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On the Sum of Digits of Numerators of Bernoulli Numbers

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Abstract. Let $b > 1$ be an integer. We prove that for almost all n, the sum of the digits in base b of the numerator of the Bernoulli number B_{2n} exceeds $c \log n$, where $c := c(b) > 0$ is some constant depending on b.

1 Introduction

Let $\{B_n\}_{n\geq 0}$ be the sequence of Bernoulli numbers given by $B_0 = 1$ and

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \qquad \text{for all} \quad n \geq 2.$$

Then $B_1 = -1/2$ and $B_{2n+1} = 0$ for all $n \geq 0$. Furthermore, we have $(-1)^{n+1}B_{2n} > 0$. Write $B_{2n} =: (-1)^{n+1}C_n/D_n$ with coprime positive integers C_n and D_n. The denominator D_n is well understood by the von Staudt–Clausen theorem, which asserts that

$$D_n = \prod_{p-1|2n} p.$$

Let $b > 1$ be an integer. For a positive integer m, put $s_b(m)$ for the sum of digits of m in base b. In [2], it was shown that there exists a positive constant c_0 depending on b such that the inequality $s_b(n!) > c_0 \log n$ holds for all positive integers n. Here, we prove that a similar inequality holds with $n!$ replaced by C_n on a set of n of asymptotic density 1.

Theorem 1.1 *The inequality*

$$s_b(C_n) > \frac{\log n}{6 \log b}$$

holds on a set of positive integers n of asymptotic density 1.

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Our main tools are the classical formula

$$\zeta(2n) = (-1)^{n+1} B_{2n} \frac{(2\pi)^{2n}}{2(2n)!} = \frac{C_n (2\pi)^{2n}}{D_n 2(2n)!} \tag{1.1}$$

valid for all $n \geq 1$ and the following result, which is [1, Theorem 2].

Lemma 1.2 *For every $\varepsilon > 0$, there is a $T := T(\varepsilon)$ such that if $x > T$, then the number of $n < x$ that have a divisor $p - 1 > T$ with p prime is less than εx.*

In what follows, we use the Landau symbols O and o and the Vinogradov symbols $\gg$ and $\ll$ with their usual meaning. We also use x_0 for a large real number, not necessarily the same from one occurrence to the next.

Proof Consider the following set of positive integers:

$$\mathcal{M}_b(x) := \Big\{ n \in [x/2, x) : s_b(C_n) < \frac{\log x}{6 \log b} \Big\}.$$

We need to show that $\#\mathcal{M}_b(x) = o(x)$ as $x \to \infty$, because after this the conclusion of Theorem 1.1 will follow by replacing x by $x/2$, then by $x/4$, and so on, and summing up the resulting estimates.

Put $y := \log x$ and consider the set

$$\mathcal{L}(x) := \{ n \in [x/2, x) : p - 1 \mid 2n \text{ for some prime } p \geq y \}.$$

It follows from Lemma 1.2 that

$$\#\mathcal{L}(x) = o(x) \qquad \text{as} \quad x \to \infty. \tag{1.2}$$

We now put $\mathcal{N}_b(x) := \mathcal{M}_b(x) \backslash \mathcal{L}(x)$. In light of (1.2), it suffices to show that $\#\mathcal{N}_b(x) = o(x)$ as $x \to \infty$. Let

$$\mathcal{D}(x) := \{ D_n : n \in \mathcal{N}_b(x) \}.$$

Since $n \notin \mathcal{L}(x)$, it follows that if $p \mid C_n$, then $p < y$. Thus,

$$\#\mathcal{D}(x) \leq 2^{\pi(y)} = x^{o(1)} \qquad \text{as} \qquad x \to \infty. \tag{1.3}$$

For $n \in \mathcal{N}_b(x)$, we write $C_n = d_1 b^{n_1} + d_2 b^{n_2} + \cdots + d_s b^{n_s}$, where $d_1, \ldots, d_s \in \{1, \ldots, b-1\}$ and $n_1 > n_2 > \cdots > n_s$. We next put $t := t(n)$ for the smallest index $i \in \{1, 2, \ldots, s-1\}$ such that $b^{n_1 - n_{i+1}} > n^2$ if it exists and set $t := s$ otherwise. From the definition of $t(n)$, we see immediately that

$$C_n = \big(d_1 b^{n_1} + \cdots + d_t b^{n_t} \big) \big(1 + O(n^{-2}) \big) := b^{m_n} E_n \big(1 + O(x^{-2}) \big), \tag{1.4}$$

where $m = m_n := n_t$ and $E_n := d_1 b^{n_1 - n_t} + d_2 b^{n_2 - n_t} + \cdots + d_t$.

Let

$$\mathcal{E}_b(x) = \{ E_n : n \in \mathcal{N}_b(x) \}.$$

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Let us find an upper bound for the cardinality of $\mathcal{E}_b(x)$. First observe that

$$E_n < b^{n_1 - n_t + 1} \le b^{2(\log x)/(\log b)+1}. \tag{1.5}$$

The positive integers $E := E_n$ bounded by the right-hand side of inequality (1.5) have at most $K := \lfloor 2(\log x)/(\log b) + 2 \rfloor$ digits in base b. As $n \in \mathcal{N}_b(x)$, the number of nonzero digits of E_n is bounded by $S := \lfloor (\log x)/(6 \log b) \rfloor$, so the number of possible values for E is at most

$$\begin{aligned}\sum_{i=0}^{S} \binom{K}{i}(b-1)^i &\le (S+1)\binom{K}{S}(b-1)^S \le (S+1)\Big(\frac{(b-1)eK}{S}\Big)^S \\ &\le \Big(\frac{\log x}{6\log b}+1\Big)\Big(\frac{(b-1)e}{3}+o(1)\Big)^{(\log x)/(6\log b)} = x^{\delta+o(1)}\end{aligned}$$

as $x \to \infty$, where

$$\delta := \frac{\log((b-1)e/3)}{6\log b} < \frac{1}{6}.$$

Thus, we get that

$$\#\mathcal{E}_b(x) = x^{\delta+o(1)} \quad \text{as} \quad x \to \infty. \tag{1.6}$$

We next use formula (1.1) as well as the aproximation

$$\zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = 1 + O\Big(\frac{1}{2^{2n}}\Big)$$

to get that

$$C_n = D_n \frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n) = D_n \frac{2(2n)!}{(2\pi)^{2n}}\left(1 + O\left(\frac{1}{2^{2n}}\right)\right). \tag{1.7}$$

We take logarithms in (1.7) to arrive at

$$\log(C_n/D_n) - \log(2(2n)!) + 2n\log(2\pi) = \log\Big(1 + O\Big(\frac{1}{2^{2n}}\Big)\Big) \ll \frac{1}{2^x}. \tag{1.8}$$

Taking logarithms in (1.4) and comparing the result with (1.8), we get

$$\log(2(2n)!) - 2n\log(2\pi) = \log(E_n/D_n) + m_n \log b + O\Big(\frac{1}{x^2}\Big). \tag{1.9}$$

Now fix a pair of numbers $(D, E) \in \mathcal{D}_b(x) \times \mathcal{E}_b(x)$ and look at the set

$$\mathcal{N}_{b,D,E}(x) := \{n \in \mathcal{N}_b(x) : (D_n, E_n) = (D, E)\}.$$

We let $z := x^{1/5}$ and show that if $x > x_0$, then every subinterval I of $(x/2, x]$ of length z contains at most two elements of $\mathcal{N}_{b,D,E}(x)$. Now, assume that this is not so and let

$n_0 < n_1 < n_2$ be all three in $\mathcal{N}_{b,D,E}(x)$, where $n_2 - n_0 \leq z$. Write $n_i = n_0 + k_i$ for $i = 1, 2$, with $0 < k_1 < k_2 \leq z$. We evaluate relation (1.9) in $n_0,\ n_1$, and n_2 getting

$$\log(2(2n_i)!) + 2n_i \log(2\pi) = \log(E/D) + m_{n_i} \log b + O\Big(\frac{1}{x^2}\Big) \tag{1.10}$$

for $i = 0,\ 1,\ 2$. We let $\Lambda := (\lambda_0, \lambda_1, \lambda_2) = (k_2 - k_1, -k_2, k_1)$. Observe that

$$\begin{cases} \lambda_0 + \lambda_1 + \lambda_2 = 0, \\ n_0\lambda_0 + n_1\lambda_1 + n_2\lambda_2 = 0, \end{cases} \tag{1.11}$$

and $\max\{|\lambda_i| : i = 0, 1, 2\} \leq z$. Taking the linear combination of the three relations (1.10) with the coefficients given by Λ and using the second equation (1.11), we get

$$\sum_{i=0}^{2} \lambda_i \log(2(2n_i)!) = \Gamma \log b + O\Big(\frac{z}{x^2}\Big), \tag{1.12}$$

where $\Gamma := \sum_{i=0}^{2} \lambda_i m_{n_i} \in \mathbb{Z}$. Write

$$2(2n_i)! = 2(2n_0)!(2n_0 + 1)\cdots(2n_i) =: 2(2n_0)!X_i, \quad (i = 0, 1, 2).$$

Hence,

$$\begin{aligned} \sum_{i=0}^{2} \lambda_i \log(2(2n_i)! &= \sum_{i=0}^{2} \lambda_i (\log(2(2n_0)!) + \log X_i) \\ &= \sum_{i=0}^{2} \lambda_i \log(2(2n_1)!) + \sum_{i=0}^{2} \lambda_i \log X_i \\ &= \sum_{i=0}^{2} \lambda_i \log X_i = \lambda_1 \log X_1 + \lambda_2 \log X_2, \end{aligned}$$

where in the above equalities we used the first equation of (1.11) as well as the fact that $X_0 = 1$. Writing

$$\begin{aligned} \log X_i = \sum_{j=1}^{2k_i} \log(2n_0 + j) &= 2k_i \log(2n_0) + \sum_{j=1}^{2k_i} \log\Big(1 + \frac{j}{n_0}\Big) \\ &= 2k_i \log(2n_0) + \sum_{j=1}^{2k_i} \Big(\frac{j}{n_0} + O\Big(\frac{j^2}{n_0^2}\Big)\Big) \\ &= 2k_i \log(2n_0) + \sum_{j=1}^{2k_i} \frac{j}{n_0} + O\Big(\sum_{j=0}^{2k_i} \frac{j^2}{n_0^2}\Big) \\ &= 2k_i \log(2n_0) + \frac{2k_i(2k_i + 1)}{n_0} + O\Big(\frac{k_i^3}{x^2}\Big), \end{aligned}$$

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for $i = 1, 2$, we get that

$$\begin{aligned}
&\lambda_1 \log X_1 + \lambda_2 \log X_2 \\
&\quad = 2(\lambda_1 k_1 + \lambda_2 k_2)\log(2n_0) + \frac{4\lambda_1 k_1^2 + 4\lambda_2 k_2^2}{n_0} + \frac{2\lambda_1 k_1 + 2\lambda_2 k_2}{n_0} + O\Big(\frac{k_2^4}{n_0^2}\Big) \\
&\quad = 4\Big(\frac{\lambda_1 k_1^2 + \lambda_2 k_2^2}{n_0}\Big) + O\Big(\frac{z^4}{x^2}\Big).
\end{aligned}$$

Inserting the above estimate into the left-hand side of (1.12), we get

$$4\Big(\frac{\lambda_1 k_1^2 + \lambda_2 k_2^2}{n_0}\Big) = \Gamma \log b + O\Big(\frac{z}{x^2} + \frac{z^4}{x^2}\Big) = \Gamma \log b + O\Big(\frac{1}{x^{6/5}}\Big). \tag{1.13}$$

If $\Gamma = 0$, then (1.13) implies that

$$|\lambda_1 k_1^2 + \lambda_2 k_2^2| = O\Big(\frac{n_0}{x^{6/5}}\Big) = O\Big(\frac{1}{x^{1/5}}\Big) = o(1)$$

as $x \to \infty$, showing that $\lambda_1 k_1^2 + \lambda_2 k_2^2 = 0$ for $x > x_0$. This last equation is equivalent to $k_1 k_2 (k_1 - k_2) = 0$, which is impossible. Thus, $\Gamma \neq 0$, showing that the right-hand side in estimate (1.13) is $\gg 1$, and since the left-hand side of estimate (1.13) is $O(k_2^3/x)$, we get that $k_2 \gg x^{1/3}$. This is not possible for large x because $k_2 \le z$. We conclude that indeed for large x, I cannot contain three numbers from $\mathcal{N}_{b,D,E}(x)$. This shows that

$$\#\mathcal{N}_{b,D,E}(x) \le \Big[\frac{x - x/2}{z}\Big] + 2 \ll x^{4/5}.$$

Hence, by estimates (1.3) and (1.6), we get that

$$\begin{aligned}
\mathcal{N}_b(x) &= \sum_{(D,E)\in \mathcal{D}_b(x)\times \mathcal{E}_b(x)} \#\mathcal{N}_{b,D,E} \ll x^{4/5} \times \#\mathcal{D}_b(x) \times \#\mathcal{E}_b(x) \\
&\le x^{4/5+\delta+o(1)} < x^{29/30+o(1)} = o(x)
\end{aligned}$$

as $x \to \infty$, which is what we wanted to prove. ∎

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