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ON THE STONE-ČECH COMPACTIFICATION OF AN ORBIT SPACE

KAVITA SRIVASTAVA

By extending the given action of a discrete group G on a Tychonoff space X to βX, it is proved that the Stone-Čech compactification of the orbit space of X is the orbit space of the Stone-Čech compactification βX of X, when G is finite. The notion of G-retractive spaces is introduced and it is proved that the orbit space of a G-retractive space with G finite, is G-retractive.

1. Introduction

By a G-space X we mean a triple (X,G,θ) consisting of a Tychonoff space X, a discrete group G and an action θ of G on X. A subspace A of a G-space X is called *invariant* if $\theta(G \times A) = A$. An action θ of a group G on a space X is called *trivial* if $g \cdot x = x$ for $g \in G$, $x \in X$ and θ is said to be *transitive* if for each $x \in X$, the orbit $G_x = \{g \cdot x | g \in G\}$ is X itself. A map f from a G-space X to a G-space Y is called *equivariant* if $f(g \cdot x) = g \cdot f(x)$, $g \in G$, $x \in X$. Denote the set of all orbits G_x of a G-space X by X/G and let $\pi : X \to X/G$ be the orbit map taking x to G_x. Then X/G endowed with

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the quotient topology relative to π is called the *orbit space* of X. Action on X/G is taken to be trivial. The orbit map is open, continuous and equivariant.

The Stone-Čech compactification of a Tychonoff space X will be denoted by βX and X^* will denote the growth $\beta X - X$ of X. The z-ultrafilter A^p on X represents the point $p \in \beta X$. The Stone extension βf of a continuous map f from a Tychonoff space X to a Tychonoff space Y is given by $(\beta f)(p) = \bigcap_{Z \in A^p} Cl_{\beta Y} f(Z)$.

By an *extension of an action* θ of a group G on a subspace A of a space X, to the space X, we mean an action of G on X, whose restriction to $G \times A$ is θ.

Extending the given action of a discrete group G on a Tychonoff space X to βX, in Section 2, we obtain the Stone-Čech compactification of the orbit space X/G in terms of the orbit space of βX; precisely, we prove that $\beta(X/G) = \beta X/G$, when G is finite. It is observed that the result need not be true in the case that G is an infinite discrete group.

Comfort [2] introduced the important concept of retractive spaces in 1965 [see 4 also]. In Section 3, we put this and other related concepts in their G-versions and prove that for a finite group G, the orbit space of a G-retractive space is G-retractive.

For terms not explained here see [1] and [3].

2. Action on βX

Let X be a G-space and for $g \in G$, let $T_g : X \to X$ be the homeomorphism defined by $T_g(x) = g \cdot x$, $x \in X$. For a subset A of X and a family $\mathcal{A}$ of subsets of X, denote $T_g(A) = \{g \cdot a \mid a \in A\}$ by $g \cdot A$ and $T_g(\mathcal{A}) = \{g \cdot A \mid A \in \mathcal{A}\}$ by $g.\mathcal{A}$. Clearly, $e.\mathcal{A} = \mathcal{A}$ and $g_1 \cdot (g_2 . \mathcal{A}) = (g_1 g_2) . \mathcal{A}$, where e is the identity in G and $g_1, g_2 \in G$. Since, for $g \in G$ and $p \in \beta X$, $g \cdot A^p$ is a z-ultrafilter on X, it corresponds to a point, say $g \cdot p$, in βX. Define $\theta_\beta : G \times \beta X \to \beta X$ by $\theta_\beta(g,p) = g \cdot p$, $g \in G$, $p \in \beta X$. Then it can be seen that θ_β is an extension to βX of the action θ on X . The continuity of θ_β follows by noting that for a zero-set Z of X and $g \in G$, $g \cdot Cl_{\beta X} Z = Cl_{\beta X}(g \cdot Z)$ so that $(\theta_\beta)^{-1}(Cl_{\beta X} Z) = \bigcup_{g \in G} \left(\{g\} \times Cl_{\beta X}(g^{-1} \cdot Z)\right)$ and that the family of closed sets $\left(\{g\} \times Cl_{\beta X}(g^{-1} \cdot Z)\right)$ is locally finite.

For a G-space X , the G-space βX will mean the triple $(\beta X, G, \theta_\beta)$.

We state the following lemma without proof.

LEMMA 2.1. *For G-spaces X and Y , we have*

(a) X and X^* are invariant subspaces of the G-space βX ,

(b) the extension θ_β on βX of a transitive action θ on X is transitive if and only if X is compact,

(c) the Stone extension βf of a continuous equivariant map $f : X \to Y$ is equivariant.

THEOREM 2.2. *If X is a G-space with G finite, then the Stone-Čech compactification of the orbit space X/G is the orbit space of the Stone-Čech compactification of X ; that is, $\beta(X/G) = \beta X/G$.*

Proof. Note that X/G is a dense subspace of the compact space $\beta X/G$. Since G is compact, $\beta X/G$ is Hausdorff [see 1; I, Theorem 3.1]. We show that the Stone extension βi of the inclusion map $i : X/G \to \beta X/G$ is a homeomorphism. Note that $i \circ \pi_X = \pi \circ i_X$, where $\pi_X : X \to X/G$ and $\pi : \beta X \to \beta X/G$ are the orbit maps and i_X is the inclusion map of X into βX . From the functorial properties of β , it follows that $\beta i \circ \beta\pi_X = \pi \circ I_{\beta X}$. For q_1 , q_2 in $\beta(X/G)$, choose $p_1 \in (\beta\pi_X)^{-1}(q_1)$ and $p_2 \in (\beta\pi_X)^{-1}(q_2)$. Then $(\beta i)(q_1) = (\beta i)(q_2)$ implies that $G_{p_1} = G_{p_2}$. Therefore $p_1 = g \cdot p_2$ for some $g \in G$. Using equivariance of $\beta\pi_X$ and that the action on $\beta(X/G)$ is trivial, we obtain that $q_1 = q_2$. For surjectivity of βi , we note that $(\beta i)((\beta\pi_X)(p)) = G_p$, where $G_p \in \beta X/G$. Thus $\beta i : \beta(X/G) \to \beta X/G$ is a homeomorphism which keeps X/G pointwise fixed.

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Remark 2.3. Consider a transitive action of a discrete group G on a non-compact Tychonoff space X. Since the orbit space X/G is the singleton space, in view of Lemma 2.1 (b), we obtain that the result of Theorem 2.2 may not be true for the case of an infinite discrete group.

3. G-retractive spaces

DEFINITION 3.1. A continuous equivariant map r from a G-space X onto an invariant subspace A of X is called a *G-retraction* of X onto A if r leaves points of A fixed. The invariant subspace A is then called a *G-retract* of X. A G-space X is said to be a G-retractive space if the invariant subspace X^* is a *G-retract* of βX.

If G is a singleton or the action is trivial, the concept of G-retraction coincides with that of retraction and consequently those of G-retracts and G-retractive spaces coincide with retracts and retractive spaces, respectively.

EXAMPLE 3.2. Let X be a non-compact Tychonoff space and let $S = \beta X - \{p\}$, where $p \in X^*$. Then $\beta S = \beta X$ [see 3; 6.7]. Denote by G the discrete group of all self homeomorphisms on S together with the binary operation as the composition of maps and consider the G-space (S, G, θ), where $\theta : G \times S \to S$ is defined by $\theta(T, s) = T(s)$, $T \in G$, $s \in S$. In view of Lemma 2.1 (a) $S^* = \{p\}$ is invariant. It is easily seen that S^* is a G-retract of βS. Consequently, the G-space (S, G, θ) is a G-retractive space.

LEMMA 3.3. *If X is a G-retractive space, then X^*/G is a G-retract of $\beta X/G$.*

Proof. Let r be a G-retraction of βX onto X^*. Then the map $r_1 : \beta X/G \to X^*/G$ defined by $r_1(G_p) = G_{r(p)}$, $p \in \beta X$, is a G-retraction.

THEOREM 3.4. *The orbit space of a G-retractive space is G-retractive, where G is finite.*

Proof. Let X be a G-retractive space with G finite, r be a G-retraction of βX onto X^* and let $\beta i : \beta(X/G) \to \beta X/G$ and $r_1 : \beta X/G \to X^*/G$ be as in the proofs of Theorem 2.2 and Lemma 3.3, respectively. Define $r_2 : \beta(X/G) \to (X/G)^*$ by $r_2(q) = f(r_1(\beta i(q)))$, where $q \in \beta(X/G)$ and $f : X^*/G \to (X/G)^*$ is given by $f(G_p) = (\beta i)^{-1}(G_p)$, $p \in X^*$. It can be checked that r_2 is a G-retraction.

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Senior Research Fellow
Allahabad Mathematical Society
Vijay Niwas
198, Mumfordganj
ALLAHABAD 211002,
India.