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IN SEARCH OF A PAPPIAN LATTICE IDENTITY

BY
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§1 Introduction In [8] and subsequent papers, Jónsson (et al) developed a lattice identity which reflects precisely Desargues Law in projective geometry in that a projective geometry satisfies Desargues Law if and only if the geometry, qua lattice, satisfies this identity. This identity, appropriately called the Arguesian law, has become exceedingly important in recent investigations in the variety of modular lattices (see for example [2], [3], [9], and [12]). In this note, we supply two possible lattice identities for the Pappus' Law of projective geometry.

The main difference between a possible pappian equation and the known arguesian one is that any pappian equation must place some restriction on the dimension of the modular lattice. This is necessary because of the well known fact that there is a lattice embedding of the non-pappian projective plane over the quaternions into the 5-dimensional projective space over the complex numbers.

Another problem is that even in a projective plane, the geometric pappian law (as opposed to the desargusean law) contains definite inequality statements (all six points on the two lines must be distinct from each other and also from the intersection point of the two lines). Moreover if equalities occur, the conclusion of Pappus is patently false. (See Heyting [5] for one possible version of a generalized Pappus proposition).

This second problem is handled by imposing configuration restrictions on four variables of the equation. We will consider two distinct configurations: 3-frames and line pairs. The first notion is due to von Neumann [11] and in the form we use, to Huhn [6]. The second notion seems new and generalizes that of a 3-frame. Our "equations" then are of the form

$$(F_3(a_0, a_1, b_0, b_1) \Rightarrow \lambda \leq \rho)$$

and

$$(LP(a_0, a_1, b_0, b_1) \Rightarrow \lambda \leq \rho).$$

The above sentences translate into real-life equations because being a 3-frame or a line-pair is, as we shall see, a projective property.

The first problem is solved by the form of the equational part, $\lambda \leq \rho$. It is not

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clear to the author precisely why this occurs but the F_3 configuration forces the affine dimension to be less than 6 and the line-pair configuration further restricts the affine dimension to less than or equal to 3.

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§2 Preliminaries. All results in this paper apply to $\mathcal{M}$, the variety of all modular lattices. All lattices are assumed to be modular.

A subset $\{a_1, \ldots, a_n, c_{12}, \ldots, c_{1n}\}$ of a (modular) lattice is called an *n-frame* if

$$(F_n 1) \quad \prod_{i}^{l,n} a_i = a_i \cdot \sum_{j \neq i}^{l,n} a_j \quad \text{(all } i)$$

$$(F_n 2) \quad a_1 + c_{1i} = a_1 + a_i = a_i + c_{1i}$$

and

$$a_1 c_{1i} = a_1 a_i = a_i c_{1i} \quad \text{(all } i)$$

The canonical example of an n-frame occurs in the subspace lattice, $L(K^n)$, of an n-dimensional vector space over a division ring K where $\{e_l, \ldots, e_n\}$ is a basis for K^n and $a_i = \langle e_i \rangle$, $c_{1i} = \langle e_l - e_i \rangle$, $i = 1, \ldots, n$, the subspaces generated by the given vectors.

We then can define $c_{ij} = (c_{1i} + c_{1j})(a_i + a_j)$, $i \neq$ to produce elements satisfying

$$a_i + c_{ij} = a_i + a_j = a_j + c_{ij}$$

$$a_i c_{ij} = a_i a_j = a_j c_{ij}.$$

We refer the reader to Von Neumann [11: Part II, Chap. V] for all necessary computational results.

An equivalent notion due to Huhn [7] is that of an *n-diamond* which is a subset $\{a_0, a_l, \ldots, a_n\}$ of a (modular) lattice satisfying

$$(D_n 1) \quad \sum_{i}^{0,n} a_i = \sum_{j \neq i}^{0,n} a_j \quad \text{(all } i)$$

$$(D_n 2) \quad a_i \cdot \sum_{\substack{k \neq i \\ k \neq j}}^{0,n} a_k = \prod_{i}^{0,n} a_i \quad \text{(all } i \neq j)$$

By Herrmann and Huhn [4], there is a natural bijective correspondence between n-frames

$$\{a_l, \ldots, a_n, c_{l2}, \ldots, c_{ln}\}$$

and n-diamonds $\{a_0, a_1, \ldots, a_n\}$. This allows one to alternate freely between these equivalent concepts.

An n-diamond (hence an n-frame) is a projective configuration in that:

2.1 Theorem ([4]). *There exist (modular) lattice polynomials $d_0, d_l, \ldots, d_n$ in $n+1$ variables such that*

(1) $d_0(x_0, \ldots, x_n), \ldots, d_n(x_0, \ldots, x_n)$ *is an n-diamond.*
(2) $\{x_0, \ldots, x_n\}$ *is an n-diamond if and only if $x_i = d_i(x_0, \ldots, x_n)$ for all i.*

A possible set of such polynomials is given by:

$$v = \prod_i^{0,n} \sum_{j\neq i}^{0,n} a_j$$

$$u = \sum_i^{0,n}\left(\sum_{j\neq i}^{0,n}\left(a_i \cdot \sum_{k\neq i,j}^{0,n} a_k\right)\right)$$

$$d_i = (u + x_i)v = u + x_i v \quad (\text{all } i)$$

A (modular) lattice is called n-distributive if for all $x, y_0, \ldots, y_n$

$$x \cdot \sum_i^{0,n} y_i = \sum_i^{o,n}\left(x \cdot \sum_{j\neq i}^{0,n} y_j\right)$$

2.2 Theorem ([7]). *A modular lattice is n-distributive if and only if it does not contain a non-trivial $(n+1)$-diamond (or $(n+1)$-frame).*

(An n-diamond $\{x_0, \ldots, x_n\}$ is trivial if two (and hence all by [4]) of the variables are equal.)

§3 Pappus' law in projective planes. We consider a projective plane as an irreducible complemented modular lattice of dimension 3, L, (see Crawley and Dilworth [1]) whose points are the atoms of L and whose lines are the coatoms.

3.1 Definition. A projective plane $(L, +, \cdot)$ is called pappian if given distinct lines $a \neq b$ and pairwise distinct points

$$a_0, a_1, a_2 \leq a, b_0, b_1, b_2 \leq b$$

with $a_i \neq ab \neq b_i$ all i, then the (distinct) points

$$(a_0 + b_1)(a_1 + b_0), (a_0 + b_2)(a_2 + b_0),$$

and $(a_1 + b_2)(a_2 + b_1)$ are collinear.

Our first result is a characterization of pappian planes that will provide our basic equation.

3.2 Theorem. *A projective plane $(L, +, \cdot)$ is pappian if and only if for any points*

$$a_0, a_1, a_2, b_0, b_1, b_2$$

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of L, $a_2 \leq a_0 + a_1$ and $b_2 \leq b_0 + b_1$ imply

$$(a_0+b_1)(a_1+b_0)(a_2+b_0+b_1)(b_2+a_0+a_1) \leq (a_2+b_0)(a_0+b_2)+(a_2+b_1)(a_1+b_2)$$

Proof. The condition is clearly sufficient for a pappian plane since the requirements of the pappian condition give

$$a_2+b_0+b_1 = b_2+a_0+a_1 = 1.$$

In order to prove necessity we must show the implication holds in all "trivial" or "degenerate" cases. We let

$$\lambda = (a_0+b_1)(a_1+b_0)(a_2+b_0+b_1)(b_2+a_0+a_1),$$
$$\rho = (a_2+b_0)(a_0+b_2)+(a_2+b_1)(a_1+b_2)$$

and assume

$$a_2 \leq a_0 + a_1 \quad \text{and} \quad b_2 \leq b_0 + b_1.$$

Case 1. $a_0 = a_1$

In this case we have $x = a_0 = a_1 = a_2$ which gives

$$\begin{aligned}\lambda &= (x+b_1)(x+b_0)(x+b_2)(x+b_0+b_1)\\ &= \prod_i^{0,2} (x+b_i)\\ \rho &= (x+b_0)(x+b_2)+(x+b_1)(x+b_2) \geq \lambda.\end{aligned}$$

By the symmetries of the implication we can now assume $a_0 \neq a_1$ and $b_0 \neq b_1$.

Case 2. $a_2 = a_1 (= x)$

The simple calculations are:

$$\begin{aligned}\lambda &= (a_0+b_1)(x+b_0)(x+b_0+b_1)(b_2+a_0+x)\\ &= (a_0+b_1)(x+b_0)(x+a_0+b_2)\\ \rho &= (x+b_0)(a_0+b_2)+(x+b_1)(x+b_2)\\ &= (x+b_0)(a_0+b_2)+x+b_1(x+b_2)\\ &= (x+b_0)(x+a_0+b_2)+b_1(x+b_2) \geq \lambda.\end{aligned}$$

Again by symmetry we can now assume for $i \neq j$ $a_i \neq a_j$ and $b_i \neq b_j$. This allows us to consider the lines

$$a = a_0+a_1 = a_0+a_2 = a_1+a_2$$

and

$$b = b_0+b_1 = b_0+b_2 = b_1+b_2$$

Case 3. $a = b$

In this case if $\rho = a (= b)$ we are done. Therefore we need only consider possibilities that $\rho = 0$ or ρ is a point. The following subcases exhaust these possibilities.

Subcase 3a. $(a_2+b_0)(a_0+b_2)=0$

Since $a=b$ this forces $a_2=b_0\neq a_0=b_2$. But then

$$\rho=(a_2+b_1)(a_1+b_2)=(b_0+b_1)(a_1+a_0)=a\geq\lambda.$$

Subcase 3b. $(a_2+b_0)(a_0+b_2)=(a_2+b_1)(a_1+b_2)$

Now joining this equality with a_2 produces:

$$\begin{aligned}a_2+b_0=(a_2+b_0)a&=(a_2+b_0)(a_2+a_0+b_2)\\&=a_2+(a_2+b_0)(a_0+b_2)\\&=a_2+(a_2+b_1)(a_1+b_2)=\cdots\\&\;\;\vdots\\&=a_2+b_1.\end{aligned}$$

Since $b_0\neq b_1$ this gives $a_2+b_0=a_2+b_1=a$. Similarly joining with b_2 gives $a_0+b_2=a_1+b_2=a$ and therefore $\rho=a\geq\lambda$.

We can now assume a and b are distinct lines each containing three distinct points. We need only show now that the intersection of the lines, ab, can be assumed distinct from the given 6 points.

Case 4. $a_2=ab$

$$\begin{aligned}\lambda&=(a_0+b_1)(a_1+b_0)(ab+b)(b_2+a)\\&=(a_0+b_1)(a_1+b_0)b(a+b_2)\\&=(b_1+a_0b)(b_0+a_1b)(a+b_2).\end{aligned}$$

But $a_2\leq b$ and the a_i's distinct gives $a_0b=a_1b=0$. $b_0\neq b_1$ implies

$$\lambda=b_0b_1(a+b_2)=0\leq\rho.$$

Case 5. $a_0=ab$

$$\begin{aligned}\lambda&=(ab+b_1)(a_1+b_0)(a_2+b)(b_2+a)\\&=b(a+b_1)(a_1+b_0)(a+b_2),\quad\text{since}\quad a_2+b=1\\&=(a+b_1)(b_0+a_1b)(a+b_2)\\&=b_0(a+b_1)(a+b_2),\quad\text{since}\quad a_1b=0\\ \rho&>(a_2+b_0)(ab+b_2)\\&=(a_2+b_0)b(a+b_2)\\&=(b_0+a_2b)(a+b_2)\\&=b_0(a+b_2),\quad\text{since}\quad a_2b=0\\&\geq\lambda.\end{aligned}$$

Again using symmetry we are left with showing the implication for the 6 points on 2 lines in a pappian configuration. This proves the necessity.

We should note that, although our condition is a universal implication for *points* in a projective plane, it cannot a universal implication for all *elements* of the plane $(L, +, \cdot)$.

To see this one need only consider a plane that has at least 4 distinct points on each line. Let a_0, a_1, b_0, b_1 be these points and let $a_2 = b_2 = 0$. Then λ is the line and $\rho = 0$. To make the implication work for all possible substitutions we will impose configuration restrictions on the four variables a_0, a_1, b_0 and b_1.

§4 The line-pair configuration. Of the two special configurations we will consider, the line-pair provides the sharper results.

4.1 Definition. A quadruple (a_0, a_1, b_0, b_1) in a (modular) lattice is called a line-pair if

$$a_0(b_0+b_1) = a_1(b_0+b_1) = b_0(a_0+a_1) = b_1(a_0+a_1).$$

Clearly any 3-diamond is a line pair for any ordering of the four elements.

4.2 Lemma. *Line-pair is a projective configuration in that there exists (modular) polynomials d_0, d_1, e_0, e_1 in four variables x_0, x_1, y_0, y_1 such that* (1) (d_0, d_1, e_0, e_1) *is a line-pair* (2) (a_0, a_1, b_0, b_1) *is a line-pair if and only if*

$$a_i = d_i(a_0, a_1, b_0, b_1) \quad \text{and} \quad b_i = e_i(a_0, a_1, b_0, b_1).$$

Proof. Let $q = x_0(y_0+y_1) + x_1(y_0+y_1) + y_0(x_0+x_1) + y_1(x_0+x_1)$ and $d_i = x_i + q$, $e_i = y_i + q$, $i = 0, 1$.

$$\begin{aligned} d_i(e_0+e_1) &= (x_i+q)(y_0+y_1+q) \\ &= (x_i+q)(y_0+y_1) \\ &= q + x_i(y_0+y_1) \\ &= q. \end{aligned}$$

Similarly $e_i(d_0+d_1) = q$ hence (d_0, d_1, e_0, e_1) is a line-pair.

Finally for a line pair (a_0, a_1, b_0, b_1) it is clear that

$$a_i = d_i(a_0, a_1, b_0, b_1) \quad \text{and} \quad b_i = e_i(a_0, a_1, b_0, b_1).$$

4.3 Notation. Let (a_0, a_1, b_0, b_1) be a quadruple in a modular lattice. We will write $\underline{LP(a_0, a_1, b_0, b_1)}$ if the quadruple is a line-pair and let 0_{LP} be the common value

$$a_0(b_0+b_1) = b_0(a_0+a_1) = a_1(b_0+b_1) = b_1(a_0+a_1).$$

Easy consequences of the definitions are

4.4 Lemma. If $LP(a_0, a_1, b_0, b_1)$ in a modular lattice L then for all i, j,

$k \in \{0, 1\}$:

(1) $a_i = (a_i + a_j)(a_i + b_k)$
(2) $b_i = (b_i + b_j)(a_k + b_i)$
(3) $0_{LP} = a_i b_j$
(4) $\langle a_i, b_j, b_k \rangle$ and $\langle a_i, a_j, b_k \rangle$ are distributive sublattices of L.

4.5 Definition. A line-pair (a_0, a_1, b_0, b_1) is called *degenerate* if it is not an anti-chain (i.e. no two are comparable).

If $a_i \leq b_j$ for some $i, j \in \{0, 1\}$ then by (4.4.3) $a_i = 0_{LP} \leq a_j, b_0, b_1$. Therefore a line-pair will be degenerate if and only if a_0 is comparable with a_1 or b_0 is comparable with b_1.

4.6 Lemma. *In a projective plane $(L, +, \cdot)$, a line-pair (a_0, a_1, b_o, b_1) is non-degenerate if and only if it is a complete quadrangle of points.*

Proof. If $a_0 + a_1 = 1$ then $b_0 = b_0(a_0 + a_1) = 0_{LP} \leq b_1$. Therefore $a_0 + a_1$ and $b_0 + b_1$ are lines in L. Moreover they must be distinct lines since otherwise

$$a_0 = a_0(a_0 + a_1) = a_0(b_0 + b_1) = 0_{LP} \leqslant a_1.$$

Similarly we must have $a_i \not\leq b_0 + b_1$ and $b_i \not\leq a_0 + a_1$.

We now are in a position to present our equation and show that it reflects the geometrical property of pappian planes.

4.7 Definition. A modular lattice L is called pappian if for all

$$a_i, b_i \in L \quad i = 0, 1, 2 : LP(a_0, a_1, b_0, b_1)$$

and $a_2 \leq a_0 + a_1$ and $b_2 \leq b_0 + b_1$ imply $\lambda \leq \rho$ where

$$\lambda = (a_0 + b_1)(a_1 + b_o)(a_2 + b_0 + b_1)(b_2 + a_0 + a_1)$$

and

$$\rho = (a_2 + b_0)(a_0 + b_2) + (a_2 + b_1)(a_1 + b_2)$$

4.8 Theorem. *Let L be a modular lattice. If (a_0, a_1, b_0, b_1) is a degenerate line-pair then for all $a_2 \leq a_0 + a_1$ and $b_2 \leq b_0 + b_1$, $\lambda \leq \rho$.*

Proof. Observe first of all that we may assume without loss of generality that $0_{LP} \leq a_2$ and $0_{LP} \leq b_2$ for replacing a_2 and b_2 by $a_2 + 0_{LP}$ and $b_2 + 0_{LP}$ does not change the values of λ and ρ. Moreover we need only consider the case where $a_0 \leq a_1$ for the others will follow by symmetry. Therefore let $a_0 \leq a_1$ hence

$$0_{LP} \leq a_2 \leq a_1 = a_0 + a_1.$$

We compute:

$$\begin{aligned}
\lambda &= (a_1+b_0)(a_0+b_1)(b_2+a_1)(a_2+b_0+b_1) \\
&= [a_0+b_1(a_1+b_0)](b_2+a_1)(a_2+b_0+b_1), \quad \text{since} \quad a_0 \le a_1 \\
&= (a_0+b_0b_1)(a_1+b_2)(a_2+b_0+b_1), \quad \text{by (4.4)} \\
&= [a_0+b_0b_1(b_0+b_1)(a_1+b_2)](a_2+b_0+b_1), \quad \text{since} \quad a_0 \le a_1 \\
&= [a_0+b_0b_1(b_2+a_1(b_0+b_1))](a_2+b_0+b_1), \quad \text{since} \quad b_2 \le b_0+b_1 \\
&= (a_0+b_0b_1b_2)(a_2+b_0+b_1), \quad \text{since} \quad 0_{LP} \le b_2 \\
&= b_0b_1b_2+a_0a_1(a_2+b_0+b_1) \\
&= b_0b_1b_2+a_0(a_2+a_1(b_0+b_1)), \quad \text{since} \quad a_2 \le a_1 \\
&= b_0b_1b_2+a_0a_2, \quad \text{since} \quad 0_{LP} \le a_2 \\
&\le \rho.
\end{aligned}$$

4.9 Theorem. *Let L be a projective plane. Then L is pappian as a geometry if and only if L is pappian as a (modular) lattice.*

Proof. If L is pappian as a geometry then by (4.8) we need only check the lattice pappian implication for non-degenerate line pairs (a_0, a_1, b_0, b_1). By (4.6) we may then assume that (a_0, a_1, b_0, b_1) is a complete quadrangle of points. By (3.2) we need only worry about a_2 and b_2 having the values 0 or a_0+a_1 (b_0+b_1 resp.). If $a_2=0$,

$$\begin{aligned}
\lambda &= (a_0+b_1)(a_1+b_0)(b_0+b_1)(b_2+a_0+a_1) \\
&= 0 \le \rho
\end{aligned}$$

since (a_0, a_1, b_0, b_1) is a complete quadrangle. If $a_2=a_0+a_1$

$$\begin{aligned}
\rho &= (a_0+a_1+b_0)(a_0+b_2)+(a_0+a_1+b_1)(a_1+b_2) \\
&= a_0+a_1+b_2 \ge \lambda
\end{aligned}$$

again since (a_0, a_1, b_0, b_1) is a complete quadrangle.

The converse is obvious.

We close this section with some remarks on our pappian "equation" ($LP(a_0, a_1, b_0, b_1)$ and $a_2 \le a_0+a_1$ and $b_2 \le b_0+b_1$) imply $\lambda \le \rho$.

The point is that this implication is an equation in disguise. One only replaces a_0, a_1, b_0, b_1 in the $\lambda \le \rho$ by the polynomials d_0, d_1, e_0 and e_1 of (4.2) respectively and then replace a_2 and b_2 by $a_2(d_0+d_1)$ and $b_2(e_0+e_1)$ respectively. Moreover as mentioned in the proof of (4.8) we may always assume $a_2, b_2 \ge 0_{LP}$ so we could replace a_2 and b_2 by $(q+a_2)(d_0+d_1)$ and $(q+b_2)(e_0+e_1)$.

§5 Dimension restrictions of the pappian identity. As mentioned in the introduction the projective space $PG_5(\mathbb{C})$ cannot satisfy the pappian identity as

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it contains the non-pappian projective plane over the quaternions. Therefore our equation must force some restriction on the size of an n-frame (or equivalently n-diamond) that can sit inside a pappian (modular) lattice.

5.1 THEOREM. *A pappian modular lattice does not contain a non-trivial 4-frame.*

Proof. Let $\{a_1, a_2, a_3, a_4, c_{12}, c_{13}, c_{14}\}$ be a 4-frame in a modular lattice L and define

$$x_0 = c_{14} \qquad x_1 = (c_{14} + c_{23})(c_{12} + c_{34})$$
$$y_0 = a_1 + a_2 \qquad y_1 = a_1 + a_3$$

We claim that $LP(x_0, x_1, y_0, y_1)$. Now standard (see [11]) n-frame calculations show that

$$x_0 + x_1 = (c_{14} + c_{23})(c_{12} + c_{34} + c_{14}) = c_{14} + c_{23}.$$

This gives

$$\begin{aligned} y_0(x_0 + x_1) &= (a_1 + a_2)(c_{14} + c_{23}) \\ &= (a_1 + a_2 + a_3)(a_1 + a_2 + a_4)(c_{14} + c_{23}) \\ &= c_{14}(a_1 + a_2 + a_3) + c_{23}(a_1 + a_2 + a_4) \\ &= 0_F, \end{aligned}$$

the zero of the 4-frame.

$$\begin{aligned} x_1(y_0 + y_1) &= (c_{14} + c_{23})(c_{12} + c_{34})(a_1 + a_2 + a_3) \\ &= (c_{23} + c_{14}(a_1 + a_2 + a_3))(c_{12} + c_{34}(a_1 + a_2 + a_3)) \\ &= c_{23}c_{12} = 0_F. \end{aligned}$$

Similarly $y_1(x_0 + x_1) = 0_F = x_0(y_0 + y_1)$.

Therefore (x_0, x_1, y_0, y_1) is a line-pair with $0_{LP} = 0_F$.

Now let $x_2 = 0_F$ and compute λ and ρ (assuming $0_F \le y_2 \le y_0 + y_1$)

$$\begin{aligned} \lambda &= (x_0 + y_1)(x_1 + y_0)(y_0 + y_1)(y_2 + x_0 + x_1) \\ &= (y_1 + x_0(y_0 + y_1))(y_0 + x_1(y_0 + y_1))(y_2 + x_0 + x_1) \\ &= y_0 y_1 (y_2 + x_0 + x_1) \\ \rho &= y_0(y_2 + x_0) + y_1(y_2 + x_1) \\ &= y_0(y_0 + y_1)(y_2 + x_0) + y_1(y_0 + y_1)(y_2 + x_1) \\ &= y_0(y_2 + x_0(y_0 + y_1)) + y_1(y_2 + x_1(y_0 + y_1)) \\ &= y_0 y_2 + y_1 y_2. \end{aligned}$$

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Now let $y_2=(a_1+c_{23})(a_3+c_{12})$ to produce:

$$\begin{aligned}\lambda &= (a_1+a_2)(a_1+a_3)[c_{14}+c_{23}+(a_1+c_{23})(a_3+c_{12})]\\ &= a_1[c_{14}+(a_1+c_{23})(a_3+c_{12}+c_{23})]\\ &= a_1(c_{14}+a_1+c_{23})\\ &= a_1\end{aligned}$$

and

$$\begin{aligned}\rho &= (a_1+a_2)(a_1+c_{23})(a_3+c_{12})+(a_1+a_3)(a_1+c_{23})(a_3+c_{12})\\ &= a_1(a_3+c_{12})+a_1(a_3+c_{12})\\ &= 0_F.\end{aligned}$$

Therefore L is not pappian.

5.2 Corollary 1. *Every pappian modular lattice is* 3-*distributive.*

5.3 Corollary 2. *Let V_K be a vector space over a division ring K. Then $L(V_K)$ is a pappian lattice if and only if* $\dim(V_K)\le 2$ *or* $\dim(V_K)=3$ *and K is a field.*

One must mention at this time the alternative special configuration for the variables (a_0, a_1, b_0, b_1). Let $F_3(a_0, a_1, b_0, b_1)$ mean that $\{a_0, a_1, b_0, b_1\}$ is a 3-diamond in a modular lattice and define a modular lattice to be frame-pappian if $F_3(a_0, a_1, b_0, b_1)$ and $a_2\le a_0+a_1$ and $b_2\le b_0+b_1$ imply $\lambda\le\rho$. Since every 3-diamond is a line pair we have that pappian implies frame-pappian. The relevant statements from section 4 apply to frame-pappian modular lattices but the two concepts are distinct.

5.4 Theorem. *A frame-pappian* (*modular*) *lattice does not contain a non-trivial* 6-*frame.*

Sketch of proof. Let L be a modular lattice and let

$$F_6=\{a_i, c_{1j}: i=1,\ldots,6; j=2,\ldots,6\}$$

be a non-trivial 6-frame in L. By letting

$$\begin{aligned}x_0&=a_1+a_2,\\ x_1&=c_{13}+c_{24},\\ y_0&=c_{35}+c_{46},\end{aligned}$$

and

$$y_1=a_5+a_6$$

we have $F_3(x_0, x_1, y_0, y_1)$. Now take $x_2=c_{14}$ and $y_2=c_{45}$. Then $\lambda=c_{15}$ and $\rho=0_F$. That is, L is not frame-pappian.

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5.5 Corollary 1. *Frame-pappian modular lattices are 5-distributive.*

5.6 Corollary 2. *Let V_K be a vector space over a division ring K. Then $L(V_K)$ is a frame-pappian lattice iff* $\dim(V_K) \le 2$ *or* $3 \le \dim(V_K) \le 5$ *and K is a field.*

Proof. Any 3-diamond (or equivalently 3-frame) in $L(V_K)$ with $\dim(V_K) \le 5$ produces an interval sublattice $[0_F, 1_F]$ isomorphic to $L(W_K)$ with $\dim(W_K) = 3$. Since we may assume $a_2, b_3 \in [0_F, 1_F]$, we are only working in $L(W_K)$.

5.7 Corollary 3. *Pappian is strictly stronger than frame-pappian.*

Now if A is a finite dimensional algebra over a field K with $\dim A_K = n$, there is a natural embedding of $L(A^m_A)$ into $L(K^{mn}_K)$. This in particular gives us an embedding of $L(\mathbb{H}^3_{\mathbb{H}})$ into $L(\mathbb{C}^6_{\mathbb{C}})$ (into $L(\mathbb{R}^{12}_{\mathbb{R}})$) or in other words $PG_2(\mathbb{H})$ is a sublattice of $PG_5(\mathbb{C})$ (is a sublattice of $PG_{11}(\mathbb{R})$). Since $PG_2(\mathbb{H})$ is not a pappian projective plane, we must have that $PG_5(\mathbb{C})$ is not pappian in any sense. Therefore the above result, 5.4, tells us that frame-pappian is probably as weak as possible for any sought for lattice identity. By 5.1, line-pair-pappian is probably as strong as possible since it restricts the geometries to projective planes.

§6 Concluding remarks. A pappian identity would be of even greater interest if one could show that for modular lattices pappian implies arguesian. Such a proof might shed more light on the situation for projective planes. The historical notes of Seidenberg in [10] show that even for projective planes this implication has caused problems. This author knows of no such lattice theory proof at this time but in a forthcoming paper with Andras Huhn several results on the relations between pappian, arguesian and 2-distributive lattices will help perhaps to clarify the problem.

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