \cap C \neq \emptyset$, then $x^p x \subset \overline{B}$.*

PROOF: Put $\overline{B} = \overline{B}(x_0, r)$. Suppose that the condition (3.1) is satisfied with x_0 in place of y. Let $y \in x^p x$. Then either $y \in x_0 x$ or $y \in x_0^p x_0$. In the first case we have $d(y, x_0) = \|y - x_0\| \leqslant \|x - x_0\| \leqslant r$ and in the latter case we obtain $d(y, x_0) = \|y - x_0\| \leqslant \|x_0^p - x_0\| \leqslant r$.

Now, suppose that the condition (3.1) with x_0 instead of y is not satisfied. Let $y \in x^p x$. Then $y = (1 - \alpha)x + \alpha x^p$ for some $\alpha \in [0, 1]$ and $\|y^p - y\| \leqslant \|x^p - x\|$. Then we obtain: $d(x_0, y) = \|x_0 - x_0^p\| + d_C(x_0^p, y^p) + \|y^p - y\| \leqslant \|x_0 - x_0^p\| + d_C(x_0^p, x^p) + \|x^p - x\| = d(x_0, x) \leqslant r$. $\square$

LEMMA 3.4. *Let $\overline{B}_0 = \overline{B}(x_0, r_0)$ and $\overline{B}_1 = \overline{B}(x_1, r_1)$ have no common points with C and let $\overline{B}_0 \cap \overline{B}_1 \neq \emptyset$. Then $\overline{B}_0$ and $\overline{B}_1$ are segments included in S_{x_0}.*

PROOF: The proof of Lemma 3.4 is obvious and therefore we omit it. $\square$

The main result of this section is the following

THEOREM 3.1. *Let us assume that (C, d_C), where $C \subset X$ is Chebyshev, is hyperconvex. Then, the metric space (X, d), where d is the metric described in Definition 3.1, is also hyperconvex.*

PROOF: First we show that (X, d) is metrically convex. Let $x, y \notin C$ and $x \neq y$. If x and y satisfy the condition (3.1), then we can set $z = 1/2(x + y)$. In any other case it is enough to apply the metrical convexity of (C, d_C) to the points x^p, y^p.

Now we show that (X, d) has the property (P). Let $\mathcal{B} = \{\overline{B}_\lambda\}_{\lambda \in \Lambda}$ be any family of closed balls in (X, d) such that every pair of these balls intersects. First suppose that one of these balls, for example, $\overline{B}_{\lambda_0} = \overline{B}(x_0, r_0)$ has no common points with C. Then $\overline{B}_{\lambda_0}$ is a closed segment contained in S_{x_0}. Then for every $\lambda \in \Lambda$ either $\overline{B}_\lambda \cap C = \emptyset$ (in this case $\overline{B}_\lambda$ is also a segment contained in S_{x_0}) or $\overline{B}_\lambda \cap C \neq \emptyset$. In the latter case, by Lemma 3.3, $\overline{B}_\lambda \cap S_{x_0}$ is a closed segment in S_{x_0}. Moreover, if two balls belonging to $\mathcal{B}$ intersect, then their common intersection with S_{x_0} is also nonempty. Indeed, supposing that none of these balls is a subset of S_{x_0} and using the fact that they have a common point with C as well as with $\overline{B}_{\lambda_0}$, by Lemma 3.3 we know that x_0^p belongs to their intersection. Hence, considering the family of intersections of the balls belonging to $\mathcal{B}$ with S_{x_0}, we infer that $\bigcap_{\lambda \in \Lambda} \overline{B}_\lambda \neq \emptyset$.

https://doi.org/10.1017/S0004972700037588 Published online by Cambridge University Press

Now assume that every ball belonging to $\mathcal{B}$ has a common point with C. Then every two of these balls have a common point in C. Thus it is enough to consider a family of intersections of balls from $\mathcal{B}$ with C. The proof is complete. ∎

REMARK 3.1. The question is if any hyperconvex metric space can be represented in the way described in Theorem 3.1. Let us note that the topology generated by the metric d described in Definition 3.1 is not linear and therefore the answer to this question is negative.

REMARK 3.2. Obviously not every $\mathbb{R}$-tree can be obtained in the way described in Definition 3.1, which requires a linear structure in addition to the metric one. On the other hand, not every hyperconvex space obtained using the described method is an $\mathbb{R}$-tree. For instance, the space described in Example 3.2 is not an $\mathbb{R}$-tree. Indeed, let us consider its subsets: $A = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, y = 0\}$ and $B = \{(x,y) \in \mathbb{R}^2 : (0 \leqslant x \leqslant 1/2 \wedge y = x) \vee (1/2 < x \leqslant 1 \wedge y = 1 - x)\}$. It is not difficult to see that both are metric segments joining the points $(0,0)$ and $(1,0)$.

REMARK 3.3. Note that every hyperconvex space with unique metric segments is an $\mathbb{R}$-tree (see [**17**, Theorem 3.2]). This leads to the conclusion that, with the assumptions of Theorem 3.1, the space (X,d) described in Definition 3.1 is an $\mathbb{R}$-tree if and only if the space (C,d_C) is an $\mathbb{R}$-tree, too.

EXAMPLE 3.1. Let us consider $\mathbb{R}^2$ with the usual Euclidean norm. Then putting $C = \{(0,0)\}$ in (3.2) defines the well-known radial metric. Further, putting $C = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ defines the "river" metric. Hence Lemmas 2 and 3 from [**6**] which state that these two metric spaces are hyperconvex follow immediately from Theorem 3.1.

EXAMPLE 3.2. Again, let us consider $\mathbb{R}^2$ with the usual Euclidean norm and let $C = [-1,1] \times [-1,1]$. Moreover, let d_C be the metric on C induced by the "maximum" norm. In this case $(\mathbb{R}^2, d)$, where d is defined by (3.2), becomes a hyperconvex metric space.

EXAMPLE 3.3. Let M_i, $i = 1,\dots,n$ be subspaces of a normed space X and let $M_n = X$. Assume that $M_i \subset M_{i+1}$ and M_i is a Chebyshev subset of M_{i+1} for $i = 1,\dots,n-1$. Then we can use the Definition 3.1 $(n-1)$ times obtaining certain hyperconvex metric space. For example, let us consider $\mathbb{R}^3$ with the usual Euclidean metric and let $C = \{(0,0,0)\}$ and $M = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = 0\}$. The set $C \subset M$ is Chebyshev and thus using (3.2) we obtain a hyperconvex metric space M. Further, $M \subset \mathbb{R}^3$ is also Chebyshev and therefore applying (3.2) again we infer that $\mathbb{R}^3$ with this metric is hyperconvex.

EXAMPLE 3.4. Let us consider the space $L^2([0,1])$ of all real square-integrable functions on $[0,1]$. Let $C = \{x \mapsto ax+b : a, b \in \mathbb{R}\} \subset L^2([0,1])$ be the subspace of all affine functions on $[0,1]$. It is easy to see that C is a nonempty, closed and convex subset of the Hilbert space $L^2([0,1])$, so it is Chebyshev. Let us define a norm $\|\cdot\|$ on C by the formula: $\|x \mapsto ax+b\| = \max\{|a|, |b|\}$. This way we obtain a space $(C, \|\cdot\|)$ which is isometric to the Euclidean plane with the "maximum" norm, so it is hyperconvex. Applying Definition 3.1 we obtain a hyperconvex metric on $L^2([0,1])$.

4. Hyperconvexity and extremal points

At the beginning of this section we shall consider the linear space $\mathbb{R}^2$. First, we prove the following

PROPOSITION 4.1. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^2$ such that the closed unit ball in this norm has exactly four extremal points. Then $\mathbb{R}^2$ with this norm is a hyperconvex metric space.*

PROOF: Let $z_i = (x_i, y_i)$, where $i = 1, \dots, 4$, denote the extremal points of the closed unit ball $\overline{B}$. We may assume that $z_1 = -z_3$ and $z_2 = -z_4$. Let us denote the maximum norm in $\mathbb{R}^2$ by $\|\cdot\|_m$. Consider the linear mapping $T\colon (\mathbb{R}^2, \|\cdot\|_m) \to (\mathbb{R}^2, \|\cdot\|)$ given by the matrix

$$M(T) = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) & \frac{1}{2}(x_1 - x_2) \\ \frac{1}{2}(y_1 + y_2) & \frac{1}{2}(y_1 - y_2) \end{bmatrix}.$$

Put $u_1 = (1,1)$, $u_2 = (1,-1)$, $u_3 = (-1,1)$, $u_4 = (-1,-1)$. Obviously, we have $Tu_i = z_i$ for $i = 1, \dots, 4$. Let us denote by $\overline{B}_m$ the closed unit ball in the space $(\mathbb{R}^2, \|\cdot\|_m)$. Then $\overline{B}_m = \operatorname{conv}\{u_i : i = 1, \dots, 4\}$ and $T(\overline{B}_m) = \overline{B}$. Thus T is a nonsingular linear mapping of norm 1. Further, there exists T^{-1} and $\|T^{-1}\| = 1$, so T is an isometry. Hence $(\mathbb{R}^2, \|\cdot\|_m)$ is hyperconvex and the proof is complete. $\square$

The question is now as follows: does there exist a norm in $\mathbb{R}^2$ such that the closed unit ball in this normed space has more than 4 extremal points and this space is hyperconvex? The following result gives the answer.

PROPOSITION 4.2. *Let there be given a norm in $\mathbb{R}^2$ such that the closed unit ball in this space has more than four extremal points. Then this space is not hyperconvex.*

PROOF: Let $\overline{B}$ be the closed unit ball in the space in question. Suppose that $\overline{B}$ has more than 4 extremal points. Let us denote projections onto axes $0x$ and $0y$ by p_x and p_y, respectively. Put $m_x = \max\{p_x(x,y) : (x,y) \in \overline{B}\}$. Let $z_m = (x_m, y_m)$ be an extremal point of $\overline{B}$ such that $x_m = m_x$. Obviously such a point exists. Without loss of generality we can assume that z_m lies on $0x$. Denote by l_1, l_2 the extremal tangent halflines to $\overline{B}$ at z_m and, analogously, by l_3, l_4 the extremal tangent halflines to $\overline{B}$ at $-z_m$. Assume that $l_1, l_4 \in \{(x,y) \in \mathbb{R}^2 : y \geqslant 0\}$. It is well known that such halflines exist.

First, assume that l_1 is not parallel to l_4. Let $\overline{B}_1 = 2z_m + \overline{B} = T_{2z_m}(\overline{B})$. It is clear that $l_1 \parallel l_3$ and $l_2 \parallel l_4$. Thus $\angle(l_2, l_3) = \angle(l_1, l_4)$; let us denote this angle by α. Let $m_y = \max\{p_y(x,y) : (x,y) \in \overline{B}\}$ and let $u_n = (x_n, y_n)$ be an extremal point of $\overline{B}$ such that $y_n = m_y$. By the assumptions we infer that $u_n \in \triangle(z_m, -z_m, s) \setminus \{s\}$, where s is the point of intersection of l_1 and l_4. Denote by β the angle between the extremal tangent halflines at u_n or put $\beta = \pi$ if these halflines are parallel. Then it is easy to see that $\alpha < \beta$.

Now, there are two possibilities. First, suppose that $u_n \in \operatorname{Int} \triangle(z_m, -z_m, s) \setminus \{s\}$. Let $\overline{B}' = T_{z_m - u_n}(\overline{B})$. Then $\overline{B}' \subset \{(x,y) \in \mathbb{R}^2 : y \leqslant 0\}$. Because $\alpha < \beta$, then $\operatorname{Int} \overline{B}' \cap \operatorname{Int} \overline{B} \neq \emptyset$ and $\operatorname{Int} \overline{B}' \cap \operatorname{Int} \overline{B}_1 \neq \emptyset$. Hence the translation $\overline{B}_2$ of $\overline{B}'$ by a sufficiently small vector $u_\varepsilon = [0, -\varepsilon]$ for some $\varepsilon > 0$ $\left(\overline{B}_2 = T_{z_m - u_n + u_\varepsilon}(\overline{B})\right)$ satisfies the conditions: $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B} \neq \emptyset$, $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B}_1 \neq \emptyset$. We also have $\overline{B}_1 \cap \overline{B} \neq \emptyset$, while $\overline{B}_1 \cap \overline{B}_2 \cap \overline{B} = \emptyset$.

Suppose now that $u_n \in l_1$ (in the case when $u_n \in l_4$ we argue similarly). Because $\overline{B}$ is convex, the segment $(-z_m)u_n$ is included in $\overline{B}$ and so is $z_m(-u_n)$. Let $\overline{B}'$ be defined as above. Since $\alpha < \beta$, we have $\overline{B}' \cap \overline{B}_1 \neq \emptyset$ and $\operatorname{Int} \overline{B}' \cap \operatorname{Int} \overline{B} \neq \emptyset$. Let u_ε be a sufficiently small vector with the beginning at z_m, parallel to the segment $u_m z_m$ and not included in l_1. Then for $\overline{B}_2 = T_{z_m - u_m + u_\varepsilon}(\overline{B})$, we have $\overline{B}_2 \cap \overline{B}_1 \neq \emptyset$, $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B} \neq \emptyset$ and $\overline{B} \cap \overline{B}_1 \neq \emptyset$, but $\overline{B}_1 \cap \overline{B}_2 \cap \overline{B} = \emptyset$.

Finally, assume that l_1 is parallel to l_4. If $\beta < \pi$, then after exchanging $0x$ and $0y$ we argue similarly as above. Hence we can assume that $\beta = \pi$. Then we have $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B} \neq \emptyset$ and $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B}_1 \neq \emptyset$. Thus, translating $\overline{B}'$ by a sufficiently small vector $u_\varepsilon = [0, -\varepsilon]$, where $\varepsilon > 0$, we obtain $\overline{B}_2 = T_{z_m - u_m + u_\varepsilon}(\overline{B})$ such that $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B}_1 \neq \emptyset$, $\operatorname{Int} \overline{B}_2 \cap \operatorname{Int} \overline{B} \neq \emptyset$; moreover $\overline{B}_1 \cap \overline{B} \neq \emptyset$ and $\overline{B}_1 \cap \overline{B}_2 \cap \overline{B} = \emptyset$, which completes the proof. $\square$

As a corollary from Propositions 4.1 and 4.2 we obtain the following characterisation.

Theorem 4.1. *A space $(\mathbb{R}^2, \|\cdot\|)$ is hyperconvex if and only if the closed unit ball in this space has exactly four extremal points.*

The space $\mathbb{R}^3$ has quite different character. Namely, we shall prove the following

Proposition 4.3. *For every even number $n \geqslant 6$ there exists a norm $\|\cdot\|_n$ in $\mathbb{R}^3$ such that:*

(i) *the closed unit ball in* $(\mathbb{R}^3, \|\cdot\|_n)$ *has exactly* n *extremal points;*
(ii) *the space* $(\mathbb{R}^3, \|\cdot\|_n)$ *is not hyperconvex.*

PROOF: For $n = 6$ it is enough to consider the norm $\|x\|_6 = \sum_{i=1}^{3} |x_i|$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Now, let $n > 6$. Then $n = 2k + 2$ for some $k \geqslant 3$. Let $e_j^k = \exp\big((2j\pi/k)i\big)$ for $j = 0, 1, \ldots, k-1$. Put $p_0 = (0,0,2)$, $q_0 = (0,0,-2)$, $p_j^k = (e_j^k, 1)$ and $q_j^k = (-e_j^k, -1)$ for $j = 0, 1, \ldots, k-1$. Let $A_k = \{p_0, q_0\} \cup \bigcup_{j=0}^{k-1} \{p_j^k, q_j^k\}$ and let $\overline{B}_k = \operatorname{conv} A_k$. It is clear that $\overline{B}_k$ is a closed unit ball in $\mathbb{R}^3$ with the norm $\|\cdot\|_n$ defined by the Minkowski functional of $\overline{B}_k$ (shortly: $\|\cdot\|_n = p_{\overline{B}_k}(\,\cdot\,)$). Moreover, let us remark that card $(\operatorname{ext} \overline{B}_k) = n$, where $\operatorname{ext} \overline{B}_k$ denotes the set of all extremal points of $\overline{B}_k$. Indeed, let us consider two cases. First, assume that k is even. Then

$$\operatorname{conv}\left(\bigcup_{j=0}^{k-1} \{p_j^k\}\right) = \operatorname{conv}\left(\bigcup_{j=0}^{k-1} \{q_j^k\}\right) + p_0.$$

Let $z_1 = (0,0,4)$ and $z_2 = (2,0,2)$ and consider closed balls $\overline{B}(0,1)$, $\overline{B}(z_1,1)$ and $\overline{B}(z_2,1)$. It can be easily seen that $\overline{B}(0,1) \cap \overline{B}(z_1,1) = \{(0,0,2)\}$, $(1,0,1) \in \overline{B}(0,1) \cap \overline{B}(z_2,1)$ and $(1,0,3) \in \overline{B}(z_1,1) \cap \overline{B}(z_2,1)$, but $\overline{B}(0,1) \cap \overline{B}(z_1,1) \cap \overline{B}(z_2,1) = \emptyset$.

Now, consider the case when k is odd. We have $e_{(k-1)/2}^k = \big(-\cos(\pi/k), \sin(\pi/k)\big)$ and $e_{(k+1)/2}^k = \big(-\cos(\pi/k), -\sin(\pi/k)\big)$. Hence it is clear that in the set $\operatorname{conv}\big(\bigcup_{j=0}^{k-1} \{p_j^k\}\big)$ the points $p_{(k-1)/2}^k$, $p_{(k+1)/2}^k$ have the smallest first coordinate and, analogously, in the set $\operatorname{conv}\big(\bigcup_{j=0}^{k-1} \{q_j^k\}\big)$ the points $q_{(k-1)/2}^k$, $q_{(k+1)/2}^k$ have the biggest first coordinate. Thus it is easily seen that $\overline{B}(0,1) \cap \overline{B}(z_1,1) \neq \emptyset$, $\overline{B}(0,1) \cap \overline{B}(z_2,1) \neq \emptyset$, $\overline{B}(z_1,1) \cap \overline{B}(z_2,1) \neq \emptyset$ and $\overline{B}(0,1) \cap \overline{B}(z_1,1) \cap \overline{B}(z_2,1) = \emptyset$, which completes the proof. ∎

REMARK 4.1. Proposition 4.3 can also be proved in another way. Let us define the set $A = \{(0,0,1), (0,0,-1), (1,1,0), (1,-1,0), (-1,-1,0), (-1,1,0)\}$. Then $\operatorname{conv} A$ is a closed unit ball for some norm in $\mathbb{R}^3$. Consider also the sets $(0,0,2) + \operatorname{conv} A$, $(1,1,1) + \operatorname{conv} A$ and $(1,-1,1) + \operatorname{conv} A$. It is easy to verify that every pair and every triple of these balls have a common point, but all these four balls have no common point (this example was suggested to us by Wośko [**30**]). Hence we obtain an example of a norm in $\mathbb{R}^3$ such that the closed unit ball in this norm has exactly 6 extremal points and that $\mathbb{R}^3$ with this norm is not hyperconvex. In fact, it is an example of a space which is 4-hyperconvex and not 5-hyperconvex (see also [**2,** Problem 1, p. 437]). Using this idea we can prove that for any $n \geqslant 6$ there exists a norm $\|\cdot\|_n$ in $\mathbb{R}^3$ satisfying

https://doi.org/10.1017/S0004972700037588 Published online by Cambridge University Press

(i) and (ii) in Proposition 4.3. However, note that the normed space constructed in the proof of Proposition 4.3 is not 4-hyperconvex.

Now we shall consider $\mathbb{R}^n$, where $n > 2$ is arbitrary. First we prove the following

Lemma 4.1. *Let $\overline{B}$ be a closed unit ball in the space $\mathbb{R}^n$, $e_{n+1}^+ = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, $e_{n+1}^- = (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$, $\overline{B}_1 = \{(x, 0) : x \in \overline{B}\}$ and let $\overline{B}_2 = \operatorname{conv}\bigl(\overline{B}_1 \cup \{e_{n+1}^+, e_{n+1}^-\}\bigr)$. Then*

(1) *$\overline{B}_2$ is a closed unit ball in $\mathbb{R}^{n+1}$, that is, its Minkowski functional $p_{\overline{B}_2}$ is a norm and $\overline{B}_2 = \{z \in \mathbb{R}^{n+1} : p_{\overline{B}_2}(z) \leqslant 1\}$.*

(2) *If $\overline{B}$ has n extremal points, then $\overline{B}_2$ has $(n+2)$ extremal points.*

(3) *Let us denote the canonical projection of the space $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ by p, that is, $p\bigl((x_1, \dots, x_n, x_{n+1})\bigr) = (x_1, \dots, x_n)$. Then $p(\overline{B}_2) = \overline{B}$.*

(4) *Let $\overline{B}(\overline{x}, r)$ be a closed ball in $(\mathbb{R}^{n+1}, p_{\overline{B}_2})$. Then $p\bigl(\overline{B}(\overline{x}, r)\bigr) = p(\overline{x}) + r\overline{B} = \Bigl\{x \in \mathbb{R}^n : p_{\overline{B}}\bigl(x - p(\overline{x})\bigr) \leqslant r\Bigr\}$.*

(5) *If $x, y \in \mathbb{R}^n$, then $p_{\overline{B}_2}\bigl((x, 0) - (y, 0)\bigr) = p_{\overline{B}}(x - y)$.*

Proof: The proofs of (1), (3), (4) and (5) are straightforward and thus we prove only (2).

In view of the Krein and Milman theorem, if $\operatorname{card}\bigl(\operatorname{ext}\overline{B}\bigr) = n$, then $\operatorname{card}\bigl(\operatorname{ext}\overline{B}_2\bigr) \leqslant n + 2$. First we show that if $x \in \operatorname{ext}\overline{B}$, then $(x, 0) \in \operatorname{ext}\overline{B}_2$. Indeed, let $x \in \operatorname{ext}\overline{B}$ and suppose that $(x, 0) = \alpha(x^1, y^1) + \beta(x^2, y^2)$, where $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $(x^1, y^1), (x^2, y^2) \in \overline{B}_2$. It is clear that $x^1, x^2 \in \overline{B}$. Further, if $(x, y) \in \overline{B}_2$ and $x \in \operatorname{ext}\overline{B}$, then $y = 0$. Thus from the equality $(x, 0) = \alpha(x^1, y^1) + \beta(x^2, y^2)$ it follows that $x^1 = x^2 = x$ and $y^1 = y^2 = 0$, so $(x, 0) \in \operatorname{ext}\overline{B}_2$. Moreover, it is clear that $e_{n+1}^+, e_{n+1}^- \in \operatorname{ext}\overline{B}_2$, so $\operatorname{card}\bigl(\operatorname{ext}\overline{B}_2\bigr) = \operatorname{card}\bigl(\operatorname{ext}\overline{B}\bigr) + 2 = n + 2$, which completes the proof. ∎

Using Lemma 4.1 we can prove

Proposition 4.4. *Let $\overline{B}$ be the closed unit ball in a normed space $(\mathbb{R}^n, p_{\overline{B}})$ and let $\overline{B}_2$ be the closed unit ball described in Lemma 4.1. If $(\mathbb{R}^n, p_{\overline{B}})$ is not hyperconvex, then $(\mathbb{R}^{n+1}, p_{\overline{B}_2})$ is not hyperconvex, either.*

Proof: Put $\|\cdot\|_1 = p_{\overline{B}}$ and $\|\cdot\|_2 = p_{\overline{B}_2}$. As $(\mathbb{R}^n, \|\cdot\|_1)$ is not hyperconvex, there exists a family $\{\overline{B}(x_\alpha, r_\alpha) : \alpha \in A\}$ such that $\|x_\alpha - x_\beta\|_1 \leqslant r_\alpha + r_\beta$ for every pair $\alpha, \beta \in A$ $(\alpha \neq \beta)$ and $\bigcap_{\alpha \in A} \overline{B}(x_\alpha, r_\alpha) = \emptyset$. Let $\overline{B}_2'\bigl((x_\alpha, 0), r_\alpha\bigr) = \Bigl\{y \in \mathbb{R}^{n+1} : \bigl\|(x_\alpha, 0) - y\bigr\|_2 \leqslant r_\alpha\Bigr\}$. By Lemma 4.1 (5), we have $\bigl\|(x_\alpha, 0) - (x_\beta, 0)\bigr\|_2 = \|x_\alpha - x_\beta\|_1 \leqslant r_\alpha + r_\beta$ for any $\alpha, \beta \in A$ such that $\alpha \neq \beta$. Supposing that $\bigcap_{\alpha \in A} \overline{B}_2'\bigl((x_\alpha, 0), r_\alpha\bigr) \neq \emptyset$, by Lemma

4.1 (4) we would have

$$\begin{aligned}\emptyset \neq p\Big(\bigcap_{\alpha\in A} \overline{B}'_2((x_\alpha,0),r_\alpha)\Big) &\subset \bigcap_{\alpha\in A} p\Big(\overline{B}'_2((x_\alpha,0),r_\alpha)\Big)\\ &= \bigcap_{\alpha\in A} p\big((x_\alpha,0)+r_\alpha\overline{B}\big) = \bigcap_{\alpha\in A}\big(x_\alpha + r_\alpha\overline{B}\big) = \bigcap_{\alpha\in A}\overline{B}(x_\alpha,r_\alpha) = \emptyset,\end{aligned}$$

which gives a contradiction. ∎

From Proposition 4.3, Lemma 4.1 (2) and Proposition 4.4 we obtain

THEOREM 4.2. *Let $n \geqslant 3$ be arbitrary. For every even $k \geqslant 2n$ there exists a norm $\|\cdot\|_k$ in $\mathbb{R}^n$ such that*

(a) *the closed unit ball in $(\mathbb{R}^n, \|\cdot\|_k)$ has exactly k extremal points and*
(b) *$(\mathbb{R}^n, \|\cdot\|_k)$ is not hyperconvex.*

REMARK 4.2. In the case of infinite-dimensional Banach spaces it is not possible to predict the hyperconvexity of a given space knowing only how many extremal points its closed unit ball possesses. For example, on one hand, the closed unit ball in l^∞ has infinitely many extremal points, and on the other hand, there are well-known examples of reflexive Banach spaces with the same property.

5. APPENDIX

Let (X,d) be a hyperconvex metric space and let $A \subset X$ be bounded. Suppose that $\operatorname{diam} A = a > 0$ and consider the family of closed balls $\overline{B}(x,(1/2)a)$, where $x \in A$. Since X is hyperconvex, we have $\bigcap_{x\in A}\overline{B}(x,(1/2)a) \neq \emptyset$. Let $x_0 \in \bigcap_{x\in A}\overline{B}(x,(1/2)a)$ be arbitrary. Then obviously $A \subset \overline{B}(x_0,(1/2)a)$.

Now we show that the converse implication does not hold. To do this suppose that E is any non-Archimedean normed space over p-adic field $\mathbb{Q}$ and let A be any bounded subset of E such that $\operatorname{diam} A = a > 0$. Fix $x \in A$ and take any $y \in A$. Then $\|x-y\| \leqslant a$, so

$$\Big\|x - \frac{1}{2}(x+y)\Big\| \leqslant \frac{1}{2}a \quad\text{and}\quad \Big\|y - \frac{1}{2}(x+y)\Big\| \leqslant \frac{1}{2}a.$$

Thus $1/2(x+y) \in \overline{B}(x,(1/2)a) \cap \overline{B}(y,(1/2)a)$. In view of the well-known property of non-Archimedean normed spaces [**22**, Chapter III, Section 4, Proposition 2], we have $\overline{B}(y,(1/2)a) \subset \overline{B}(x,(1/2)a)$. In particular $y \in \overline{B}(x,(1/2)a)$, so $A \subset \overline{B}(x,(1/2)a)$.

But, in view of [**4**, Theorem 6] it is known that no non-Archimedean normed space is hyperconvex.

REMARK 5.1. Denote by α the Kuratowski's measure of noncompactness and by ω the measure of weak noncompactness introduced by De Blasi (see [8, 19] for the definitions). Let E be a Banach space, $\overline{B}$ be its closed unit ball and let A be a bounded subset of E. De Blasi [8, Theorem 2] proved that for every $r \geqslant 0$ the following equality holds:

$$\omega\big(A + r\overline{B}\big) = \omega(A) + r\omega\big(\overline{B}\big)$$

and asked whether the same property holds for the α index. Note that in the case of hyperconvex spaces the answer to this question is positive.

REFERENCES

[1] N. Aronszajn, *On metric and metrization*, Ph.D. thesis (University of Warsaw, 1930).

[2] N. Aronszajn and P. Panitchpakdi, 'Extensions of uniformly continuous transformations and hyperconvex metric spaces', *Pacific J. Math.* **6** (1956), 405–439.

[3] J.-B. Baillon, 'Nonexpansive mappings and hyperconvex spaces', *Contemp. Math.* **72** (1988), 11-19.

[4] D. Bugajewski, 'Fixed-point theorems in hyperconvex spaces revisited', in *Nonlinear Operator Theory*, (R.P. Agarwal and D.O'Regan, Editors), Math. Comput. Modelling **32**, 2000, pp. 1457–1461.

[5] D. Bugajewski and R. Espínola, 'Remarks on some fixed point theorems for hyperconvex spaces and absolute retracts', in *Function Spaces*, Lecture Notes in Pure and Appl. Math. Series **213** (Marcel Dekker, New York), pp. 85–92.

[6] D. Bugajewski and E. Grzelaczyk, 'A fixed point theorem in hyperconvex spaces', *Arch. Math.* **75** (2000), 395–400.

[7] W.J. Davis, 'A characterization of $\mathcal{P}_1$ spaces', *J. Approximation Theory* **21** (1977), 315–318.

[8] F.S. De Blasi, 'On a property of a unit sphere in a Banach space', *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **21** (1977), 315–318.

[9] R. Espínola, 'Darbo–Sadovski's theorem in hyperconvex metric spaces', in *Proceedings of the Workshop "Functional Analysis: methods and applications"*, Rend. Circ. Mat. Palermo Ser. II Suppl. **40**, 1996, pp. 129–137.

[10] R. Espínola, W. A. Kirk and G. López, 'Nonexpansive retractions in hyperconvex spaces', *J. Math Anal. Appl.* **251** 2 (2000), 557–570.

[11] R. Espínola and G. López, 'Ultimately compact operators in hyperconvex metric spaces', in *Nonlinear Analysis and Convex Analysis* (World Sci. Publishing, River Edge, NJ, 1998), pp. 142–149.

[12] E. Jawhari, D. Misane and M. Pouzet, 'Retracts: graphs and ordered sets from the metric point of view', *Contemp. Math.* **57** (1986), 175–226.

[13] J.L. Kelley, 'Banach spaces with the extension property', *Trans. Amer. Math. Soc.* **72** (1952), 323–326.

[14] M. A. Khamsi, 'KKM and Ky Fan theorems in hyperconvex metric spaces', *J. Math. Anal. Appl.* **204** (1996), 298–306.

[15] M.A. Khamsi and S. Reich, 'Nonexpansive mappings and semigroups in hyperconvex spaces', *Math. Japon.* **35** (1990), 467–471.

[16] W.A. Kirk, 'Continuous mappings in compact hyperconvex metric spaces', *Numer. Funct. Anal. Optim.* **17** (1996), 599–603.

[17] W.A. Kirk, 'Hyperconvexity of $\mathbb{R}$-trees', *Fund. Math.* **156** (1998), 67–72.

[18] W.A. Kirk and S.S. Shin, 'Fixed points theorems in hyperconvex spaces', *Houston J. Math.* **23** (1997), 175–188.

[19] K. Kuratowski, 'Sur les espaces complets', *Fund. Math.* **15** (1930), 301–309.

[20] H.E. Lacey, *The isometric theory of classical Banach spaces* (Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[21] M. Lin and R. Sine, 'Retractions on the fixed point set of semigroups of nonexpansive maps in hyperconvex spaces', *Nonlinear Anal.* **15** (1990), 943–954.

[22] A. F. Monna, *Analyse non-archimédienne* (Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[23] L. Nachbin, 'A theorem of Hahn–Banach type', *Trans. Amer. Math. Soc.* **68** (1950), 28–46.

[24] S. Park, 'Fixed point theorems in hyperconvex metric spaces', *Nonlinear Anal.* **37** (1999), 467–472.

[25] R.C. Sine, 'On nonlinear contraction semigroups in sup norm spaces', *Nonlinear Anal.* **3** (1979), 885–890.

[26] R.C. Sine, 'Hyperconvexity and approximate fixed points', *Nonlinear Anal.* **13** (1989), 863–869.

[27] R.C. Sine, 'Hyperconvexity and nonexpansive multifunctions', *Trans. Amer. Math. Soc.* **315** (1989), 755–767.

[28] P. Soardi, 'Existence of fixed points of nonexpansive mappings in certain Banach lattices', *Proc. Amer. Math. Soc.* **73** (1979), 25–29.

[29] E. Tarafdar and G. X.-Z. Yuan, 'Some applications of the Knaster–Kuratowski and Mazurkiewicz principle in hyperconvex metric spaces', Nonlinear Operator Theory, *Math. Comput. Modelling* (2000), 1311–1320.

[30] J. Wośko, personal communication.

Optimisation and Control Theory Department
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Poznań
Poland
e-mail: mbork@venus.wmid.amu.edu.pl
ddbb@main.amu.edu.pl
hubert@main.amu.edu.pl