REGULAR-NILPOTENT GROUPS OF AUTOMORPHISMS

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(Received 30 May, 2000; accepted 7 May, 2002)

Abstract. Using the definition of regular p-group given by M. Hall [1], a new class of finite groups called regular-nilpotent has been defined. The action of these groups as automorphisms of compact Riemann surfaces has been investigated. It is proved that a necessary and sufficient condition for a Fuchsian group to cover a regular-nilpotent group is that its orbit genus be zero and its periods satisfy the least common multiple condition, first defined by Harvey [2] and Maclachlan [4].

2000 Mathematics Subject Classification. 20H10, 20F28.

1. Introduction. This paper is another sequel to [6] in which I used the notion of p-localization defined in [3] to obtain the best possible bound 16 (g − 1) for the order of a finite nilpotent group acting as the group of automorphisms of a Riemann surface of genus g ≥ 2. Marshall Hall [1] has defined a particular class of p-groups called regular p-groups. These p-groups have the property that, for any two elements a, b and any integer of the form n = pa, the identity (ab)n = a-na2...anR1S2...Rt is satisfied. Here S1, S2, ..., St are appropriate elements from the commutator subgroup of the group generated by a and b. In this paper, we define yet another class of finite groups called regular-nilpotent: namely, the nilpotent groups all of whose p-Sylow subgroups are regular p-groups. After establishing the existence of such finite groups, we then look for necessary and sufficient conditions under which this particular type of nilpotent group is covered by a co-compact Fuchsian group. We also generalize a property, described in [8], of 3-groups of automorphisms of Riemann surfaces to any other odd prime. More precisely, we shall show that for any odd prime p ≥ 3, the smallest p-group covered by the Fuchsian group (0; p, p, p2) is an irregular group Cp wr Cp of order pp+1. Here wr denotes wreath product.

2. Notation and terminology. Let P be a finite p-group whose order is pa, where p is a prime. If P is of class less than p and n = pa, P = ⟨a1, a2, ..., ar⟩, then we have

\[(a_1a_2...a_r)^n = a_1^n a_2^n ... a_r^n S_1^n S_2^n ... S_t^n,\]

where S1, S2, ..., St are elements of the commutator subgroup of the group P

Part (A). Known facts about regular p-groups.
1. Every p-group of class less than p is regular.
2. Every p-group of order at most pp is regular.
3. The group $P$ is regular if every subgroup generated by two elements in $P$ is regular.

4. Every subgroup and every factor group of a regular $p$-group are also regular.

5. If $P$ is a finite $p$-group, $P$ is regular if and only if for any $a, b$ in $P$ we have $(ab)^p = a^p b^p S^p$, where $S \in [P_1, P_1]$ and $P_1 = \langle a, b \rangle$.

6. If $P$ is regular with $n = p^a$, then the following conditions are satisfied.

   (i) $[a^n, b] = 1$ if and only if $[a, b]^n = 1$, where $a \in P$, $b \in P$.

   (ii) If $[a^n, b] = 1$, then $[a, b^n] = 1$.

   (iii) A commutator $S$ involving an element $u$ has order at most that of $u$ modulo the centre of $P$.

   (iv) The order of the elements $a_1 a_2 \cdots a_r$ is at most $\max \{|a_i| : 1 \leq i \leq r \}$.

7. If in a regular $p$-group we have $a^p = b^p = 1$, then $ab$ has order at most $n$.

**Part (B). Well-known facts about Fuchsian groups and $p$-localization.**

1. A Fuchsian group is a discrete subgroup $\Gamma$ of orientation preserving isometries of the upper half-plane $D$ with hyperbolic structure. Moreover, if $D/\Gamma$ is a compact surface, then $\Gamma$ has the following presentation:

   $$
   \Gamma(S) = \left\langle x_1, x_2, \ldots, x_s, a_1, b_1, \ldots, a_g b_g \right| x_j^{m_j}(i = 1, \ldots, s), \prod_{i=1}^{s} x_i \prod_{j=1}^{g} [a_j, b_j] \rightrangle
   $$

   where $S = (g : m_1, m_2, \ldots, m_s)$ is the signature of the group $\Gamma$. The integers $m_1, m_2, \ldots, m_s$ are the periods of $\Gamma$ and $g$ is its orbit genus. The groups of automorphisms of compact Riemann surfaces are the quotient groups of Fuchsian groups. Every Fuchsian group has an associated fundamental region whose hyperbolic area $\mu(\Gamma)$ depends only on the group itself. Suppose that $\Gamma$ has the signature $S$ defined above. Then

   $$
   \mu(\Gamma) = 2\pi \left[ (2g - 2) + \sum_{k=1}^{s} (1 - m_k^{-1}) \right]
   $$

2. If $\Gamma_1$ is a subgroup of finite index in the group $\Gamma(S)$, where $S$ is a non-degenerate signature, then there is a signature $S_1$ such that $\Gamma_1 \cong \Gamma(S_1)$ and we have the following Riemann–Hurwitz index formula

   $$
   [\Gamma : \Gamma_1] = \mu(\Gamma(S)) / \mu(\Gamma(S)).
   $$

3. It is well known that a compact Riemann surface of genus $g \geq 2$ can be represented as the quotient group $D/\Gamma$, where $\Gamma$ is a Fuchsian group with signature $(g; -)$ called the surface group of genus $g$. Here $-$ denotes the empty set of periods.

4. A finite group $G$ acts as the group of automorphisms of a given surface group if and only if there is a Fuchsian group $\Pi$ and a homomorphism $\Phi$ from $\Pi$ onto $G$ having $\Gamma$ as its kernel. Such a homomorphism will be smooth; (that is, has torsion-free kernel). Also $\Pi$ admits $G$ as a smooth factor group. Moreover, the homomorphism $\Phi : \Pi \rightarrow G$ is smooth if and only if it preserves the periods of the Fuchsian group $\Pi$.

5. Suppose that $S = (g; m_1, m_2, \ldots, m_s)$ is the signature of the Fuchsian group $\Gamma$. Let $a_i$ be the largest number such that $p^{a_i} | m_i (i = 1, \ldots, s)$. The signature...
$S_p = (g; p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_s})$ is called the $p$-localization signature of $S$. We have

$$
\Gamma(S_p) = \left\{ x_1^{p^{\alpha_1}}, \ldots, x_s^{p^{\alpha_s}}, y_1^{p^{\beta_1}}, \ldots, y_t^{p^{\beta_t}} \middle| x_i^{p^{\alpha_i}} = x_i^{p^{\alpha_j}} = \cdots = x_i^{p^{\alpha_s}} = 1 \right\}.
$$

6. A signature $S$ is called $p$-local if every period of $S$ is already a power of the same prime $p$ so that $S = S_p$. We also call the homomorphism $\iota_p : \Gamma(S) \to \Gamma(S_p)$, obtained by extending the function defined on the generating sets by

$$
x_i \to x_i', \quad a_j \to a_j', \quad b_k \to b_k' \quad (i = 1, 2, \ldots, s; j, k = 1, \ldots, g),
$$

the $p$-localization homomorphism.

7. Each smooth homomorphism $\Phi : \Gamma(S) \to G = G_{p_1} \times G_{p_2} \times \cdots G_{p_s}$, determined by the function defined on the Sylow subgroups $G_{pi}(i = 1, 2, \ldots, s)$, determines a set of homomorphisms of the form $\Phi_{pi} : \Gamma(S_p) \to G_{pi}$ such that if $y \in \Gamma(S)$ and $g_i = \Phi_{pi}(l_{pi}(y))$, then we have $\Phi(y) = g_1 g_2 \cdots g_s$. Therefore, we can obtain all possible smooth homomorphisms from the Sylow $p$-subgroups of $G$.

**3. Regular-nilpotent groups of automorphisms.** We shall now introduce and study the action of a class of finite nilpotent groups that we call regular-nilpotent as automorphisms of a compact Riemann surface of genus $g \geq 2$.

**Definition 3.1.** $P$ is said to be a regular $p$-group if for any pair of elements $a, b$ in $P$ and $n = p^\alpha$ we have

$$(ab)^n = a^n b^n S_1^n S_2^n \cdots S_t^n,$$

where the $S_i$ are elements of the commutator subgroup $[P_1, P_1]$ of the group $P_1 = \langle a, b \rangle$.

**Definition 3.2.** A finite group $G$ is called regular-nilpotent if it is nilpotent and all of its Sylow subgroups are regular $p$-groups.

**Theorem 3.1.** Let $S_p = (g; p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_s})$ be a $p$-local signature. The Fuchsian group $\Gamma(S_p)$ covers a regular $p$-group $G_p$ if $S_p$ has orbit genus $g = 0$ and its periods are all equal; that is, $\alpha_1 = \alpha_2 = \cdots = \alpha_s = \alpha$.

**Proof.** By Corollary 6.7 of [3], $\Gamma(S_p)$ is residually a finite $p$-group, since $S_p$ is a $p$-local signature. Since $g = 0$, every nilpotent automorphism group covered by $\Gamma(S_p)$ is a finite $p$-group by Theorem 2.1.1 of [6]. Hence we have

$$
\Gamma(S_p) = \left\{ x_1, x_2, \ldots, x_s \middle| x_1^{p^{\alpha_1}} = x_1^{p^{\alpha_2}} = \cdots = x_1^{p^{\alpha_s}} = x_1 x_2 \cdots x_s = 1 \right\}.
$$

Now suppose that $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. On the one hand we have the inequalities $p^{\alpha_j} \leq p^\alpha (j = 1, \ldots, s)$. On the other hand, if we let $x_N$ be an element of order $p^\alpha$, then from the long relation of $\Gamma(S_p)$ given above we obtain

$$(x_N^{-1})^{p^\alpha} = (x_{N+1} x_{N+2} \cdots x_s x_1 x_2 \cdots x_{N-1})^{p^\alpha} = 1.$$
However, if $G_p$ is a regular $p$-group, then by Property 6(iv) of § 2 Part A we have
\[ |x_{N+1}x_{N+2} \cdots x_1x_2 \cdots x_{N-1}| = p^\alpha \leq |y| = p^\alpha \quad (j \neq N), \]
where $|y|$ denotes the order of the element $y$ in $G$. Hence $p^\alpha = p^\alpha (j \neq N)$. The result now follows.

We note that the result is not true in general for signatures $S_p$ with non-zero orbit genus. Moreover, since $\Gamma(S_p)$ covers the regular $p$-group $G_p$ smoothly, the signature $S_p$ cannot be degenerate. See [3]. This implies that $s \geq 2$. For $s = 2$, $S_p$ is still a non-degenerate signature, since here we have $m_1 = m_2$.

The next theorem generalizes the observation in [7, p. 240] on 3-groups of automorphisms. This explains why the smallest 3-group of automorphisms must have order 81 and must be $Z_3 \wr Z_3$. Note that here, however, we are merely dealing with the Fuchsian groups $(p, p, p^2)$, which do not necessarily give an upper bound for the rest of the $p$-groups of automorphisms.

**Theorem 3.2.** Let $p$ be an odd prime. Consider the Fuchsian group $\Gamma = \Gamma(0; p, p, p^2)$. The smallest nilpotent group $G$ covered by $\Gamma$ has the following properties.

(i) $G \cong$ Sylow $p$-subgroup of $S_p \cong C_p \wr C_p$.

(ii) $G = \langle x, y | x^p = y^p = (xy)^{p^2} = [x, [x, y]] = 1 \rangle$ \quad $(n = 1, 2, \ldots, p - 1)$.

(iii) The order of $G$ is given by $|G| = p^\alpha + 1$.

(iv) $G$ is not a regular $p$-group.

**Proof.** First, we shall investigate the $p$-subgroup of the symmetric group $S_{p^2}$ on $p^2$ letters. We shall show that this group is generated by two elements of order $p$ whose product has order $p^2$. The Sylow subgroups of $S_{p^2}$ can be easily constructed by means of the wreath product. Observe that the factors of $(p^2)!$ divisible by $p$ are $p, 2p, 3p, \ldots, (p - 1)p, p^2$. Hence $(p^2)!$ is divisible by $p^\alpha + 1$ and this is the highest power of $p$ dividing $(p^2)!$.

Now $S_{p^2}$ has a subgroup which is the direct product of the cyclic groups generated by the $p$-cycles $a_1 = (1, 2, \ldots, p), a_2 = (p + 1, p + 2, \ldots, 2p), a_3 = (2p + 1, \ldots, 3p), \ldots, a_p = (p^2 - p + 1, \ldots, p^2)$. Consider another element of order $p$; for example
\[ b = (1, p + 1, 2p + 1, \ldots, p^2 - p + 1)(2, p + 2, 2p + 2, \ldots, p^2 - p + 2) \cdots \times (p, 2p, 3p, \ldots, p^2). \]

It can be checked that
\[ a_{k+1} = b a_k b^{-1} = b^2 a_{k-1} b^{-2} = \cdots = b^k a_1 b^{-k}. \]

Therefore, we have
\[ a_k = b^{k-1} a_1 b^{1-k} \quad (k = 2, \ldots, p), \quad G = \langle a_1, a_2, \ldots, a_p, b \rangle = \langle a_1, b \rangle \cong C_p \wr C_p \]
and the last group is a $p$-Sylow subgroup of $S_{p^2}$. Moreover $|G| = p^\alpha + 1$.

To find a presentation for this group we consider the relations $a_1^p = b^p = 1$ and note that
\[ a_1 b = (1, p + 2, 2p + 2, \ldots, p(p - 1) + 2, 2, p + 3, \ldots, p^2) \]
is a $p^2$-cycle. To get the extra relation we note that

$$a_{i+1}a_{j+1} = a_{j+1}a_{i+1} \Rightarrow (b' a_1 b^{-i})(b' a_1 b^{-j}) = (b' a_1 b^{-j})(a' a_1 b^{-i}).$$

Setting $j - i = k$, the relation above becomes

$$a_1 b^{-k} a_1 b^k = b^{-k} a_1 b^k \Rightarrow [a_1, [a_1, b^k]] = 1.$$

By §2, Item 7 of Part (A), the order of $a_1 b$ is at most $p$ and so this group is irregular.

**THEOREM 3.3.** A Fuchsian group $\Gamma(0; m_1, m_2, \ldots, m_n)$ with orbit genus zero can cover a regular nilpotent group $G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_k}$ if and only if its periods satisfy

$$\text{l.c.m } \{m_1, m_2, \ldots, m_j, m_{j+1}, \ldots, m_k\} = \text{l.c.m } \{m_1, m_2, \ldots, m_k\}.$$

Here $\hat{m}_j$ means that $m_j$ should be omitted from the list.

**Proof.** We use an idea from Section 5 of [3] to localize the given signatures $S_{p_i} = (0; p_i^{\alpha_1}, p_i^{\alpha_2}, \ldots, p_i^{\alpha_k})$. Then we use our Theorem 3.1 to show that for each of the localized signatures $S_{p_i}$ we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha_i$ ($i = 1, 2, \ldots, k$). From this we deduce that $S$ can be localized into the $p$-localization signatures $S_{p_i} = (0; p_i^{\alpha_1}, \ldots, p_i^{\alpha_k}), S_{p_j} = (0; p_j^{\beta_1}, \ldots, p_j^{\beta_k}), \ldots, S_{p_k} = (0; p_k^{\delta_1}, \ldots, p_k^{\delta_k}).$

Obviously, the signatures above each might have a different number of periods, but each must contain at least two periods in order to be a non-degenerate signature. Hence, if $p_i | \prod_j m_j$, then the same power $p_i^{m_j}$ of the prime $p_i$ must divide at least two of the periods $m_j$. We can now conclude that

$$\text{l.c.m } \{m_1, m_2, \ldots, \hat{m}_j, m_{j+1}, \ldots, m_k\} = \text{l.c.m } \{m_1, m_2, \ldots, m_k\}.$$

$$= p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}.$$

The proof is complete, since the argument can easily be reversed.

**Remark.** The least common multiple condition has earlier been used in [2] and [4].

**REFERENCES**