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NON-LINEAR $\mathcal{A}$-PROPER MAPPINGS OF THE ANALYTIC TYPE

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Introduction. Let Y be a complex Banach space, U an open subset of Y, f a mapping of U into Y. Then f is said to be complex analytic if for each pair of elements x and y of Y with x in U, the function $f(x + \xi y)$ of the single complex variable ξ is analytic in ξ on some neighbourhood of the origin.

In [**5**], J. Cronin used the Leray-Schauder degree theory to establish existence and uniqueness theorems for a class of compact complex analytic operators. Her techniques were subsequently systematized and extended to general compact analytic operators by J. T. Schwartz in [**10**]. In [**1**] F. E. Browder introduced analytic type mappings. These were defined to be mappings with the same general properties, of a topological and differential character, as those that hold for complex analytic mappings if considered as operating on the real space X obtained from the complex space Y by ignoring the complex structure. Using the Leray-Schauder degree theory Browder was able to establish various mapping theorems for analytic type compact operators and from these deduced various uniqueness theorems for complex analytic compact operators.

A large proportion of the development of non-linear functional analysis in the last few years has been concentrated on extending the classical theorems of compact operators to non-compact operators. To this end various generalized degree theories have been developed. So it is natural to ask whether the results on compact analytic operators can be extended to non-compact operators. This problem was tackled by Browder and C. P. Gupta in [**3**]. Using one of the generalized degree theories they were able to extend Schwartz's theorems to large classes of non-compact operators.

In [**7**] W. V. Petryshyn introduced the class of A-proper mappings. It is the purpose of this paper to show that Browder's results on analytic type compact operators can be extended to analytic type A-proper mappings. We use the A-proper degree theory as developed by Browder and Petryshyn in [**4**] in place of the Leray-Schauder degree theory. A consequence of this work is a uniqueness theorem for k-ball contractions ($k < 1$).

Since this work was completed it has come to our attention that Nussbaum has proved a similar theorem for complex analytic Φ_0-maps (see [**6**, Theorem 6]) from which our theorem could be derived. However by using a restrictive class of Banach spaces we have avoided the necessity of developing a degree

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theory for complex analytic Φ_0-maps (as in [**6**]) and at the same time have provided mapping theorems for mappings of a real Banach space. It should be noted that our theorems are generalizations of theorems which appear in [**1**].

1. Preliminaries. Throughout this paper X denotes a real Π_1-space (i.e., X is a Banach space with a sequence of finite dimensional subspaces $\{X_n\}$ and a corresponding sequence of projections $\{P_n\}$ (linear idempotent mappings) such that

(a) $X_n \subset X_{n+1}$ for all n;
(b) $P_n : X \to X_n$, $P_n(X) = X_n$ and $||P_n|| = 1$ for all n;
(c) $P_n x \to x$ as $n \to \infty$ for all $x \in X$).

We use the notation $\bar{D}$, ∂D to denote the closure, boundary respectively of a subset D of X.

Definition 1.1. Let G be a bounded open subset of X. A mapping $T : \bar{G} \to X$ is said to be A-proper (with respect to the approximation scheme $\{X_n, P_n\}$) if, whenever $\{x_{n_j}\}$ is a bounded sequence such that $x_{n_j} \in X_{n_j} \cap \bar{G}$ and $P_{n_j} T x_{n_j} \to z$ as $j \to \infty$, there exists a convergent subsequent $\{x_{n_{j(k)}}\}$ such that $x_{n_{j(k)}} \to x$ as $k \to \infty$ and $Tx = z$.

Remark. This is the usual definition of A-properness (see for instance [**4**, Definition 1.2]).

Let G be a bounded open subset of X, $T : \bar{G} \to X$ a continuous A-proper mapping and suppose $y \in X \backslash T(\partial G)$. In [**4**] Browder and Petryshyn defined $\mathrm{Deg}(T, G, y)$ to be the "limit points" of the sequence

$$\{\deg(P_n T | X_n, G \cap X_n, P_n y)\}$$

(where "deg" denotes is the Brouwer degree). Although "Deg" is possibly multivalued, it retains most of the properties associated with a degree function (e.g. homotopy invariance and decomposition of domain). We refer the reader to [**4**] for a discussion of the A-proper degree theory.

Definition 1.2. Let X be a real Π_1-space, U an open subset of X, T a continuous mapping of U into X. Then T is said to be a mapping of the analytic type (with respect to $\{X_n, P_n\}$) if the mappings

$$T_n = P_n T : U \cap X_n \to X_n$$

are of the analytic type in the sense of Browder (see [**1**, Definitions 1, 2, 3]).

In the next section we generalize the following theorem, which was proved by Browder [**1**, Theorem 1], to A-proper mappings of the analytic type.

THEOREM 1.3 (Browder). *Let X_n be a finite dimensional Banach space, U an open subset of X_n, T a continuous mapping of U into X_n which is of the analytic type. Suppose G is a bounded open subset of U such that $\bar{G} \subset U$ and let $y \in X_n \backslash T(\partial G)$. Then*

(a) $\deg(T, G, y) \geqq 0$;
(b) $\deg(T, G, y) > 0$ *if and only if* $y \in T(G)$;
(c) $\deg(T, G, y) = 1$ *implies* $T^{-1}(y) \cap G$ *is connected;*
(d) *if* $T(G)$ *contains a point of a given component* C *of* $X_n \backslash T(\partial G)$ *then* $C \subset T(G)$.

2. On A-proper mappings of the analytic type. The first theorem we prove is the extension of Theorem 1.3 to A-proper mappings.

THEOREM 2.1. *Suppose* U *is an open subset of* X, G *a bounded open subset of* U *such that* $\bar{G} \subset U$. *Let* $T : U \to X$ *be a continuous mapping of the analytic type which is* A*-proper on* $\bar{G}$. *Let* $y \in X \backslash T(\partial G)$. *Then,*
(a) $\mathrm{Deg}(T, G, y)$ *contains no negative integers;*
(b) $0 \notin \mathrm{Deg}(T, G, y)$ *if and only if* $y \in T(G)$;
(c) *if* $1 \in \mathrm{Deg}(T, G, y)$ *then* $T^{-1}(y) \cap G$ *is connected;*
(d) *for* C *a component of* $X \backslash T(\partial G)$ *such that* $T(G) \cap C \neq \emptyset$, *we have* $C \subset T(G)$.

Proof. (a) We know that $\deg(P_n T|X_n, G \cap X_n, P_{ny})$ is well-defined for large enough n. Hence by Theorem 1.3(a) it must be greater than or equal to zero. Thus by definition, $\mathrm{Deg}(T, G, y)$ cannot contain negative integers.

(b) Suppose $0 \notin \mathrm{Deg}(T, G, y)$. Then by [**4**, Theorem 1(b)] there exists $x \in G$ such that $Tx = y$.

Suppose on the other hand $y \in T(G)$. As $T(\partial G)$ is closed (see for example [**8**, Lemma 1]) and $y \in X \backslash T(\partial G)$ it follows that $2d = \mathrm{dist}(y, T(\partial G)) > 0$. Let $x \in G$ be such that $Tx = y$. Hence there exists $x_1 \in G \cap X_n$ for all $n \geqq N'$ (say) such that

$$\|P_n Tx - P_n Tx_1\| \leqq \|Tx - Tx_1\| < d. \tag{1}$$

(This follows from the continuity of T and the denseness of $\bigcup_{n=1}^{\infty} X_n$ in X.)

For all $n \geqq N'$ consider the paths

$$(1 - t)P_n Tx_1 + tP_n y \qquad \text{where } t \in [0, 1].$$

Suppose there is a sequence $\{n(j)\}$ of integers such that for each $n(j)$ there exists $t_{n(j)} \in [0, 1]$ such that

$$(1 - t_{n(j)})P_{n(j)} Tx_1 + t_{n(j)} P_{n(j)} y = P_{n(j)} Tz_{n(j)}$$

where $z_{n(j)} \in \partial(G \cap X_{n(j)})$. (To be correct we should make some distinction between the boundary of $G \cap X_{n(j)}$ in $X_{n(j)}$ and its boundary in X. As we feel there can be no confusion in doing so we will use ∂ to denote both boundaries. Note also that $\partial(G \cap X_{n(j)}) \subset \partial G$.)

We may assume, passing to a subsequence if necessary, that $t_{n(j)} \to t$ as $j \to \infty$ and thus

$$P_{n(j)} Tz_{n(j)} \to (1 - t)Tx_1 + ty = y_0 \text{ (say) as } j \to \infty.$$

By the A-properness of T, there exists a subsequence $\{n(k)\}$ of $\{n(j)\}$ such that $z_{n(k)} \to z$ as $k \to \infty$ and $Tz = y_0$. As $z_{n(k)} \in \partial G$ and ∂G is closed, $y_0 \in T(\partial G)$.

But

$$\begin{aligned} 2d \leqq \|y_0 - y\| &= \|(1-t)(y - Tx_1) + t(y-y)\| \\ &= (1-t)\|y - Tx_1\| \\ &= (1-t)\|Tx - Tx_1\| \\ &< d \qquad \text{by (1).} \end{aligned}$$

This is a contradiction and so there exists an integer N such that $(1-t)P_nTx_1 + tP_ny \notin P_nT(\partial(G \cap X_n))$ for all $t \in [0, 1]$ and for all $n \geqq N$.

By the homotopy property of Brouwer degree we see that

$$(2) \qquad \deg(P_nT|X_n, G \cap X_n, P_ny) = \deg(P_nT|X_n, G \cap X_n, P_nTx_1)$$

for all $n \geqq N$. As $P_nTx_1 \in P_nT(G \cap X_n)$ for all $n \geqq N$ (we assume $N \geqq N'$) the right hand side of equation (2) is an integer not less than 1 (see Theorem 1.3(b)). It then follows from the definition of $\text{Deg}(T, G, y)$ that $0 \notin \text{Deg}(T, G, y)$.

(c) Suppose for some point $y \in X \backslash T(\partial G)$ with $1 \in \text{Deg}(T, G, y)$ that $T^{-1}(y) \cap G$ is not connected. Since $T^{-1}(y) \cap G = T^{-1}(y) \cap \bar{G}$ is compact (see for example [**8**, Lemma 1]) there are two non-empty compact sets H and K such that

$$T^{-1}(y) \cap G = H \cup K \quad \text{and} \quad H \cap K = \emptyset.$$

As $\text{dist}(H, K) > 0$ we may find open neighbourhoods N, M of H, K respectively such that $N \cap M = \emptyset$. But

$$\text{Deg}(T, G, y) \subseteq \text{Deg}(T, N, y) + \text{Deg}(T, M, y),$$

(this is a trivial consequence of [**4**, Theorem 1(d)]) where for two subsets $D_1; D_2$ of $\mathbf{Z}' = \mathbf{Z} \cup \{+\infty, -\infty\}$ we set

$$D_1 + D_2 = \{\gamma : \gamma_1 + \gamma_2 \text{ with } \gamma_1 \text{ in } D_1, \gamma_2 \text{ in } D_2\}$$

and apply the convention that $\infty + (-\infty) = \gamma$ for all γ in $\mathbf{Z}'$.

As $\text{Deg}(T, N, y)$ and $\text{Deg}(T, M, y)$ contain only strictly positive integers (see (a) and (b)) the least integer in $\text{Deg}(T, G, y)$ must be greater than or equal to 2. This contradicts our hypothesis and so the theorem is proved.

(d) If $y \in C \cap T(G)$ then $0 \notin \text{Deg}(T, G, y)$ by (b). If z belongs to the same connected component of $X \backslash T(\partial G)$ as y it is easy to show $\text{Deg}(T, G, y) = \text{Deg}(T, G, z)$. Hence $0 \notin \text{Deg}(T, G, z)$ and so $z \in T(G)$.

This completes the proof of the theorem.

The next theorem can be deduced from Theorem 2.1 in exactly the same manner that Browder deduced Theorem 3 from Theorem 2 in [**1**].

THEOREM 2.2. *Let U, G and T be defined as in Theorem 2.1. Suppose for each $y \in X \backslash T(\partial G)$ that $T^{-1}(y) \cap G$ is totally disconnected. Then for any point $y \in X \backslash T(\partial G)$ such that $1 \in \mathrm{Deg}(T, G, y)$, $T^{-1}(y) \cap G$ is a single point. And for the component C of $X \backslash T(\partial G)$ containing y, T is a homeomorphism of $T^{-1}(C) \cap G$ onto C.*

This completes the extension of Browder's results to A-proper analytic type mappings. In the next section we show how they may be used to prove uniqueness theorems for operator equations.

3. A uniqueness theorem. Theorem 2.2 plays an important part in our uniqueness theorem. To apply the theorem we must find conditions which imply that the solution set of an operator equation is totally disconnected. In this direction Browder has proved the following (see [**2**, Theorems 15.3 and 15.5]).

THEOREM 3.1. *Let Y be a complex Banach space, G an open subset of Y and $T : G \rightarrow Y$ a complex analytic Fredholm mapping. Suppose $T^{-1}(y) \cap G$ is compact for some $y \in Y$. Then $T^{-1}(y) \cap G$ is totally disconnected. Also, if* $\mathrm{ind}(T) = 0$, *then $T^{-1}(y) \cap G$ is finite.*

For a proof of this theorem and precise definitions of "Fredholm" and "ind(T)" we refer the reader to [**2**, Section 15]. An A-proper mapping is apparently not necessarily Fredholm. However an important subclass of the A-proper mappings have this property. We now define this subclass.

Definition 3.2. Let Y be a Banach space and suppose $f : D(f) \subset Y \rightarrow Y$ is continuous and satisfies

$$\beta_Y(f(\Omega)) \leqq k\beta_Y(\Omega),$$

for all bounded subsets Ω of $D(f)$, where

$$\beta_Y(\Omega) = \inf\{d \geqq 0 : \Omega \text{ is contained in the union of a finite number of balls (of } Y) \text{ with diameter } d\}.$$

Then we say $\beta_Y(\Omega)$ is the ball measure of non-compactness of Ω and f is a k-ball contraction.

THEOREM 3.3. *Let Y be a complex Banach space, U an open subset of Y and G an open subset of U such that $\bar{G} \subset U$. Suppose $f : U \rightarrow Y$ is complex analytic and $f : G \rightarrow Y$ a k-ball contraction ($k < 1$). Then $I - f : G \rightarrow Y$ is a Fredholm mapping of index zero.*

Proof. Notice that our definition of k-ball contraction presupposes that f is continuous. This together with the complex analyticity assumption implies that the Fréchet derivative of f at y (in G), f_y', is a bounded linear mapping of Y into Y, and also that f is infinitely differentiable, which, *a fortiori*, implies that f is once continuously differentiable. Thus to prove our theorem it is sufficient to show that at any point $y \in G$, $I - f_y'$ satisfies the Fredholm Alternative. It is a routine matter to show that f_y' is a k-ball contraction (same k as for f) and so [**9**, Theorem 11] implies that $I - f_y'$ satisfies the Fredholm Alternative. Our theorem is proved.

Before we prove the main theorem of this section let us make the following remarks. Let Y be a complex Π_1-space and let X be the real Banach space obtained from Y by ignoring the complex structure. Obviously X is a Π_1-space. If $f : D(f) \subset Y \rightarrow Y$ is a k-ball contraction then it is also a k-ball contraction when considered as operating in X. Webb proved the following theorem [**11**, Theorem 1].

THEOREM 3.4. *Let X be a real Π_1-space, G an open subset of X and $f : \bar{G} \rightarrow X$ a k-ball contraction ($k < 1$). Then $I - f$ is A-proper.*

Browder proved that a complex analytic mapping, if considered as operating in the real space obtained from the complex space by ignoring the complex structure, is of the analytic type [**1**, Proposition 1] (indeed this was the motivation for the name "analytic type"). Thus the results of Section 2 can be applied to complex analytic k-ball contractions. In particular we obtain the following theorem.

THEOREM 3.5. *Let Y be a complex Π_1-space, U an open subset of Y and G an open convex bounded subset of U such that $\bar{G} \subset U$. Suppose that $f : U \rightarrow Y$ is complex analytic, $f : \bar{G} \rightarrow Y$ is a k-ball contraction ($k < 1$) such that $f(\partial G) \subset G$ and f has no fixed points on the boundary of G. Then f has one and only one fixed point in G.*

Proof. Let X be the real Π_1-space obtained from Y by ignoring the complex structures. Put $T = I - f$. Then considering T as operating in X we see that $T : U \rightarrow X$ is of the analytic type and $T : \bar{G} \rightarrow X$ is A-proper. We have assumed that $0 \notin T(\partial G)$ and so $\text{Deg}(T, G, 0)$ is well-defined. Also $0 \notin T(G)$ implies $T^{-1}(0) \cap G$ is compact and so Theorem 3.1 is applicable. We deduce $T^{-1}(0) \cap G$ is finite and thus totally disconnected. A standard argument proves $\text{Deg}(T, G, 0) = \{1\}$ and so our theorem follows from Theorem 2.2.

This theorem is comparable to [**3**, Theorem 2] where it was assumed that f was of the form $H + C$, H a k-contraction and C a compact operator. We have generalized this result to k-ball contractions although we have had to restrict our attention to Π_1-spaces.

It should be remarked that the conditions "$f(\partial G) \subset G$ and f has no fixed points on the boundary of G" of Theorem 3.5 can be replaced by other well-known conditions, which imply that $\text{Deg}(T, G, 0) = \{1\}$, without affecting the conclusion of the theorem.

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