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A REMARK ON COLORING INTEGERS

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In memory of Leo Moser

Color the integers $1, 2, \ldots, n$ red and blue. In this note we consider arithmetic sequences in which the discrepancy between red and blue integers is maximized.

More formally, we represent the coloring by a function $\lambda:\{1, \ldots, n\} \to \{+1, -1\}$. Set

$$(1) \qquad G(n) = \min_{\lambda} \max_{a,d,m} \left| \sum_{k=0}^{m} \lambda(a+kd) \right|, \quad (a+md \le n).$$

It is known that

$$(2) \qquad O(n^{1/4}) \le G(n) \le O(n^{1/2}).$$

The lower bound is due to Roth [2] and the upper bound to Erdös [1]. We here improve the upper bound by showing

$$(3) \qquad G(n) \le C\sqrt{n}\,\sqrt{\log\log n}/\sqrt{\log n}.$$

($C, c_1, c_2, \ldots$ will always signify suitably chosen absolute constants.) The proof is an extension of the method of Erdös.

LEMMA. *Let $M(m, t)$ denote the fraction of functions $\lambda:\{1, \ldots, m\} \to \{+1, -1\}$ such that*

$$\max_{1 \le u \le v \le m} \left| \sum_{k=u}^{v} \lambda(k) \right| > tm^{1/2}.$$

Then there exist absolute constants c_1, c_2 such that

$$M(m, t) < c_1 e^{-c_2 t^2}.$$

Proof. The proof is given in [1]. It uses an inequality of Kolmogoroff and the theory of binomial expansion.

Now let n be fixed. Set $b = [(\log n)/3]$, $N = gcd(1, 2, \ldots, b)$. By the Prime Number Theorem $N \sim e^b \sim n^{1/3}$. (Actually, we could take $N = b!$ with similar results.) Let T be the set of functions $\lambda:\{1, \ldots, n\} \to \{+1, -1\}$ satisfying

$$(4) \qquad \lambda(x+N) = -\lambda(x) \quad \text{for} \quad \begin{aligned} 1 &\le x \le N \\ 2N+1 &\le x \le 3N \\ 4N+1 &\le x \le 5N \\ &\vdots \\ 2rN+1 &\le x \le (2r+1)N \\ &\vdots \end{aligned}$$

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We count the number of $\lambda \in T$ such that there exist d, a, m,

$$\left|\sum_{k=0}^{m} \lambda(a+kd)\right| > C\sqrt{n}\sqrt{\log\log n}/\sqrt{\log n}. \tag{5}$$

For $d \le b$, and any a, there are no such λ. For then $d \mid N$ and the values $\lambda(x)$, $\lambda(x+N)$ will cancel for those x satisfying (4). There will be at most $2N \sim 2n^{1/3}$ end points that might not cancel.

Now say $d > b$. There are d congruence classes w, modulo d. By the lemma, for each w the fraction of $\lambda \in T$ satisfying (5) where $a \equiv w \pmod{d}$ is bounded by $c_1 e^{-c_2 t^2}$ where

$$t = (C\sqrt{n}\sqrt{\log\log n}/\sqrt{\log n})/\sqrt{n/d} = C\sqrt{d}\sqrt{\log\log n}/\sqrt{\log n}.$$

Thus the total fraction of $\lambda \in T$ satisfying (5) is bounded by

$$\sum_{d=b+1}^{n} dc_1 e^{-c_2 C^2 (\log\log n)(d/\log n)}.$$

It is easy to show that for C sufficiently large (independent of n), this quantity is less than unity. For that C there exist $\lambda \in T$ not satisfying (5), thus proving (3).

Roth suspects that $G(n) > n^{1/2-\epsilon}$. It appears likely that the bound given here is very close to the true value of $G(n)$.

References

1. P. Erdös, *Szamelmeleti megjegyzesek V. Extremalis problemak a szamelmeletben, II*, Mat. Lapok (1966), 135–155.

2. K. F. Roth, *Remark concerning integer sequences*, Acta Arith. **IX** (1964), 257–260.

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