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Abstract
We study Cox rings of K3 surfaces. A first result is that a K3 surface has a finitely generated Cox ring if and only if its effective cone is rational polyhedral. Moreover, we investigate degrees of generators and relations for Cox rings of K3 surfaces of Picard number two, and explicitly compute the Cox rings of generic K3 surfaces with a nonsymplectic involution that have Picard number 2 to 5 or occur as double covers of del Pezzo surfaces.

1. Introduction
The Cox ring $R(X)$ of a normal complete algebraic variety $X$ with a finitely generated divisor class group $\text{Cl}(X)$ is the multigraded algebra

$$R(X) := \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

For a toric variety $X$, the Cox ring $R(X)$ is a polynomial ring and its multgrading can be explicitly determined in terms of a defining fan; see [Cox95]. Moreover, for del Pezzo surfaces $X$ there are recent results on generators and relations of the Cox ring; see [BP04, Der, LV09, STV07, SX, TVV]. In the present paper, we investigate Cox rings of K3 surfaces $X$, i.e. smooth complex projective surfaces $X$ with $b_1(X) = 0$ that admit a nowhere vanishing holomorphic 2-form $\omega_X$.

A first basic problem is to decide if the Cox ring $R(X)$ is finitely generated. In §2, we first discuss this question in general, and extend results of Hu and Keel on $\mathbb{Q}$-factorial projective varieties [HK00] to normal complete ones; a consequence is that every normal complete surface with finitely generated Cox ring is $\mathbb{Q}$-factorial and projective; see Theorem 2.5. For K3 surfaces, we obtain the following; see Theorem 2.7.

**Theorem 1.** A K3 surface has finitely generated Cox ring if and only if its cone of effective rational divisor classes is polyhedral.

The same characterization holds for Enriques surfaces; see Theorem 2.10. The second basic problem is to describe the Cox ring $R(X)$ in terms of generators and relations. We first consider K3 surfaces $X$ having Picard number $\rho(X) = 2$; see §3. In this setting, if the effective cone is polyhedral, then it is known that its generators are of self intersection zero or minus two; see §2 for this and some more background. For the case that both generators are of self intersection zero, we obtain the following; see Proposition 3.1 and Theorem 3.2.

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Theorem 2. Let $X$ be a K3 surface with $\text{Cl}(X) \cong Zw_1 \oplus Zw_2$, where $w_1$, $w_2$ are effective, and intersection form given by $w_1^2 = w_2^2 = 0$ and $w_1 \cdot w_2 = k \geq 3$.

(i) The effective cone of $X$ is generated by $w_1$ and $w_2$ and it coincides with the semiample cone of $X$.

(ii) The Cox ring $\mathcal{R}(X)$ is generated in degrees $w_1$, $w_2$, $w_1 + w_2$, and one has

$$\dim(\mathcal{R}(X)_{w_1}) = 2, \quad \dim(\mathcal{R}(X)_{w_1+w_2}) = k + 2.$$ 

Moreover, every minimal system of generators of $\mathcal{R}(X)$ has $k + 2$ members.

(iii) For $k = 3$, the Cox ring $\mathcal{R}(X)$ is of the form $\mathbb{C}[T_1, \ldots, T_5]/\langle f \rangle$ and the degrees of the generators and the relation are given by

$$\deg(T_1) = \deg(T_2) = w_1, \quad \deg(T_4) = \deg(T_5) = w_2,$$

$$\deg(T_3) = w_1 + w_2, \quad \deg(f) = 3w_1 + 3w_2.$$ 

(iv) For $k \geq 4$, any minimal ideal $\mathcal{I}(X)$ of relations of $\mathcal{R}(X)$ is generated in degree $2w_1 + 2w_2$, and we have

$$\dim(\mathcal{I}(X)_{2w_1+2w_2}) = \frac{k(k-3)}{2}.$$ 

The statements on the generators are directly obtained and, for the relations, we use the techniques developed in [LV09]. When at least one of the generators of the effective cone is a $(-2)$-curve, then the semiample cone is a proper subset of the effective cone. We show that in this case the number of degrees needed to generate the Cox ring can be arbitrarily big; Propositions 3.6 and 3.7 give a lower bound for this number in terms of the intersection form of $\text{Cl}(X)$.

For the K3 surfaces $X$ with Picard number $\rho(X) \geq 3$, we use a different approach. Many K3 surfaces $X$ with $\rho(X) \geq 3$ and polyhedral effective cone admit a non-symplectic involution, i.e. an automorphism $\sigma : X \to X$ of order two with $\sigma^* \omega_X \neq \omega_X$. The associated quotient map $\pi : X \to Y$ is a double cover. If it is unramified then $Y := X/\langle \sigma \rangle$ is an Enriques surface, otherwise $Y$ is a smooth rational surface. In the latter case, one may use known results and techniques to obtain the Cox ring of $Y$.

This observation suggests studying the behavior of Cox rings under double coverings $\pi : X \to Y$. As it may be of independent interest, we consider in §4 more general, e.g. cyclic, coverings $\pi : X \to Y$ of arbitrary normal varieties $X$ and $Y$. We relate finite generation of the Cox rings of $X$ and $Y$ to each other; Propositions 4.6 and 4.3 provide generators and relations for the Cox ring of $X$ in terms of $\pi$ and the Cox ring of $Y$ for the case that $\pi$ induces an isomorphism on the level of divisor class groups. This enables us to compute Cox rings of K3 surfaces that are general double covers of $\mathbb{P}^2$ or of del Pezzo surfaces.

Besides $\mathbb{P}^2$ and the del Pezzo surfaces, other rational surfaces $Y = X/\langle \sigma \rangle$ can occur. For $2 \leq \rho(X) \leq 5$, these turn out to be blow ups of the fourth Hirzebruch surface $\mathbb{F}_4$ in at most three general points, and we are in this setting if and only if the branch divisor of the covering $\pi : X \to Y$ has two components. Then, in order to determine the Cox ring of $X$, we have to solve two problems. Firstly, the computation of the Cox ring of $Y$. While blowing up one or two points gives a toric surface, the blow up of $\mathbb{F}_4$ in three general points is non-toric; we compute its Cox ring in §5 using the technique of toric ambient modifications developed in [Han08]. The second problem is that $\pi : X \to Y$ no longer induces an isomorphism on the divisor class groups. Here, Proposition 6.4 provides a general result.

Putting all this together, we obtain the following results in the case of Picard number $2 \leq \rho(X) \leq 5$; see Propositions 6.5–6.8 for the complete statements.
Theorem 3. Let $X$ be a generic K3 surface with a non-symplectic involution and associated double cover $X \to Y$ and Picard number $2 \leq \varrho(X) \leq 5$. Then the Cox ring $\mathcal{R}(X)$ is given as follows.

(i) For $\varrho(X) = 2$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_5]/(T_5^2 - f)$ and the degree of $T_i$ is the $i$th column of

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 2
\end{bmatrix} \quad \text{if } Y = \mathbb{F}_0,$$

$$\begin{bmatrix}
1 & 0 & -1 & -1 & -1 \\
0 & 1 & 1 & 1 & 3
\end{bmatrix} \quad \text{if } Y = \mathbb{F}_1,$$

$$\begin{bmatrix}
1 & 0 & 2 & 0 & 3 \\
0 & 1 & 4 & 1 & 6
\end{bmatrix} \quad \text{if } Y = \mathbb{F}_4.$$

(ii) For $\varrho(X) = 3$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_6]/(T_6^2 - f)$ and the degree of $T_i$ is the $i$th column of

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 1 & 1 & 3
\end{bmatrix} \quad \text{if } Y = \text{Bl}_1(\mathbb{F}_0),$$

$$\begin{bmatrix}
1 & 0 & 0 & 2 & 0 & 3 \\
0 & 1 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & 3 & 1 & 5
\end{bmatrix} \quad \text{if } Y = \text{Bl}_1(\mathbb{F}_4).$$

(iii) For $\varrho(X) = 4$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_7]/(T_7^2 - f)$ and the degree of $T_i$ is the $i$th column of

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 1 & -1 & -1 & -1
\end{bmatrix} \quad \text{if } Y = \text{Bl}_2(\mathbb{F}_0),$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 2 & 0 & 3 \\
0 & 1 & 0 & 0 & 3 & 1 & 5 \\
0 & 0 & 1 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 & 4
\end{bmatrix} \quad \text{if } Y = \text{Bl}_2(\mathbb{F}_4).$$

(iv) For $\varrho(X) = 5$ one has the following two cases.

(a) The surface $Y$ is the blow up of $\mathbb{F}_0$ at three general points. Then the Cox ring $\mathcal{R}(X)$ of $X$ is

$$\mathbb{C}[T_1, \ldots, T_{11}]/(f_1, \ldots, f_5, T_{11}^5 - g),$$

where $f_1, \ldots, f_5, g \in \mathbb{C}[T_1, \ldots, T_{10}]$ and $f_1, \ldots, f_5$ are the Plücker relations in $T_1, \ldots, T_{10}$. The degree of $T_i$ is the $i$th column of

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\
1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & 1 & 1
\end{bmatrix}. $$
ON Cox rings of K3 surfaces

(b) The surface $Y$ is the blow up of $\mathbb{F}_4$ at three general points. Then the Cox ring $\mathcal{R}(X)$ of $X$ is

$$\mathbb{C}[T_1, \ldots, T_9]/(T_2T_5 + T_4T_6 + T_7T_8, T_6^2 - f),$$

where $f \in \mathbb{C}[T_1, \ldots, T_8]$ is a prime polynomial and the degree of $T_i \in \mathcal{R}(X)$ is the $i$th column of

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 & 4 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & 4 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1
\end{bmatrix}.$$
If the divisor class group Cl(X) is free, then the Cox ring \( R(X) \) admits unique factorization; see [BH03]. If Cl(X) has torsion, then the unique factorization property is replaced by a graded version: every non-trivial homogeneous non-unit is a product of homogeneous primes, where the latter refers to non-trivial homogeneous non-units \( f \) such that \( f|gh \) with \( g, h \) homogeneous implies that \( f|g \) or \( f|h \); see [Hau08]. If the Cox ring \( R(X) \) is finitely generated as a \( \mathbb{K} \)-algebra, then one may define the total coordinate space \( \overline{X} = \text{Spec} \, R(X) \) and realize \( X \) as the quotient of an open subset \( \overline{X} = \text{Spec}_X R(X) \) by the action of the diagonalizable group \( \text{Spec} \, \text{Cl}(X) \) defined by the \( \text{Cl}(X) \)-grading of the sheaf \( \mathcal{R} \) of \( \mathcal{O}_X \)-algebras. For smooth \( X \), the map \( \overline{X} \to X \) is also known as the universal torsor of \( X \).

Let \( \text{Cl}_Q(X) = \text{Cl}(X) \otimes \mathbb{Q} \) denote the rational divisor class group of \( X \). A first step is to give descriptions of the cones \( \text{Eff}(X) \subseteq \text{Cl}_Q(X) \) of effective classes and \( \text{Mov}(X) \subseteq \text{Cl}_Q(X) \) of movable classes, i.e. classes having a stable base locus of codimension at least two. We call a cone in a rational vector space \( V \) polyhedral if it is the positive hull cone \( \mathcal{P}(v_1, \ldots, v_r) \) of finitely many vectors \( v_i \in V \). The following statement generalizes part of [Hau08, Proposition 4.1].

**Proposition 2.1.** Let \( X \) be a normal complete variety with finitely generated Cox ring \( R(X) \). Then the cones of effective and movable divisor classes in \( \text{Cl}_Q(X) \) are polyhedral. Moreover, if \( f_1, \ldots, f_r \in R(X) \) is any system of pairwise non-associated homogeneous prime generators, then one has

\[
\text{Eff}(X) = \bigcap_{i=1}^r \text{cone}(\deg(f_i); i = 1, \ldots, r),
\]

\[
\text{Mov}(X) = \bigcap_{i=1}^r \text{cone}(\deg(f_j); j \neq i).
\]

For the proof and also for later use, we have to fix some notation. On a normal variety \( X \), let a class \( w \in \text{Cl}(X) \) be represented by a divisor \( D \in \text{WDiv}(X) \). Then, as usual, we write

\[
H^i(D) := H^i(X, D) := H^i(X, \mathcal{O}_X(D)), \quad h^i(w) := h^i(D) := \dim_{\mathbb{K}}(H^i(D)).
\]

**Lemma 2.2.** Let \( X \) be a normal complete variety with \( \text{Cl}(X) \) finitely generated and let \( w \in \text{Cl}(X) \) be effective. Then the following two statements are equivalent.

\begin{enumerate}[(i)]
  \item The stable base locus of the class \( w \in \text{Cl}(X) \) contains a divisor.
  \item There exists a class \( w_0 \in \text{Cl}(X) \) with the following properties:
    \begin{itemize}
      \item the class \( w_0 \) generates an extremal ray of \( \text{Eff}(X) \) and \( h^0(nw_0) = 1 \) holds for any \( n \in \mathbb{N} \);
      \item there is an \( f_0 \in R(X)w_0 \) such that for any \( n \in \mathbb{N} \) and \( f \in R(X)_{nw} \) one has \( f = f'f_0 \) with some \( f' \in R(X)_{nw-w_0} \).
    \end{itemize}
\end{enumerate}

**Proof.** The implication ‘(iii) \( \Rightarrow \) (i)’ is obvious. So, assume that (i) holds. The class \( w \in \text{Cl}(X) \) is represented by some non-negative divisor \( D \). Let \( D_0 \) be a prime component of \( D \), which occurs in the fixed part of any positive multiple \( nD \), and let \( w_0 \in \text{Cl}(X) \) be the class of \( D_0 \). Then the canonical section of \( D_0 \) defines an element \( f_0 \in R(X)w_0 \), which divides any \( f \in R(X)_{nw} \). Note that \( h^0(nD_0) = 1 \) holds for every \( n \in \mathbb{Z}_{\geq 0} \); because otherwise \( H^0(na_0D_0) \subseteq R(X)_{nw} \), where \( a_0 > 0 \) is the multiplicity of \( D_0 \) in \( D \), would provide enough sections to move \( na_0D_0 \). Moreover, \( \text{cone}(w_0) \) is an extremal ray of \( \text{Eff}(X) \), because otherwise we would have \( nD_0 \sim D_1 + D_2 \) with some \( n \in \mathbb{Z}_{\geq 0} \) and non-negative divisors \( D_1, D_2 \), none of which is a multiple of \( D_0 \); this contradicts \( h^0(nD_0) = 1 \).

**Proof of Proposition 2.1.** Only for the description of the moving cone there is something to show. For this, set \( w_i := \deg(f_i) \) and note that the extremal rays of the effective cone occur.
Let \( w \in \text{Mov}(X) \). Then Lemma 2.2 tells us that for any \( i = 1, \ldots, d \) there must be a monomial of the form \( \prod_{j \neq i} f_j^{n_j} \) in some \( \mathcal{R}(X)_{nw} \). Consequently, \( w \) lies in the cone of the right-hand side. Conversely, consider an element \( w \) of the cone of the right-hand side. Then, for every \( i = 1, \ldots, d \), a product \( \prod_{j \neq i} f_j^{n_j} \) belongs to some \( \mathcal{R}(X)_{nw} \). Hence, none of the \( f_1, \ldots, f_d \) divides all elements of \( \mathcal{R}(X)_{nw} \). Again by Lemma 2.2, we conclude that \( w \in \text{Mov}(X) \). \( \square \)

We characterize finite generation of the Cox ring. By \( \text{SAmp}(X) \subseteq \text{Cl}_Q(X) \) we denote the cone of semiample divisor classes of a variety \( X \), i.e. classes having a base point free positive multiple. Moreover, by a small birational map \( X \to Y \), we mean a rational map that defines an isomorphism \( U \to V \) of open subsets \( U \subseteq X \) and \( V \subseteq Y \) such that the respective complements \( X \setminus U \) and \( Y \setminus V \) are of codimension at least two.

**Theorem 2.3.** Let \( X \) be a normal complete variety with finitely generated divisor class group. Then the following statements are equivalent.

(i) The Cox ring \( \mathcal{R}(X) \) is finitely generated.

(ii) The effective cone \( \text{Eff}(X) \subseteq \text{Cl}_Q(X) \) is polyhedral and there are small birational maps \( \pi_i: X \to X_i \), where \( i = 1, \ldots, r \), such that each semiample cone \( \text{SAmp}(X_i) \subseteq \text{Cl}_Q(X_i) \) is polyhedral and one has

\[
\text{Mov}(X) = \pi_1^*(\text{SAmp}(X_1)) \cup \cdots \cup \pi_r^*(\text{SAmp}(X_r)).
\]

Moreover, if one of these two statements holds, then there is a small birational map \( X \to X' \) with a \( \mathbb{Q} \)-factorial projective variety \( X' \).

In the proof we use the fact that the moving cone of any normal complete variety is of full dimension; we are grateful to Jenia Tevelev for providing us with the following statement and proof.

**Lemma 2.4.** Let \( X \) be a normal complete variety with \( \text{Cl}(X) \) finitely generated. Then the moving cone \( \text{Mov}(X) \) is of full dimension in the rational divisor class group \( \text{Cl}_Q(X) \).

**Proof.** Using Chow’s lemma and resolution of singularities, we obtain a birational morphism \( \pi: X' \to X \) with a smooth projective variety \( X' \). Let \( D_1, \ldots, D_r \in \text{WDiv}(X) \) be prime divisors generating \( \text{Cl}(X) \), and consider their proper transforms \( D'_1, \ldots, D'_r \in \text{WDiv}(X') \). Moreover, let \( E' \in \text{CaDiv}(X') \) be very ample such that all \( E' + D'_i \) are also very ample, and denote by \( E \in \text{WDiv}(X) \) its push-forward. Then the classes \( E \) and \( E + D_i \) generate a full-dimensional cone \( \tau \subseteq \text{Cl}_Q(X) \) and, since \( E' \) and the \( E' + D'_i \) are movable, we have \( \tau \subseteq \text{Mov}(X) \). \( \square \)

**Proof of Theorem 2.3.** Suppose that (i) holds. Then Proposition 2.1 tells us that \( \text{Eff}(X) \) is polyhedral. Let \( \mathfrak{f} = (f_1, \ldots, f_r) \) be a system of pairwise non-associated homogeneous prime generators of \( R := \mathcal{R}(X) \) and set \( w_i := \deg(f_i) \).

By [Hau08, Proposition 2.2], the group \( H = \text{Spec} \mathbb{K} [\text{Cl}(X)] \) acts freely on an open subset \( W \subseteq X \) of \( X = \text{Spec} \mathcal{R}(X) \) such that \( X \setminus W \) is of codimension at least two in \( X \). Thus, we can choose a point \( z \in W \) with \( f_i(z) = 0 \) and \( f_j(z) \neq 0 \) for \( j \neq i \). Consequently, the weights \( w_i \), where \( j \neq i \), generate \( \text{Cl}(X) \) and hence the system of generators \( \mathfrak{f} \) is admissible in the sense of [Hau08, Definition 3.4]. Moreover, by Lemma 2.4, the moving cone of \( X \) is of full dimension and, by
Proposition 2.1, it is given as

\[ \text{Mov}(X) = \bigcap_{i=1}^{r} \text{cone}(w_j; j \neq i). \]

Thus, we are in the setting of [Hau08, Corollary 4.3]. That means that Mov(X) is a union of full-dimensional GIT chambers \( \lambda_1, \ldots, \lambda_r \), the relative interiors of which are contained in the relative interior of Mov(X) and the associated projective varieties \( X_i := \tilde{X}_i / H \), where \( \tilde{X}_i := X^{ss}(\lambda_i) \) are \( \mathbb{Q} \)-factorial and have \( \mathcal{R}(X) \) as their Cox ring and \( \lambda_i \) as their semiample cone.

Moreover, if \( f : \tilde{X} \rightarrow X \) and \( q_i : \tilde{X}_i \rightarrow X_i \) denote the associated universal torsors, then the desired small birational maps \( \pi_i : X \rightarrow X_i \) are obtained as follows. Let \( X' \subseteq X \) and \( X'_i \subseteq X_i \) be the respective sets of smooth points. Then, by [Hau08, Proposition 2.2], the sets \( q_i^{-1}(X'_i) \) have a small complement in \( X \) and thus we obtain open embeddings with a small complement

\[ X \leftarrow (q_i^{-1}(X'_i) \cap q_i^{-1}(X'_i)) / H \rightarrow X_i. \]

Now suppose that (ii) holds. Let \( w_1, \ldots, w_d \in \text{Eff}(X) \) be those primitive generators of extremal rays of Eff(X) that satisfy \( h^0(nw_i) \leq 1 \) for any \( n \in \mathbb{Z}_{\geq 0} \) and fix \( 0 \neq f_i \in \mathcal{R}(X)_{nw_i} \) with \( n_i \) minimal. Then we have

\[ \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{R}(X)_{nw_i} = \mathbb{K}[f_i]. \]

Set \( \lambda_i := \pi_i^*(\text{SAample}(X_i)) \). Then, by Gordon’s lemma and [HK00, Lemma 1.8], we have another finitely generated subalgebra of the Cox ring, namely

\[ S(X) := \bigoplus_{w \in \text{Mov}(X)} \mathcal{R}(X)_{nw} = \biggoplus_{i=1}^{r} \left( \bigoplus_{w \in \lambda_i} \mathcal{R}(X)_w \right). \]

We show that \( \mathcal{R}(X) \) is generated by \( S(X) \) and the \( f_i \in \mathcal{R}(X)_{nw_i} \). Consider any \( 0 \neq f \in \mathcal{R}(X)_w \) with \( w \notin \text{Mov}(X) \). Then, by Lemma 2.2, we have \( f = f^{(1)} f_i \) for some \( 1 \leq i \leq d \) and some \( f^{(1)} \in \mathcal{R}(X) \) homogeneous of degree \( w(1) := w - n_i w_i \). If \( w(1) \notin \text{Mov}(X) \) holds, then we repeat this procedure with \( f^{(1)} \) and obtain \( f = f^{(2)} f_i f_j \) with \( f^{(2)} \) homogeneous of degree \( w(2) \). At some point, we must end with \( w(n) = \deg(f(n)) \in \text{Mov}(X) \), because otherwise the sequence of the \( w(n) \) would leave the effective cone. \( \square \)

**Theorem 2.5.** Let \( X \) be a normal complete surface with finitely generated divisor class group \( \text{Cl}(X) \). Then the following statements are equivalent.

(i) The Cox ring \( \mathcal{R}(X) \) is finitely generated.

(ii) The effective cone \( \text{Eff}(X) \subseteq \text{Cl}_\mathbb{Q}(X) \) and the moving cone \( \text{Mov}(X) \subseteq \text{Cl}_\mathbb{Q}(X) \) are polyhedral and \( \text{Mov}(X) = \text{SAample}(X) \) holds.

Moreover, if one of these two statements holds, then the surface \( X \) is \( \mathbb{Q} \)-factorial and projective.

**Proof.** We verify the implication ‘(i) ⇒ (ii)’. By Proposition 2.1, we only have to show that the moving cone coincides with the semiample cone. Clearly, we have \( \text{SAample}(X) \subseteq \text{Mov}(X) \).

Suppose that \( \text{SAample}(X) \neq \text{Mov}(X) \) holds. Then \( \text{Mov}(X) \) is properly subdivided into GIT chambers; see [Hau08, Corollary 4.3]. In particular, we find two chambers \( \lambda' \) and \( \lambda \) both intersecting the relative interior of \( \text{Mov}(X) \) such that \( \lambda' \) is a proper face of \( \lambda \). The associated GIT quotients \( Y' \) and \( Y \) of the total coordinate space \( \overline{X} \) have \( \lambda' \) and \( \lambda \) as their respective
On Cox rings of K3 surfaces

semiample cones. Moreover, the inclusion \( \lambda' \subseteq \lambda \) gives rise to a proper morphism \( Y \to Y' \), which is an isomorphism in codimension one. As \( Y \) and \( Y' \) are normal surfaces, we obtain \( Y \cong Y' \), which contradicts the fact that the semiample cones of \( Y \) and \( Y' \) are of different dimension.

The verification of ‘(ii) \( \Rightarrow \) (i)’ runs as in the preceding proof; this time one uses the finitely generated subalgebra

\[ S(X) := \bigoplus_{w \in \text{Mov}(X)} \mathcal{R}(X)_w = \bigoplus_{w \in \text{SAmple}(X)} \mathcal{R}(X)_w. \]

Moreover, by Theorem 2.3, there is a small birational map \( X \to X' \) with \( X' \) projective and \( \mathbb{Q} \)-factorial. As \( X \) and \( X' \) are complete surfaces, this map already defines an isomorphism.

In the case of a \( \mathbb{Q} \)-factorial surface \( X \), we obtain the following simpler characterization involving the cone \( \text{Nef}(X) \subseteq \text{Cl}_{\mathbb{Q}}(X) \) of numerically effective divisor classes; note that the implication ‘(ii) \( \Rightarrow \) (i)’ was obtained for smooth surfaces in [GM05, Corollary 1].

**Corollary 2.6.** Let \( X \) be a \( \mathbb{Q} \)-factorial projective surface with finitely generated divisor class group \( \text{Cl}(X) \). Then the following statements are equivalent.

(i) The Cox ring \( \mathcal{R}(X) \) is finitely generated.

(ii) The effective cone \( \text{Eff}(X) \subseteq \text{Cl}_{\mathbb{Q}}(X) \) is polyhedral and \( \text{Nef}(X) = \text{SAmple}(X) \) holds.

**Proof of Corollary 2.6.** If (i) holds, then we infer from [BH07, Corollary 7.4] that the semiample cone and the nef cone of \( X \) coincide. Now suppose that (ii) holds. From

\[ \text{SAmple}(X) \subseteq \text{Mov}(X) \subseteq \text{Nef}(X), \]

we then conclude that \( \text{Mov}(X) = \text{Nef}(X) \). Moreover, since \( \text{Eff}(X) \) is polyhedral, \( \text{Nef}(X) \) is given by a finite number of inequalities and hence is also polyhedral. Thus, we can apply Theorem 2.5.\( \square \)

We turn to K3 surfaces \( X \). Recall that, by definition, \( X \) is a smooth complete complex surface with \( b_1(X) = 0 \) and trivial canonical class. We always assume a K3 surface \( X \) to be algebraic. As a sublattice of \( H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22} \), the divisor class group \( \text{Cl}(X) \) is finitely generated and free. In particular, we can define a Cox ring \( \mathcal{R}(X) \) as above. Our first result characterizes finite generation of \( \mathcal{R}(X) \).

**Theorem 2.7.** For any complex algebraic K3 surface \( X \), the following statements are equivalent.

(i) The Cox ring \( \mathcal{R}(X) \) is finitely generated.

(ii) \( \text{Eff}(X) \) is polyhedral.

**Lemma 2.8.** Let \( X \) be a K3 surface and \( D \) be a non-principal divisor on \( X \). If we have \( h^0(D) = 1 \), then \( D^2 < 0 \) holds.

**Proof.** Since the canonical divisor of \( X \) is principal, Serre’s duality theorem gives us \( h^2(D) = h^0(-D) = 0 \). The Riemann–Roch theorem then yields \( 1 \geq D^2/2 + 2 \). The assertion follows. \( \square \)

**Proof of Theorem 2.7.** Only for ‘(ii) \( \Rightarrow \) (i)’ there is something to show. So, assume that \( \text{Eff}(X) \) is polyhedral. Then, by Corollary 2.6, we have to show that for every numerically effective divisor \( D \) on \( X \), some positive multiple is semiample.

By a result of Kleiman, the class \([D] \in \text{Cl}_{\mathbb{Q}}(X)\) lies in the closure of the cone of ample divisor classes. Since any ample class is effective and \( \text{Eff}(X) \) as a polyhedral cone is closed in \( \text{Cl}_{\mathbb{Q}}(X) \), we obtain \([D] \in \text{Eff}(X)\). Thus, we may assume that \( D \) is non-negative.
Since $D$ is numerically effective, we have $D^2 \geq 0$. If $D^2 > 0$ holds, then [May72, Corollary, p. 11] tells us that the linear system $|3D|$ is base point free, i.e. that $3D$ is semiample. If we have $D^2 = 0$, then we write $D = D_0 + D_1$, where $D_0$ denotes the fixed part of $D$ and $D_0, D_1$ are non-negative. Then we have

$$0 = D^2 = D \cdot D_0 + D \cdot D_1.$$ 

Since $D_0$ and $D_1$ are non-negative, we conclude that $D \cdot D_0 = 0$ and $D \cdot D_1 = 0$. We show that $D_0 = 0$ must hold. Otherwise, using Lemma 2.8, we obtain

$$D_1^2 = (D - D_0)^2 < 0.$$ 

On the other hand, $D_1$ has no fixed components. Thus, according to [Sai74, Corollary 3.2], the divisor $D_1$ is base point free and hence numerically effective, a contradiction. Thus, we see that $D = D_1$ holds, and thus $D$ is semiample.

**Corollary 2.9.** Let $X$ be a K3 surface. If the cone $\text{Eff}(X) \subseteq \text{Cl}_Q(X)$ of effective divisor classes is polyhedral, then the cone $\text{SAmple}(X) \subseteq \text{Cl}_Q(X)$ of semiample divisor classes is also polyhedral.

For Enriques surfaces, i.e. smooth projective surfaces $X$ with $q(X) = 0$ and $2K_X$ trivial but $K_X$ non-trivial, we obtain the following analogue of Theorem 2.7.

**Theorem 2.10.** For any Enriques surface $X$, the following statements are equivalent.

(i) The Cox ring $\mathcal{R}(X)$ is finitely generated.

(ii) $\text{Eff}(X)$ is polyhedral.

**Proof.** Only for ‘(ii) $\Rightarrow$ (i)’ there is something to show. So, assume that $\text{Eff}(X)$ is polyhedral. Then, by Corollary 2.6, we have to show that for every given numerically effective divisor $D$ on $X$, some positive multiple is semiample. Since $\text{Eff}(X)$ is polyhedral, we obtain $\text{Nef}(X) \subseteq \text{Eff}(X)$ and hence we may assume that $D$ is non-negative.

Let $\pi: S \to X$ be the universal covering. Then $S$ is a K3 surface and $\pi$ is an unramified double covering. The pull-back $\pi^*D$ on $S$ is effective and numerically effective. As in the proof of Theorem 2.7, we see that some positive multiple $\pi^*nD$ is semiample. From [BHPV04, Lemma 17.2], we infer that

$$H^0(S, \pi^*nD) = \pi^*H^0(X, nD) + \pi^*H^0(X, nD + K_X).$$

If $x \in X$ is a base point of $nD$, then there is a $g \in H^0(X, nD + K_X)$ such that $g(x) \neq 0$ holds; otherwise $\pi^{-1}(x)$ would be in the base locus of $\pi^*nD$, which is a contradiction. Since $2K_X$ is trivial, we deduce that $2nD$ has no base points.

We conclude the section with recalling some classical statements on algebraic K3 surfaces $X$ characterizing the case of a polyhedral effective cone and thus providing further criteria for finite generation of the Cox ring. Consider the lattice $\text{Cl}(X) = \text{Pic}(X)$ with the intersection pairing; denote by $O(\text{Cl}(X))$ the group of its isometries and by $W(\text{Cl}(X))$ the Weyl group, i.e. the subgroup generated by reflections with respect to $\delta \in \text{Cl}(X)$ with $\delta^2 = -2$.

**Theorem 2.11.** See [Kov94, Theorem 2, Remark 7.2] and [PS71, § 7, Corollary]. For any algebraic K3 surface $X$, the following statements are equivalent.

(i) The cone $\text{Eff}(X) \subseteq \text{Cl}_Q(X)$ is polyhedral.

(ii) The set $O(\text{Cl}(X))/W(\text{Cl}(X))$ is finite.

(iii) The automorphism group $\text{Aut}(X)$ is finite.
Moreover, if the Picard number is at least three, then (i) is equivalent to the property that $X$ contains only finitely many smooth rational curves. In this case, the classes of such curves generate the effective cone.

The hyperbolic lattices satisfying (ii) have been classified in [PS71] and a series of papers by Nikulin; see [Nik79, Nik83, Nik85]. In particular, it has been proved that there are only finitely many of them having rank at least three. The results of these papers, together with Theorem 2.11, give the following.

**Theorem 2.12.** See [Nik79, Nik83, Nik85]. Let $X$ be an algebraic K3 surface with Picard number $\varrho(X)$.

(i) Suppose that $\varrho(X) = 2$ holds. Then $\text{Eff}(X)$ is polyhedral if and only if $\text{Cl}(X)$ contains a class of self intersection 0 or $-2$.

(ii) Suppose that $\varrho(X) \geq 3$ holds. Then $\text{Eff}(X)$ is polyhedral if and only if $\text{Cl}(X)$ belongs to a finite list of hyperbolic lattices. The following table gives the number $n$ of these lattices for any Picard number.

<table>
<thead>
<tr>
<th>$\varrho(X)$</th>
<th>3</th>
<th>4</th>
<th>5–6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11–12</th>
<th>13–14</th>
<th>15–19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>27</td>
<td>17</td>
<td>10</td>
<td>9</td>
<td>12</td>
<td>10</td>
<td>9</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The following statement is a consequence of the results mentioned above or of [Kov94, Theorem 2].

**Proposition 2.13.** Let $X$ be an algebraic K3 surface such that $\text{Eff}(X)$ is polyhedral. Then the generators of $\text{Eff}(X)$ are described in the following table.

<table>
<thead>
<tr>
<th>$\varrho(X)$</th>
<th>$\text{Eff}(X)$</th>
<th>Type of generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{Q}^+ [H]$</td>
<td>Ample divisor</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{Q}^+ [E_1] + \mathbb{Q}^+ [E_2]$</td>
<td>$(-2)$- or (0)-curves</td>
</tr>
<tr>
<td>3–19</td>
<td>$\sum \mathbb{Q}^+ [E_i]$</td>
<td>$(-2)$-curves</td>
</tr>
</tbody>
</table>

Note that for $\varrho(X) = 1$, the Cox ring of $X$ coincides with its usual homogeneous coordinate ring, whose generators have been studied in [Sai74].

### 3. K3 surfaces of Picard number two

We consider (complex algebraic) K3 surfaces $X$ with divisor class group $\text{Cl}(X) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where $w_i^2 \in \{0, -2\}$ and, as we may assume then, $w_1 \cdot w_2 \geq 1$ hold; recall from [Mor84, Corollary 2.9(i)] that any even lattice of rank two with signature $(1, 1)$ is the Picard lattice of an algebraic K3 surface. According to Theorem 2.7 and the characterization of $\text{Eff}(X)$ being polyhedral provided in Theorem 2.12, such surfaces $X$ have a finitely generated Cox ring $\mathcal{R}(X)$. We investigate the possible degrees of generators and relations for $\mathcal{R}(X)$. An explicit computation of $\mathcal{R}(X)$ for the cases $w_1 \cdot w_2 = 1, 2$ is given in § 6. A first observation concerns the effective cone.

**Proposition 3.1.** Let $X$ be a K3 surface with $\text{Cl}(X) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where $w_1, w_2$ are effective such that $w_i^2 \in \{0, -2\}$ and $w_1 \cdot w_2 \geq 2$ hold. Then $w_1$ and $w_2$ generate $\text{Eff}(X)$ as a cone.

**Proof.** Suppose that $\text{cone}(w_1, w_2) \subseteq \text{Eff}(X)$ holds. Then we may assume that $w_1$ does not lie on the boundary of $\text{Eff}(X)$. Thus, one of the generators of $\text{Eff}(X)$ is of the form $w = aw_1 - bw_2$ for
some $a, b \in \mathbb{N}$, where $a > 0$. Proposition 2.13 gives
\[ w^2 = a^2w_1^2 + b^2w_2^2 - 2ab w_1 \cdot w_2 \in \{0, -2\}. \]
This can only be realized for $b = 0$, because we assumed that $w_i^2 \in \{0, -2\}$ and $w_1 \cdot w_2 \geq 2$. Thus, 
\[ w = aw_1 \text{ holds and, consequently, } w_1 \text{ lies on the boundary of } \text{Eff}(X); \] a contradiction. \hfill \Box

Note that the assumption of $w_1$ and $w_2$ being effective in Proposition 3.1 can always be achieved: the Riemann–Roch theorem and $w_i^2 \in \{0, -2\}$ show that either $w_i$ or $-w_i$ is effective.

Our next result settles the case $w_i^2 = 0$ and $w_1 \cdot w_2 \geq 3$. In order to state it, we first have to fix our usage. Consider any finitely generated $\mathbb{C}$-algebra $R$, graded by a lattice $K$. We say that a system of homogeneous generators $f_1, \ldots, f_r$ of $R$ is minimal if no $f_i$ can be expressed as a polynomial in the remaining $f_j$. Moreover, we say that $R$ has a generator in degree $w \in K$ if any minimal system of generators for $R$ contains a non-trivial element of $R_w$. Given a system $f_1, \ldots, f_r$ of generators, we have the surjection
\[ \mathbb{C}[T_1, \ldots, T_r] \twoheadrightarrow R, \quad T_i \mapsto f_i. \]
The ideal of relations determined by $f_1, \ldots, f_r$ is the kernel $I \subseteq \mathbb{C}[T_1, \ldots, T_r]$ of this map; it is homogeneous with respect to the $K$-grading of $\mathbb{C}[T_1, \ldots, T_r]$ defined by $\deg(T_i) := \deg(f_i)$. By a minimal ideal of relations, we mean the ideal of relations determined by a minimal system of generators.

**Theorem 3.2.** Let $X$ be a $K3$ surface with $\Cl(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where $w_1$, $w_2$ are effective, and intersection form given by $w_1^2 = w_2^2 = 0$ and $w_1 \cdot w_2 = k \geq 3$.

(i) The semialample cone of $X$ coincides with its effective cone.

(ii) The Cox ring $\mathcal{R}(X)$ is generated in degrees $w_1, w_2, w_1 + w_2$, and one has
\[ \dim(\mathcal{R}(X)_{w_1}) = 2, \quad \dim(\mathcal{R}(X)_{w_1+w_2}) = k + 2. \]
Moreover, any minimal system of generators of $\mathcal{R}(X)$ has $k + 2$ members.

(iii) For $k = 3$, the Cox ring $\mathcal{R}(X)$ is of the form $\mathbb{C}[T_1, \ldots, T_5]/\langle f \rangle$ and the degrees of the generators and the relation are given by
\[ \deg(T_1) = \deg(T_2) = w_1, \quad \deg(T_4) = \deg(T_5) = w_2, \]
\[ \deg(T_3) = w_1 + w_2, \quad \deg(f) = 3w_1 + 3w_2. \]

(iv) For $k \geq 4$, any minimal ideal $\mathcal{I}(X)$ of relations of $\mathcal{R}(X)$ is generated in degree $2w_1 + 2w_2$, and we have
\[ \dim(\mathcal{I}(X)_{2w_1+2w_2}) = \frac{k(k-3)}{2}. \]
Note that for $k = 3, 4$ the Cox ring of $X$ is a complete intersection, while for $k \geq 5$ this no longer holds. Before giving the proof of the above theorem, we briefly provide the necessary ingredients.

**Lemma 3.3.** Let $X$ be a smooth surface, assume that $D, D_1, D_2 \in \text{WDiv}(X)$ satisfy $D_1 \cdot D_2 = 0$ and $h^1(D - D_1 - D_2) = 0$, and let $0 \neq f_i \in H^0(D_i)$ be such that $\text{div}(f_1) + D_1$ and $\text{div}(f_2) + D_2$ have no common components. Then one has a surjection
\[ H^0(D-D_1) \oplus H^0(D-D_2) \twoheadrightarrow H^0(D), \]
\[ (g_1, g_2) \mapsto g_1f_1 + g_2f_2. \]

974
ON COX RINGS OF K3 SURFACES

Proof. First note that due to the assumptions, the assignments \( \iota: h \mapsto (hf_2, -hf_1) \) and \( \varphi: (h_1, h_2) \mapsto h_1f_1 + h_2f_2 \) give rise to an exact sequence of sheaves

\[
0 \longrightarrow \mathcal{O}_X(-D_1 - D_2) \xrightarrow{\iota} \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0.
\]

Tensoring this sequence with \( \mathcal{O}_X(D) \) and looking at the associated cohomology sequence, we obtain the assertion. \( \Box \)

PROPOSITION 3.4. Let \( X \) be a K3 surface, \( w \in \text{Cl}(X) \) be the class of a smooth irreducible curve \( D \subseteq X \) of genus \( g \) and consider the Veronese algebra

\[
\mathcal{R}(X, w) := \bigoplus_{n \in \mathbb{N}} \mathcal{R}(X)_{nw}.
\]

Then the algebra \( \mathcal{R}(X, w) \) is generated in degree one if \( D \) is not hyperelliptic or \( g \leq 1 \), in degrees one and three if \( g = 2 \) and in degrees one and two if \( D \) is hyperelliptic of genus \( g \geq 3 \).

Proof. If \( g = 0 \) holds, then \( \mathcal{R}(X, w) = \mathbb{C}[s] \) with \( s \in H^0(D) \), since \( D \) is irreducible with negative self intersection. Thus, \( \mathcal{R}(X, w) \) is generated in degree one.

For non-rational \( D \), the canonical algebra \( \oplus H^0(D, nK_D) \) is generated in degree one if \( D \) is not hyperelliptic or \( g = 1 \), in degrees one and three if \( g = 2 \) and in degrees one and two if \( D \) is hyperelliptic and \( g \geq 3 \); see [ACGH85, p. 117]. By the adjunction formula we have \( \mathcal{O}_X(D)|_D \cong K_D \). Thus, we obtain the exact sequence

\[
0 \longrightarrow H^0(X, (n-1)D) \longrightarrow H^0(X, nD) \longrightarrow H^0(D, nK_D) \longrightarrow 0,
\]

where the last zero is due to the Kawamata–Viehweg vanishing theorem. This gives the assertion. \( \Box \)

In order to prove Theorem 3.2(iii), we use the techniques introduced in [LV09]. We say that a degree \( w \in R \) is not essential for a minimal ideal \( I \) of relations of a \( K \)-graded algebra \( R \) if no minimal system of homogeneous generators of \( I \) has members of degree \( w \).

THEOREM 3.5. See [LV09]. Let \( f_1, \ldots, f_r \in \mathcal{R}(X) \) be a minimal system of generators for the Cox ring of a surface \( X \) and set \( w_i := \deg(f_i) \in \text{Cl}(X) \). Consider the maps

\[
\varphi_{w,i}: \mathcal{R}(X)_{w-w_1-w_i} \oplus \mathcal{R}(X)_{w-w_2-w_i} \rightarrow \mathcal{R}(X)_{w-w_i},
\]

\[
(g_1, g_2) \mapsto g_1f_1 + g_2f_2,
\]

where \( w \in \text{Cl}(X) \) and \( i = 3, \ldots, r \). If \( w_1 \cdot w_2 = 0 \) holds and \( \varphi_{w,i} \) is surjective for \( i = 3, \ldots, r \), then \( w \) is not essential for the ideal of relations arising from \( f_1, \ldots, f_r \).

Proof of Theorem 3.2. Let \( D_i \in \text{WDiv}(X) \) represent \( w_i \in \text{Cl}(X) \). Then \( D_i^2 = 0 \) implies that the complete linear system of \( D_i \) defines a fibration, which in turn gives \( h^0(w_i) = 2 \). In particular, we have bases \( (f_{i1}, f_{i2}) \) for \( \mathcal{R}(X)_{w_i} \), where \( i = 1, 2 \). Moreover, applying the Riemann–Roch theorem yields \( h^1(w_i) = 0 \).

By Proposition 3.1, the classes \( w_1 \) and \( w_2 \) generate the effective cone. Moreover, \( h^0(w_i) = 2 \) and \( w_i^2 = 0 \) show that \( w_i \) is semiample, i.e. we have \( \text{Eff}(X) = \text{SAmp}(X) \). Consequently, any divisor class \( w = aw_1 + bw_2 \) with \( a, b > 0 \) is ample and the Kawamata–Viehweg vanishing theorem gives \( h^1(w) = 0 \).

We show now that \( \mathcal{R}(X) \) is generated in degrees \( w_1, w_2 \) and \( w_1 + w_2 \). Consider a class \( w = aw_1 + bw_2 \). If \( a \geq 3 \) and \( b \geq 1 \) or \( (a, b) = (2, 1) \) holds, then we have \( h^1(w - 2w_1) = 0 \).
Thus, Lemma 3.3 provides a surjective map
\[ \varphi: \mathcal{R}(X)_{w-w_1} \oplus \mathcal{R}(X)_{w-w_2} \rightarrow \mathcal{R}(X)_w, \quad (g_1, g_2) \mapsto g_1f_{11} + g_2f_{12}. \]
If \( b = 0 \) holds, then the complete linear system of any representative of \( w \) is composed with a pencil. This implies again surjectivity of the above map \( \varphi \).

Iterating this procedure, we see that for any \( w = aw_1 + bw_2 \) with \( a \geq 3 \) and \( b \geq 1 \) or \( (a, b) = (2, 1) \) or \( b = 0 \), the elements of \( \mathcal{R}(X)_w \) are polynomials in \( f_{11}, f_{12} \) and the elements of \( \mathcal{R}(X)_u \), where
\[ u = 2w_1 + bw_2 \text{ if } b \geq 2, \quad u = w_1 + bw_2 \text{ if } b = 0, 1. \]
Interchanging the roles of \( a \) and \( b \), in this reasoning, we finally see that any element of \( \mathcal{R}(X)_w \) is a polynomial in \( f_{11}, f_{12}, f_{21}, f_{22} \) and elements of \( \mathcal{R}(X)_{nu} \), where \( u := w_1 + w_2 \) and \( n \leq 2 \).

Thus, we are left with describing the elements of \( \mathcal{R}(X)_{nu} \), where \( u := w_1 + w_2 \). Observe that no complete linear system on \( X \) has fixed components, because, by Lemma 2.8 and the adjunction formula, any such component would be a \((-2)\)-curve and, by our assumptions, there are no classes of self intersection \(-2\) in \( \text{Cl}(X) \). Moreover, note that \( w_1 \) and \( w_2 \) are the only classes of elliptic curves in \( \text{Cl}(X) \).

It follows that \( u = w_1 + w_2 \) is represented by a smooth irreducible curve \( D \subseteq X \) of genus \( u^2/2 + 1 > 3 \). Since \( u^2 \geq 6 \) holds, \( u \) is a primitive class in \( \text{Cl}(X) \) and we have \( u \cdot w_1 \geq 3 \); the curve \( D \) is not hyperelliptic, see [Sai74, Theorem 5.2]. According to Proposition 3.4, the elements of \( \mathcal{R}(X)_{nu} \) are polynomials in those of \( \mathcal{R}(X)_u \).

Thus, we obtained that \( \mathcal{R}(X) \) is generated in the degrees \( w_1, w_2 \) and \( u := w_1 + w_2 \). Moreover, the Riemann–Roch theorem gives us
\[ \dim(\mathcal{R}(X)_{w_1}) = 2, \quad \dim(\mathcal{R}(X)_{w_1+w_2}) = k + 2. \]

We now turn to the relations. First note that any minimal system of generators must comprise a basis \( (f_{11}, f_{12}) \) of \( \mathcal{R}(X)_{w_1} \) and a basis \( (f_{21}, f_{22}) \) of \( \mathcal{R}(X)_{w_2} \). Now, consider any degree \( w = aw_1 + bw_2 \). If \( a \geq 4 \) and \( b \geq 2 \) or \( (a, b) = (3, 2) \) holds, then we have
\[ h^1(w - 2w_1 - w_2) = h^1(w - 3w_1 - w_2) = 0. \]
Thus, taking \( f_1 = f_{11} \) and \( f_2 = f_{12} \) in Theorem 3.5 and using Lemma 3.3, we see that \( w \) is not essential for any minimal ideal of relations of \( \mathcal{R}(X) \). If \( b = 1 \) holds, then
\[ \mathcal{R}(X)_{(c-1)w_1} \oplus \mathcal{R}(X)_{(c-1)w_2} \rightarrow \mathcal{R}(X)_{cw_1}, \quad (g_1, g_2) \mapsto g_1f_{11} + g_2f_{12} \]
is surjective for \( c = a, a - 1 \), because \( \mathcal{R}(X) \) is generated in degrees \( w_1, w_2 \) and \( w_1 + w_2 \). Thus, Theorem 3.5 shows that \( w \) is not essential for \( b = 1 \). Eventually, there are no relations of degree \( w \) for \( b = 0 \). In fact, then \( \mathcal{R}(X)_w \) is generated by \( f_{11} \) and \( f_{12} \) and hence any such relation defines a relation among \( f_{11} \) and \( f_{12} \), which contradicts the fact that \( f_{11}, f_{12} \) define a surjection \( X \rightarrow \mathbb{P}^1 \).

Exchanging the roles of \( w_1 \) and \( w_2 \) in this consideration, we obtain that essential relations can only occur in degrees \( 2u \) and \( 3u \), where \( u = w_1 + w_2 \).

In the case \( k = 3 \), the statements proven so far give that any minimal system of generators has five members and their degrees are \( w_1, w_2, w_3 \) and \( u \). Hence, there must be exactly one relation in \( \mathcal{R}(X) \). The degree of this relation minus the sum of the degrees of the generators gives the canonical class, see [BH07, Proposition 8.5], and hence vanishes. Thus, our relation must have degree \( 3u \).
Finally, let $k \geq 4$. As observed before, $\mathcal{R}(X)_{nu}$ is generated by $\mathcal{R}(X)_u$. Hence, any relation in degree $nu$ is also a relation of

$$
\bigoplus_{n \in \mathbb{N}} \mathcal{R}(X)_{nu}.
$$

Since $u \cdot w_i > 3$ holds and $X$ does not contain smooth rational curves, [Sai74, Theorem 7.2] tells us that the ideal of relations of this algebra is generated in degree two. Thus, there are only essential relations of degree $2u$ in $\mathcal{R}(X)$.

In order to determine the dimension of $\mathcal{I}(X)_{2u}$ for a minimal ideal of relations $\mathcal{I}(X)$, note that we have the four generators $f_{ij}$, where $1 \leq i, j \leq 2$, of degree $w_i$, and $k - 2$ generators of degree $u = w_1 + w_2$. Using the Riemann–Roch theorem, we obtain that $\mathcal{R}(X)_{2u}$ is of dimension $4k + 2$. Thus, denoting by $\mathbb{C}[T]$ the polynomial ring in the above generators and by $V \subseteq \mathcal{R}(X)_u$ the vector space spanned by the $k - 2$ generators of degree $u$, we obtain

$$
\dim(\mathcal{I}(X)_{2u}) = \dim(\mathbb{C}[T]_{2u}) - \dim(\mathcal{R}(X)_{2u})
= \dim(\text{Sym}^2 V) + 4(k - 2) + 9
= \frac{k(k - 3)}{2}.
$$

We now turn to the cases $w_1^2 = -2$ and $w_2^2 = 0, -2$. In contrast to the previous cases, the number of degrees occurring in a minimal system of generators for the Cox ring becomes arbitrarily large when $w_1 \cdot w_2$ increases.

**Proposition 3.6.** Let $X$ be a K3 surface with $\text{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and intersection form given by $w_1^2 = -2$, $w_2^2 = 0$ and $w_1 \cdot w_2 = k \in \mathbb{N}$.

(i) The semiample cone $\text{SAample}(X)$ of $X$ is generated by the classes $kw_1 + 2w_2$ and $w_2$.

(ii) The Cox ring $\mathcal{R}(X)$ has generators in degrees $w_1$ and $aw_1 + w_2$, where $0 \leq a \leq \lfloor k/2 \rfloor$.

(iii) If $k > 1$ holds and $k$ is odd, then the Cox ring $\mathcal{R}(X)$ has, in addition to those of (ii),

- generators in degree $kw_1 + 2w_2$.

**Proof.** To verify (i), note that $(aw_1 + bw_2) \cdot w_1 = -2a + kb$ and $(aw_1 + bw_2) \cdot w_2 = ka$ hold. These intersection products are both non-negative if $0 \leq a \leq kb/2$ holds. So, the nef cone of $X$ is generated by the classes $kw_1 + 2w_2$ and $w_2$. Since $\text{SAample}(X)$ is polyhedral, the claim follows.

We prove (ii). By Proposition 3.1, the classes $w_1$ and $w_2$ generate the effective cone. Thus, $\mathcal{R}(X)$ has generators in the degrees $w_1$ and $w_2$. Note that $h^0(w_1) = 1$ and, fixing a generator $f_1 \in \mathcal{R}(X)_{w_1}$, we obtain $\mathcal{R}(X)_{nu} = \mathbb{C}f_1^n$. Moreover, up to a constant, $f_1$ occurs in any minimal system of generators of $\mathcal{R}(X)$; we fix such a system $f_1, \ldots, f_r$.

We now show that $\mathcal{R}(X)$ has generators in degree $aw_1 + w_2$ for any $1 \leq a \leq \lfloor k/2 \rfloor$. By assertion (i), the class $aw_1 + w_2$ is big and nef for $1 \leq a \leq \lfloor k/2 \rfloor$. Using the Riemann–Roch theorem and the Kawamata–Viehweg vanishing theorem, we obtain

$$
h^0((a - 1)w_1 + w_2) < h^0(aw_1 + w_2) \text{ for } 1 < a \leq \lfloor k/2 \rfloor.
$$

This implies that there exists an $f \in \mathcal{R}(X)_{aw_1 + w_2}$, which is not a multiple of $f_1 \in \mathcal{R}(X)_{w_1}$. The same holds for $a = 1$, because then we have $(k = 1$ implies that $a = 0)$

$$
h^0(w_2) = 2, \quad h^0(w_1 + w_2) = k + 1 > 2.
$$

Suppose that every monomial $m \in \mathcal{R}(X)_{aw_1 + w_2}$ in the $f_i$ is a product $m = m_1m_2$ of non-constant $m_i$. Then $m_1$ or $m_2$ belongs to $\mathcal{R}(X)_{bw_1} = \mathbb{C}f_1^b$, where $1 \leq b \leq a$. Then
every \( f \in \mathcal{R}(X)_{aw_1 + w_2} \) is a multiple of \( f_1 \), contradicting the previous statement. Hence, some \( f_i \), where \( i = 2, \ldots, r \), has degree \( aw_1 + w_2 \).

We turn to (iii). Then \( kw_1 + 2w_2 \) is big and nef. Reasoning as before, we obtain that there is an \( f \in \mathcal{R}(X)_{kw_1 + 2w_2} \), which is not a multiple of \( f_1 \). Suppose that every monomial \( m \in \mathcal{R}(X)_{kw_1 + 2w_2} \) in the \( f_1 \) is a product \( m = m_1m_2 \) of non-constant \( m_i \). Then \( m_1 \) or \( m_2 \) belongs to \( \mathcal{R}(X)_{bw_1 + cw_2} \), where \( b/c \geq [k/2] + 1 \). Since \( (bw_1 + cw_2) \cdot w_1 = -2b + kc < 0 \) holds, every element of \( \mathcal{R}(X)_{bw_1 + cw_2} \) is divisible by \( f_1 \); a contradiction. Hence, some \( f_i \), where \( i = 2, \ldots, r \), has degree \( kw_1 + 2w_2 \). \( \square \)

**Proposition 3.7.** Let \( X \) be a K3 surface with \( \text{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \) and intersection form given by \( w_1^2 = w_2^2 = -2 \) and \( w_1 \cdot w_2 = k \in \mathbb{N} \).

(i) We have \( k \geq 3 \) and the semiample cone \( \text{SAmple}(X) \) is generated by the classes \( kw_1 + 2w_2 \) and \( 2w_1 + kw_2 \).

(ii) The Cox ring \( \mathcal{R}(X) \) has generators in degrees \( aw_1 + w_2 \) and \( w_1 + aw_2 \), where \( 0 \leq a \leq [k/2] \).

(iii) If \( k > 1 \) holds and \( k \) is odd, then the Cox ring \( \mathcal{R}(X) \) has, in addition to those of (ii), generators in degrees \( kw_1 + 2w_2 \) and \( 2w_1 + kw_2 \).

**Proof.** By the Hodge index theorem, \( \text{Cl}(X) \) has signature \((1,1)\), which implies that \( k \geq 3 \). Determining the semiample cone runs as in the proof of Proposition 3.6.

As to the remaining statements, note that the semigroup \( \text{SAmple}(X) \cap \mathbb{Z}^2 \) is generated by \( aw_1 + w_2 \) and \( w_1 + aw_2 \) with \( 0 \leq a \leq [k/2] \) if \( k \) is even, and by the same classes plus the two extremal rays if \( k \) is odd. Reasoning as in the proof of Proposition 3.6, we obtain

\[
h^0(aw_1 + w_2) < h^0(aw_1 + w_2)
\]

whenever \( 0 \leq a \leq [k/2] \) holds. This formula also holds when \( w_1 \) and \( w_2 \) are exchanged. Now the same arguments as used in the proof of Proposition 3.6 give the assertion. \( \square \)

**Example 3.8.** Let \( X \) be a K3 surface with \( \text{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \) and intersection form given by \( w_1^2 = w_2^2 = -2 \) and \( w_1 \cdot w_2 = 3 \). Then the Cox ring \( \mathcal{R}(X) \) has generators

\[f_{1,0}, f_{0,1}, f_{1,1}, g_{1,1}, f_{2,3}, f_{3,2}\]

in the corresponding degrees by Proposition 3.7 and its proof. A monomial basis of \( \text{Sym}^3 \mathcal{R}(X)_{w_1 + w_2} \), plus \( f_{2,3}f_{0,1} \) and \( f_{3,2}f_{1,0} \), give 12 linearly dependent elements of \( \mathcal{R}(X)_{3w_1 + 3w_2} \), since this space has dimension 11 by the Riemann–Roch theorem. This means that \( \mathcal{R}(X) \) has a relation in degree \( 3w_1 + 3w_2 \).

Similarly, a monomial basis of \( \text{Sym}^5 \mathcal{R}(X)_{w_1 + w_2} \), plus \( f_{2,3}f_{3,2} \) and the product of \( f_{2,3}f_{1,0} \) for a monomial basis of \( \text{Sym}^2 \mathcal{R}(X)_{w_1 + w_2} \), give 28 monomials. These are linearly dependent, since the dimension of \( \mathcal{R}(X)_{5w_1 + 5w_2} \) is 27 by the Riemann–Roch theorem. This means that \( \mathcal{R}(X) \) has a relation in degree \( 5w_1 + 5w_2 \).

We now give a geometric interpretation for generators and relations. The map \( \pi : X \to \mathbb{P}^2 \) associated with \( w_1 + w_2 \) is a double cover branched along a smooth plane sextic; see [Sai74]. Observe that \( f_{1,0}f_{0,1} = \pi^*(s) \) and \( f_{2,3}f_{3,2} = \pi^*(t) \), where \( s = 0 \) is a line and \( t = 0 \) is a quintic in \( \mathbb{P}^2 \). The second equality gives a relation in degree \( 5w_1 + 5w_2 \).

The assumptions \( w_i^2 \in \{0, -2\} \) made in Theorem 3.2 imply that the primitive generators of the effective cone form a basis of the divisor class group. However, the techniques of its proof
allow us as well to treat, for example, the following case, where the primitive generators of the effective cone span a sublattice of index two in the divisor class group.

**Proposition 3.9.** Let $X$ be a K3 surface with $\text{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and intersection form given by $w_1^2 = 4$, $w_2^2 = -4$ and $w_1 \cdot w_2 = 0$.

(i) The effective cone of $X$ is generated by $u_1 := w_1 + w_2$ and $u_2 := w_1 - w_2$.

(ii) The Cox ring $\mathcal{R}(X)$ is generated in degrees $u_1$, $u_2$ and $w_1$.

(iii) Any minimal ideal of relations of $\mathcal{R}(X)$ is generated in degree $2w_1$.

**Proof.** Note that we have $u_1^2 = u_2^2 = 0$. Thus, by the Riemann–Roch theorem, we can assume that $u_1$ and $u_2$ are effective. Moreover, Theorem 2.12 tells us that $\text{Eff}(X)$ is polyhedral; we denote by $v_1$ and $v_2$ its primitive generators. Then we have $v_i^2 \in 4\mathbb{Z}$ and thus Proposition 2.13 gives $v_i^2 = v_i^2 = 0$. Choosing presentations $u_i = a_i v_1 + b_i v_2$ with non-negative $a_i$, $b_i \in \mathbb{Q}$, we obtain

$$
8 = u_1 \cdot u_2 = (a_1 v_1 + b_1 v_2) \cdot (a_2 v_1 + b_2 v_2) = (a_1 b_2 + a_2 b_1) \cdot v_1 \cdot v_2,
$$

$$0 = u_i^2 = (a_i v_1 + b_i v_2)^2 = 2a_i b_i v_1 \cdot v_2.
$$

The first identity gives $v_1 \cdot v_2 \neq 0$ and, thus, the second one shows that $a_i b_i = 0$. As a consequence, we obtain $\{u_1, u_2\} = \{v_1, v_2\}$. This proves the first assertion.

As to the second one, note that any effective divisor class $w \in \text{Cl}(X)$ can be written as

$$w = au_1 + bu_2 + cw_1,$$

where $a, b \in \mathbb{N}$, $c = 0, 1$.

Observe that $u_1$ and $u_2$ are classes of elliptic curves. Moreover, we have $h^0(u_i) = 2$ and thus $\mathcal{R}(X)_{u_i}$ has a basis of the form $(f_{i1}, f_{i2})$.

We now proceed as in the proof of Theorem 3.2. If $a \geq 3$ and $b \geq 1$ hold, then $w - 2u_1$ is nef and big, and thus we have $h^1(w - 2u_1) = 0$. Since $u_1^2 = 0$ holds, Lemma 3.3 shows that the sections of $w$ are polynomials in $f_{i1}$, $f_{i2}$ and elements of $\mathcal{R}(X)_{w-u_1}$.

Iterating this procedure and interchanging the roles of $a$ and $b$, we reduce to the study of classes $w$ with $a + b \leq 4$. A case by case analysis now shows that $\mathcal{R}(X)$ is generated in degrees $u_1$, $u_2$ and $w_1$. In the following table we briefly provide the reason why $h^1(w - 2u_1) = 0$ holds, when $a \geq b$ and $w - 2u_1$ is not nef and big.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$h^1(w - 2w_1) = 0$ because</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\mathcal{R}(X, w_1)$ is 1-generated</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$h^1(-w_2) = 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\mathcal{R}(X, w_1)$ is 1-generated</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\mathcal{R}(X, u_1)$ is 1-generated</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>$h^1(u_2) = 0$</td>
</tr>
</tbody>
</table>

In a similar way, Theorem 3.5 and Lemma 3.3 imply that $w$ is not essential unless $a = b = 1$. Since $w_1$ is the class of a smooth irreducible curve of genus at most one, Proposition 3.4 yields that $\mathcal{R}(X)_{w_1}$ is generated in degree one. This implies that the monomials $f_{i1}f_{2k}$ are quadratic functions in the $f_i \in \mathcal{R}(X)_{w_1}$:

$$f_{i1}f_{2k} = q_{ik}(f_0, f_1, f_2, f_3),$$

where $q_{ik}$ is a homogeneous polynomial of degree two. This gives four independent relations in degree $(2, 0)$, as can be checked since $h^0(2w_1) = 10$ and the number of monomials of type $f_if_j$ and $f_{i1}f_{2k}$ is 14. \[\square\]
4. Cox rings and coverings

In this section, we investigate the effect of certain, e.g. cyclic, coverings \( \pi: X \to Y \) on the Cox ring. Among other things, we obtain that finite generation of the Cox ring is preserved, provided that \( \pi^*(\text{Cl}(Y)) \) is of finite index in \( \text{Cl}(X) \); see Proposition 4.6. In the whole section, we work over an algebraically closed field \( \mathbb{K} \) of characteristic zero. First, we make precise which type of coverings we will treat.

**Construction 4.1.** Let \( Y \) be a normal variety and \( D_1, \ldots, D_r \in \text{CaDiv}(Y) \) be a linearly independent collection of Cartier divisors. Denote by \( M^+ \subseteq \text{CaDiv}(Y) \) the semigroup generated by the divisors \( D_1, \ldots, D_r \) and set

\[
Y(D_1, \ldots, D_r) := \text{Spec}_Y(A), \quad A := \bigoplus_{D \in M^+} \mathcal{O}_Y(-D).
\]

Then the inclusion \( \mathcal{O}_Y \to A \) defines a morphism \( \alpha: Y(D_1, \ldots, D_r) \to Y \), which is a (split) vector bundle of rank \( r \) over \( Y \). Similarly, with \( n_1, \ldots, n_r \in \mathbb{Z}_{>0} \) and \( E_i := n_i D_i \), denote by \( N^+ \subseteq \text{CaDiv}(Y) \) the semigroup generated by \( E_1, \ldots, E_r \). Setting

\[
Y(E_1, \ldots, E_r) := \text{Spec}_Y(B), \quad B := \bigoplus_{E \in N^+} \mathcal{O}_Y(-E)
\]

gives a further (split) vector bundle \( \beta: Y(E_1, \ldots, E_r) \to Y \) of rank \( r \) over \( Y \). The inclusion \( B \subseteq A \) defines a morphism \( \kappa: Y(D_1, \ldots, D_r) \to Y(E_1, \ldots, E_r) \). Now, let \( \sigma: Y \to Y(E_1, \ldots, E_r) \) be a section such that all projections of \( \sigma \) to the factors \( Y(E_i) \) are non-trivial. Then we define

\[
X := \kappa^{-1}(\sigma(Y)) \subseteq Y(D_1, \ldots, D_r).
\]

Restricting \( \alpha \) gives a morphism \( \pi: X \to Y \), which we call an abelian covering of \( Y \). Note that \( \pi: X \to Y \) is the quotient for the action of the abelian group \( \mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z} \) on \( X \) defined by the inclusion \( B \subseteq A \) of graded algebras.

**Remark 4.2.** For a smooth variety \( Y \), Cartier divisors \( D \) and \( E := nD \) on \( Y \) and a section \( \sigma: Y \to Y(E) \) with non-trivial reduced divisor \( B \), Construction 4.1 gives a branched \( n \)-cyclic covering of \( Y \); see [BHPV04, §1.17].

In order to formulate our first result, we need the following pull-back construction for Weil divisors under an abelian covering \( \pi: X \to Y \) of normal varieties. Given \( D \in \text{WDiv}(Y) \), consider the restriction \( D' \) of \( D \) to the set \( Y' \subseteq Y \) of smooth points and take the usual pull-back \( \pi^*(D') \) on \( \pi^{-1}(Y') \). Since \( \pi \) is finite, the complement \( X \setminus \pi^{-1}(Y') \) is of codimension at least two in \( X \) and hence \( \pi^*(D') \) uniquely extends to a Weil divisor \( \pi^*(D) \) on \( X \).

**Proposition 4.3.** Let \( \pi: X \to Y \) be an abelian covering as in Construction 4.1, assume that \( X \) is normal and let \( K \subseteq \text{WDiv}(Y) \) be a subgroup containing \( D_1, \ldots, D_r \) of Construction 4.1. Set

\[
\mathcal{S}_Y := \bigoplus_{D \in K} \mathcal{O}_Y(D), \quad \mathcal{S}_X := \bigoplus_{D \in K} \mathcal{O}_X(\pi^*D).
\]
Then, setting $\deg(T_i) := D_i$ turns $S_Y[T_1, \ldots, T_r]$ into a $K$-graded sheaf of $O_Y$-algebras and there is a $K$-graded isomorphism of sheaves

$$\pi_*S_X \cong S_Y[T_1, \ldots, T_r]/\langle T_1^{n_1} - g_1, \ldots, T_r^{n_r} - g_r \rangle,$$

where $g_i \in \Gamma(Y, O(E_i))$ are sections such that the branch divisor $B$ of the covering $\pi: X \to Y$ is given as

$$B = \text{div}(g_1) + \cdots + \text{div}(g_r).$$

**Proof.** Note that for any open set $V \subseteq Y$ and its intersection $V' := V \cap Y'$ with the set $Y' \subseteq Y$ of smooth points, the sections of $S_Y$ over $V$ and $V'$ coincide and also the sections of $\pi_*S_X$ over $V$ and $V'$ coincide. Hence, we may assume that $K \subseteq \text{WDiv}(Y)$ consists of Cartier divisors.

A first step is to express the direct image $\pi_*S_X$ in terms of $\pi_*O_X$ and data living on $Y$. Using the projection formula, we obtain

$$\pi_*S_X = \pi_* \bigoplus_{D \in K} O_X(\pi^*D) \cong \bigoplus_{D \in K} O_Y(D) \otimes_{O_Y} \pi_*O_X \cong S_Y \otimes_{O_Y} \pi_*O_X. \quad (1)$$

Now we have to investigate $\pi_*O_X$. Denote by $q: \bar{Y} \to Y$ the torsor associated with $S_Y$, i.e. we have $\bar{Y} = \text{Spec}_Y(S_Y)$. Moreover, in Construction 4.1 we constructed the rank $r$ vector bundles

$$\alpha: Y(D_1, \ldots, D_r) \to Y, \quad \beta: Y(E_1, \ldots, E_r) \to Y.$$

Using the pull-back divisors $q^*(D_i)$ and $q^*(E_i)$, we obtain the respective pull-back bundles

$$\bar{\alpha}: \bar{Y}(q^*D_1, \ldots, q^*D_r) \to \bar{Y}, \quad \bar{\beta}: \bar{Y}(q^*E_1, \ldots, q^*E_r) \to \bar{Y}.$$

Set for short $Y(D) := Y(D_1, \ldots, D_r)$ and $\bar{Y}(q^*D) := \bar{Y}(q^*D_1, \ldots, q^*D_r)$. Similarly, define $Y(E)$ and $\bar{Y}(q^*E)$. Then we have a commutative diagram

$$\begin{array}{c}
\bar{X} \\
\downarrow \bar{\gamma} \\
\bar{Y}(q^*D) \\
\downarrow qD \\
X \end{array} \quad \begin{array}{c}
\bar{\kappa} \\
\downarrow \kappa \\
Y(D) \\
\downarrow \pi \\
X \end{array}$$

where $p, qD$ and $qE$ are the canonical morphisms, we set $\bar{X} := qD^{-1}(X)$ and $\bar{\sigma} := q^*\sigma$ is the pull-back section.

Recall that $q: \bar{Y} \to Y$ is the quotient for the free action of the torus $H := \text{Spec}(K[K])$ defined by the grading of $q_*O_{\bar{Y}} = S_Y$. Thus, $\bar{Y}(D)$ and $\bar{X}$ inherit free $H$-actions having $qD$ and $p$ as their respective quotients. Moreover, let $\bar{I}$ denote the ideal sheaf of $\bar{X}$ in $\bar{Y}(q^*D)$. Then $\bar{I}$ is homogeneous, and we have

$$\pi_*O_X \cong (\pi_*p_*O_{\bar{X}})_0 \cong (\pi_*p_*\bar{\gamma}^*(O_{\bar{Y}(q^*D)}/\bar{I}))_0 = (q_*\bar{\alpha}_*(O_{\bar{Y}(q^*D)}/\bar{I}))_0. \quad (2)$$

To proceed, we need a suitable trivialization of the bundle $\bar{\alpha}: \bar{Y}(q^*D) \to \bar{Y}$. For this, consider an open affine subset $V \subseteq Y$ such that on $V$ we have $D_i = \text{div}(h_{i,V})$ for $1 \leq i \leq r$.
This gives us sections
\[
\eta_{i,V} := \frac{h_{i,V}^{-1}}{D_{i,V}} \in \Gamma(V, \mathcal{O}_Y(D_i)) \subseteq \Gamma(q^{-1}(V), \mathcal{O}_{\tilde{Y}})_{D_i},
\]

\[
g_D(h_{i,V}) \in g_D^*(\Gamma(\alpha^{-1}(V), \mathcal{O}_Y(D))) \subseteq \Gamma(\alpha^{-1}(q^{-1}(V)), \mathcal{O}_X(q^*D))_0.
\]

Given another open affine subset \( W \subseteq Y \) such that on \( W \) we have \( D_i = \text{div}(h_{i,W}) \) for \( 1 \leq i \leq r \), we obtain \( V \cap W \) for the corresponding sections:

\[
\frac{\alpha^*(\eta_{i,W})}{\alpha^*(\eta_{i,V})} = \alpha^*(\eta_{i,W}) = g_D^*(\frac{h_{i,V}}{\eta_{i,V}}) = \frac{g_D^*(h_{i,V})}{g_D^*(\eta_{i,V})}.
\]

Covering \( Y \) with \( V \)'s as above, we obtain that the functions \( \alpha^*(\eta_{i,V}) \cdot g_D^*(h_{i,V}) \) living on \( \alpha^{-1}(q^{-1}(V)) \) glue together to a global regular function \( f_i \) of degree \( D_i \) on \( Y(q^*D_i) \) generating \( \mathcal{O}_Y(q^*D_i) \) over \( \mathcal{O}_Y \). Thus, the \( f_i \) define a trivialization,

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\iota}} & \tilde{Y}(q^*D) \\
\tilde{\alpha} \times f & \cong & \tilde{\beta} \times f^n \cong \Gamma(\eta, z) - (\tilde{g}, z^n) \\
\end{array}
\]

\[
\tilde{Y} \times \mathbb{K}^r \xrightarrow{\text{id} \times g} \tilde{Y} \times \mathbb{K}^r \xrightarrow{\tilde{\alpha} \times f} \tilde{Y}(q^*D) \xrightarrow{\tilde{\beta} \times f^n} \tilde{Y}
\]

where we write \( z \) for \( (z_1, \ldots, z_r) \) and \( z^n \) for \( (z_1^n, \ldots, z_r^n) \), etc. Since \( \tilde{\sigma} = \eta \cdot \sigma \) is \( H \)-equivariant, each component \( g_i \) of \( g \) is homogeneous with \( \deg(g_i) = E_i \). Note that the divisors \( \text{div}(g_i) \) describe the branch divisor, as claimed.

Denote by \( \tilde{J} \subseteq \mathcal{O}_Y[T_1, \ldots, T_r] \) the ideal sheaf of the image of \( \tilde{X} \) in \( \tilde{Y} \times \mathbb{K}^r \). Then, using \( \tilde{X} = \tilde{\kappa}^{-1}(\tilde{\sigma}(\tilde{Y})) \), we obtain

\[
\tilde{I} = (f_1^n - \tilde{\alpha}^* g_1, \ldots, f_r^n - \tilde{\alpha}^* g_r), \quad \tilde{J} = (T_1^n - g_1, \ldots, T_r^n - g_r).
\]

Thus, using the isomorphism \( q_! \alpha_* \mathcal{O}_Y(q^*D) \cong S_Y[T_1, \ldots, T_r] \) established by the above commutative diagram, we may continue Equations (2) as

\[
\pi_* \mathcal{O}_X \cong (S_Y[T_1, \ldots, T_r]/J)_0 \cong S_Y[T_1, \ldots, T_r]_0/\tilde{J}_0.
\]

The homogeneous ideal sheaf \( \tilde{J} \) is locally, over \( Y \), generated in degree zero in the sense that we have \( \tilde{I} = \tilde{\alpha}^* \mathcal{O}_Y \cdot \tilde{J}_0 \). The same holds for \( \tilde{J} \), and we obtain

\[
\pi_* \mathcal{S}_X \cong S_Y \otimes \mathcal{O}_Y \quad \pi_* \mathcal{O}_X \cong S_Y \otimes \mathcal{O}_Y \quad S_Y[T_1, \ldots, T_r]_0/\tilde{J}_0 \cong S_Y[T_1, \ldots, T_r]/\tilde{J}.
\]

**Proposition 4.4.** Consider a normal variety \( X \), a finitely generated subgroup \( K \subseteq \text{WDiv}(X) \) mapping onto \( \text{Cl}(X) \), a subgroup \( L \subseteq K \) and the algebras

\[
R := \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)), \quad A := \bigoplus_{D \in L} \Gamma(X, \mathcal{O}_X(D)).
\]

If the subgroup \( L \subseteq K \) is of finite index and the algebra \( A \) is finitely generated, then the algebra \( R \) is also finitely generated.

**Lemma 4.5.** Let \( X \) be a normal variety, \( K \) be a finitely generated abelian group, \( \mathcal{R} \) be a quasicoherent sheaf of normal \( K \)-graded \( \mathcal{O}_X \)-algebras and \( Z \) be its relative spectrum,

\[
\mathcal{R} = \bigoplus_{w \in K} \mathcal{R}_w, \quad Z := \text{Spec}_X(\mathcal{R}),
\]
where we assume $\mathcal{R}$ to be locally of finite type. Let $L \subseteq K$ be a subgroup of finite index and consider the associated Veronese subalgebra of the algebra of global sections

$$A := \bigoplus_{w \in L} \Gamma(X, \mathcal{R}_w) \subseteq \bigoplus_{w \in K} \Gamma(X, \mathcal{R}_w) =: R.$$ 

Suppose that there are homogeneous sections $f_1, \ldots, f_r \in A$ such that each $R_{f_i}$ is finitely generated, each $Z_{f_i} = Z \setminus V(f_i)$ is an affine variety and we have

$$Z = Z_{f_1} \cup \cdots \cup Z_{f_r}.$$ 

If $A$ is finitely generated and for $Y := \text{Spec}(A)$ the complement $Y \setminus (Y_{f_1} \cup \cdots \cup Y_{f_r})$ is of codimension at least two in $Y$, then $R$ is finitely generated.

Proof. First note that $Z$ is a variety with $\Gamma(Z, \mathcal{O}) = R$ and that $\Gamma(Z_{f_i}, \mathcal{O}) = R_{f_i}$ holds. Since each $R_{f_i}$ is finitely generated, we may construct a finitely generated $K$-graded subalgebra $S \subseteq R$ with

$$A \subseteq S, \quad S_{f_i} = R_{f_i} \text{ for } 1 \leq i \leq r.$$ 

Set $Z' := \text{Spec}(S)$. Then the inclusion $S \subseteq R$ defines a canonical morphism $\iota: Z \to Z'$. Moreover, $S$ is canonically graded by the factor group $K/L$ and hence $Z'$ comes with an action of the finite abelian group $G := \text{Spec}(\mathbb{K}[K/L])$. The inclusion $A \subseteq S$ defines the quotient map $\pi: Z' \to Y$ for the action of $G$. By construction, we have

$$Z_{f_i} = \iota^{-1}(Z'_{f_i}) \cong Z'_{f_i}, \quad Z'_{f_i} = \pi^{-1}(Y_{f_i}).$$ 

Consequently, $\iota: Z \to Z'$ is an open embedding, and we may regard $Z$ as a subset of $Z'$. By our assumptions, setting $V := Y_{f_1} \cup \cdots \cup Y_{f_r}$, we obtain $Z = \pi^{-1}(V)$. Since $\pi: Z' \to Y$ is a finite map, we can conclude that

$$\dim(Z \setminus Z) = \dim(Y \setminus V) \leq \dim(Y) - 2 = \dim(Z') - 2.$$ 

Let $Z'' \to Z'$ be the normalization. Since $Z$ is normal, we have $Z \subseteq Z''$. We conclude that $R = \Gamma(Z, \mathcal{O}) = \Gamma(Z'', \mathcal{O})$ holds, and thus $R$ is finitely generated.

Proof of Proposition 4.4. First note that the rings $R$ and $A$ do not change if we replace $X$ with the set of its smooth points. Thus, we may assume that $X$ is smooth. Then we obtain graded sheaves of normal $\mathcal{O}_X$-algebras

$$\mathcal{R} = \bigoplus_{w \in K} \mathcal{R}_w, \quad A = \bigoplus_{w \in L} \mathcal{R}_w,$$

which are locally of finite type. Our task is to verify the assumptions of Lemma 4.5 for $\mathcal{R}$ and $A = \Gamma(X, A)$. Setting $Z := \text{Spec}_X(\mathcal{R})$ and $\tilde{X} := \text{Spec}_X(A)$, we obtain normal varieties, and we have a commutative diagram of affine morphisms

$$\begin{array}{ccc}
Z & \xrightarrow{\kappa} & \tilde{X} \\
\downarrow p & & \downarrow q \\
X & \xrightarrow{q} & Y
\end{array}$$

where $\kappa$ is the quotient for the action of the finite abelian group $G := \text{Spec}(\mathbb{K}[K/L])$ on $Z$ defined by the canonical $(K/L)$-grading of $\mathcal{R}$. Moreover, we have an affine variety $Y := \text{Spec}(A)$, and there is a canonical morphism $\iota: \tilde{X} \to Y$.  

983
The Cox ring

Proposition 4.6. Let \( \pi : X \to Y \) be an abelian covering of normal varieties with finitely generated free divisor class groups such that \( \pi^*(\operatorname{Cl}(Y)) \) is of finite index in \( \operatorname{Cl}(X) \). Then the following statements are equivalent.

(i) The Cox ring \( \mathcal{R}(X) \) is a finitely generated \( K \)-algebra.

(ii) The Cox ring \( \mathcal{R}(Y) \) is a finitely generated \( K \)-algebra.

Proof. Let \( M \subseteq \operatorname{WDiv}(Y) \) and \( K \subseteq \operatorname{WDiv}(X) \) be subgroups mapping isomorphically to the respective divisor class groups \( \operatorname{Cl}(Y) \) and \( \operatorname{Cl}(X) \). Then the Cox rings are given as

\[
\mathcal{R}(Y) = \bigoplus_{E \in M} \Gamma(Y, \mathcal{O}_Y(E)), \quad \mathcal{R}(X) = \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)).
\]

Since \( \operatorname{Cl}(Y) \) is free and \( \pi : X \to Y \) is the quotient for a finite group action, the pull-back \( \pi^* : \operatorname{Cl}(Y) \to \operatorname{Cl}(X) \) is injective; see [Ful98, Example 1.7.6]. Consequently, there are a unique subgroup \( L \subseteq K \) and an isomorphism \( \pi^*(M) \to L \) inducing the identity on \( \pi^*(\operatorname{Cl}(Y)) \). By our assumption, \( L \) is of finite index in \( K \). Moreover, we have canonical identifications

\[
\mathcal{R}(Y) \subseteq \bigoplus_{E \in M} \Gamma(X, \mathcal{O}_X(\pi^*(E))) = S := \bigoplus_{D \in L} \Gamma(X, \mathcal{O}_X(D)) \subseteq \mathcal{R}(X).
\]

Suppose that \( \mathcal{R}(X) \) is finitely generated over \( K \). Then the Veronese subalgebra \( S \subseteq \mathcal{R}(X) \) is also finitely generated over \( K \). Moreover, by Proposition 4.3, the algebra \( S \) is a finite module over \( \mathcal{R}(Y) \). Thus, the tower \( K \subseteq \mathcal{R}(Y) \subseteq S \) fulfills the assumptions of the Artin–Tate lemma [AM69, Proposition 7.8], and we obtain that \( \mathcal{R}(Y) \) is a finitely generated \( K \)-algebra.

Now let \( \mathcal{R}(Y) \) be finitely generated over \( K \). Then Proposition 4.3 tells us that \( S \) is finitely generated over \( K \). Thus, Proposition 4.4 shows that \( \mathcal{R}(X) \) is finitely generated over \( K \). \( \square \)

5. Cox rings and blowing up

In this section, we compute the Cox ring of the fourth Hirzebruch surface blown up at three general points. As in the preceding section, we work over an algebraically closed field \( K \) of characteristic zero. We use the technique of toric ambient modifications provided in [Hau08], and begin with giving a short outline of this technique. A basic ingredient is the following construction of the Cox ring and the universal torsor of a toric variety given in [Cox95].

Construction 5.1. Let \( Z \) be the toric variety arising from a complete fan \( \Sigma \) in a lattice \( N \), and suppose that the primitive generators \( v_1, \ldots, v_r \) of \( \Sigma \) span the lattice \( N \). Then we have mutually
ON COX RINGS OF K3 SURFACES

dual exact sequences

\[
\begin{array}{cccc}
0 & \longrightarrow & L & \longrightarrow \mathbb{Z}^r \\
& & & \overset{P: e_i \mapsto v_i}{\longrightarrow} N & \longrightarrow 0,
\end{array}
\]

\[
\begin{array}{cccc}
0 & \longleftarrow & K & \longleftarrow \mathbb{Z}^r \\
& & & \overset{Q}{\longleftarrow} M & \longleftarrow 0,
\end{array}
\]

where the lattice \( K \) is isomorphic to the divisor class group \( \text{Cl}(Z) \). The Cox ring of \( Z \) is the polynomial ring \( \mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_r] \) with the \( K \)-grading defined by \( \deg(T_i) := Q(e_i) \). Moreover, denoting by \( \delta \subseteq \mathbb{Q}^r \) the positive orthant, we obtain a fan in \( \mathbb{Z}^r \) consisting of certain faces of \( \delta \), namely

\[
\hat{\Sigma} := \{ \hat{\sigma} \preceq \delta; P(\hat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma \}.
\]

The associated toric variety \( \tilde{Z} \) is an open toric subvariety of \( \mathbb{Z} := \mathbb{K}^r \). The toric morphism \( P: \tilde{Z} \rightarrow Z \) defined by \( P: \mathbb{Z}^r \rightarrow N \) is a universal torsor; it is a quotient for the action of the torus \( \text{Spec}(\mathbb{K}[K]) \) on \( \mathbb{Z} \) defined by the \( K \)-grading of \( \mathcal{R}(Z) \).

Given a variety \( X_0 \), the rough idea of [Hau08] is to work with a suitable embedding \( X_0 \subseteq Z_0 \) into a toric variety, consider the proper transform \( X_1 \subseteq Z_1 \) under suitable toric modifications \( Z_1 \rightarrow Z_0 \) and then compute the Cox ring \( \mathcal{R}(X_1) \) in terms of \( \mathcal{R}(X_0) \) using the toric universal torsors over \( Z_1 \) and \( Z_0 \). In our outline, we restrict to the case of blowing up a smooth projective surface \( X_0 \) with divisor class group \( K_0 \cong \mathbb{Z}^{k_0} \) and a Cox ring, which admits a representation

\[
\mathcal{R}(X_0) = \mathbb{K}[T_1, \ldots, T_r]/(f_0), \quad \deg(T_i) = w_i \in K_0,
\]

where \( f_0 \in \mathbb{K}[T_1, \ldots, T_r] \) is a homogeneous polynomial and the \( T_i \) define pairwise non-associated prime elements in \( \mathcal{R}(X_0) \). We use this presentation to embed \( X_0 \) into a toric variety. First note that we have mutually dual sequences

\[
\begin{array}{cccc}
0 & \longrightarrow & M & \longrightarrow \mathbb{Z}^r \\
& & & \overset{Q_0: e_i \mapsto w_i}{\longrightarrow} K_0 & \longrightarrow 0,
\end{array}
\]

\[
\begin{array}{cccc}
0 & \longleftarrow & N & \longleftarrow \mathbb{Z}^r \\
& & & \overset{P_0}{\longleftarrow} L_0 & \longleftarrow 0.
\end{array}
\]

Consider any complete simplicial fan \( \Sigma_0 \) in \( N \) having the images \( v_i := P_0(e_i) \in N \) of the canonical base vectors \( e_i \in \mathbb{Z}^r \) as the generators of its rays. Let \( \hat{\Sigma}_0 \) be the fan consisting of faces of the positive orthant \( \delta \subseteq \mathbb{Q}^r \) provided by Construction 5.1. Then the toric variety \( \tilde{Z}_0 \) is an open toric subvariety of \( Z_0 := \mathbb{K}^r \), and the toric morphism \( P_0: \tilde{Z}_0 \rightarrow Z_0 \) defined by \( P_0: \mathbb{Z}^r \rightarrow N \) is a universal torsor. Moreover, setting

\[
\bar{X}_0 := V(\bar{Z}_0, f_0), \quad \tilde{X}_0 := \bar{X}_0 \cap \tilde{Z}_0,
\]

we obtain \( X_0 \cong p_0(\tilde{X}_0) \), which allows us to view \( X_0 \) as a closed subvariety of the toric variety \( Z_0 \); see [Hau08, Proposition 3.14]. Note that our freedom in choosing the fan \( \Sigma_0 \) essentially relies on the assumption that \( X_0 \) is a surface; in general one has to proceed more carefully, as \( X_0 \) and \( p_0(\tilde{X}_0) \) may differ by a small birational transformation.

Now we perform a toric modification. Suppose that for some \( d \geq 2 \) the cone \( \sigma_0 \) spanned by \( v_1, \ldots, v_d \) belongs to \( \Sigma_0 \) and that we have a primitive lattice vector

\[
v_{\infty} := v_1 + \cdots + v_d \in N.
\]
Recall that \( \sigma_0 \) corresponds to a toric orbit \( T_0 \cdot z_0 \subseteq Z_0 \). Moreover, the stellar subdivision \( \Sigma_1 \) of \( \Sigma_0 \) at \( v_\infty \) defines a modification \( Z_1 \rightarrow Z_0 \) of toric varieties having the closure of the toric orbit \( T_0 \cdot z_0 \subseteq Z_0 \) as its center. Then we have commutative diagrams

\[
\begin{array}{ccc}
Z_1 \xrightarrow{\pi} Z_0 & & \overline{X}_1 \xrightarrow{\pi} \overline{X}_0 \\
\downarrow & & \downarrow \\
\hat{Z}_1 \xrightarrow{p_1} \hat{Z}_0 & & \hat{X}_1 \xrightarrow{p_0} \hat{X}_0 \\
\end{array}
\]

where \( p_1 : \hat{Z}_1 \rightarrow Z_1 \) denotes the toric universal torsor, \( X_1 \subseteq Z_1 \) the proper transform of \( X_0 \subseteq Z_0 \) and we write \( \hat{X}_1 = p_1^{-1}(X_1) \) for the inverse image and \( \overline{X}_1 \) for the closure of \( \hat{X}_1 \) in \( \overline{Z}_1 = \mathbb{K}^{r+1} \).

Note that we have

\[
\pi(z_1, \ldots, z_\infty) = (z_1 z_\infty, \ldots, z_d z_\infty, z_{d+1}, \ldots, z_r)
\]

for the lifting \( \pi : \overline{Z}_1 \rightarrow \overline{Z}_0 \) of the toric modification \( \pi : Z_1 \rightarrow Z_0 \) to the total coordinate spaces; see [Hau08, Lemma 5.3]. Moreover, \( p_1 : \hat{Z}_1 \rightarrow Z_1 \) defines another pair of dual sequences

\[
\begin{array}{cccccccccccc}
0 & \rightarrow & L_1 & \rightarrow & \mathbb{Z}^{r+1} & \rightarrow & N & \rightarrow & 0, \\
0 & \leftarrow & K_1 & \leftarrow & \mathbb{Z}^{r+1} & \leftarrow & M & \leftarrow & 0.
\end{array}
\]

Now, the basic observation is that under some mild assumptions, \( \overline{X}_1 \) is the total coordinate space of \( X_1 \) and the explicit description of \( \overline{\pi} \) given above enables us to compute moreover the Cox ring. For the precise statement, consider the \( \mathbb{Z}_{\geq 0} \)-grading of \( \mathbb{K}[T_1, \ldots, T_r] \) given by

\[
\deg(T_i) := \begin{cases} 1, & 1 \leq i \leq d, \\ 0, & d + 1 \leq i \leq r. \end{cases}
\]

Then we can write \( f_0 = g_{k_0} + \cdots + g_{k_m} \) with \( g_{k_0} \) homogeneous of degree \( k_0 \in \mathbb{Z}_{\geq 0} \) and \( k_0 < \cdots < k_m \). We call \( f_0 \in \mathbb{K}[T_1, \ldots, T_r] \) admissible, if \( g_{k_0} \) is an irreducible polynomial in at least two variables and, moreover, \( \overline{X}_0 = V(f_0) \) intersects the toric orbit \( 0 \times \mathbb{T}^{r-d} \) of \( \mathbb{K}^r \). Then [Hau08, Proposition 7.2] says the following.

**Proposition 5.2.** Suppose that the polynomial \( f_0 \in \mathbb{K}[T_1, \ldots, T_r] \) is admissible. Then the proper transform \( X_1 \) has \( \overline{X}_1 \) as its total coordinate space, and the Cox ring of \( X_1 \) is given as

\[
\mathcal{R}(X_1) = \mathbb{K}[T_1, \ldots, T_r, T_\infty]/\langle f_1 \rangle, \quad f_1 := \frac{f_0(T_1 T_\infty, \ldots, T_d T_\infty, T_{d+1}, \ldots, T_r)}{T_\infty^a},
\]

where \( a \in \mathbb{N} \) is maximal such that \( f_1 \) stays a polynomial. The divisor class group of \( X_1 \) is given by \( \text{Cl}(X_1) \cong K_1 \) and the degrees of the variables \( T_1, \ldots, T_r, T_\infty \) are given by \( \deg(T_i) = Q_1(e_i) \).

We turn to the fourth Hirzebruch surface \( F_4 \). If \( q : F_4 \rightarrow \mathbb{P}_1 \) denotes the bundle projection, we write \( C_1, C_2, C_3, C_5 \subseteq F_4 \) for the section at infinity, the fiber \( q^{-1}(0) \), the fiber \( q^{-1}(\infty) \) and the zero section, respectively. As a toric variety, \( F_4 \) arises from the fan

986
and the curves $C_1, C_2, C_3, C_5$ are the toric curves corresponding to the rays through $v_1, v_2, v_3, v_5$, respectively. In the following, we consider blow ups of $F_4$ and we will denote any proper transform of some $C_i$ again by $C_i$.

**Proposition 5.3.** Let $X$ be the blow up of $F_4$ at points $c_0, c_\infty, c_1 \in F_4 \setminus C_1$, no two of them lying in a common fiber of $q: F_4 \to \mathbb{P}_1$, and let $C_4 \subseteq X$ be the exceptional divisor over $c_0$. Then $X$ is a smooth surface with

$$\text{Cl}(X) \cong \mathbb{Z} \cdot w_1 \oplus \cdots \oplus \mathbb{Z} \cdot w_5,$$

where $w_i \in \text{Cl}(X)$ denotes the class of the curve $C_i \subseteq X$. Moreover, the Cox ring of $X$ is the polynomial ring

$$R(X) = \mathbb{K}[T_1, \ldots, T_8]/(T_2T_4 + T_3T_6 + T_7T_8)$$

and, with respect to the basis $(w_1, \ldots, w_5)$ of $\text{Cl}(X)$, the degree of the generator $T_i \in R(X)$ is the $i$th column of the matrix

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & -2 & 3 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}.$$

**Proof.** We first reduce to the case that our three points $c_0, c_1, c_\infty$ belong to the zero section $C_5 \subseteq F_4$ and within that $c_0, c_\infty$ are toric fixed points, whereas $c_1$ is the distinguished point of the non-trivial toric orbit; note that $C_5$ is the closure of the toric orbit corresponding to $\text{cone}(v_5)$.

We proceed in two steps. First choose an automorphism of $F_4$ that moves $c_0, c_1$ and $c_\infty$ into the fibers over $0, 1$ and $\infty$, respectively. Then construct a section $s: \mathbb{P}_1 \to F_4$ mapping $0, 1$ and $\infty$ to $c_0, c_1$ and $c_\infty$, and apply the automorphism $x \mapsto x - s(\pi(x))$, where $\pi: F_4 \to \mathbb{P}_1$ denotes the projection.

Blowing up the Hirzebruch surface $F_4$ at the points $c_0$ and $c_\infty$ gives a toric variety $X_0$; its fan looks as follows.
The matrix $P$ having $v_1, \ldots, v_6$ as its columns defines a surjection $\mathbb{Z}^6 \rightarrow \mathbb{Z}^2$ and hence an exact sequence. As a matrix for the projection $\mathbb{Z}^6 \rightarrow \mathbb{Z}^4$ in the dual sequence, we may take

$$Q := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}. $$

Assigning to $T_1, \ldots, T_6$ the columns $w_1, \ldots, w_6$ of $Q$ as their degrees, we obtain an action of the four torus $\mathbb{T}^4 = (\mathbb{C}^*)^4$ on $\mathbb{R}^6$. Note that $X_0$ is obtained as the GIT quotient of the set of $\mathbb{T}^4$-semistable points associated with the $\mathbb{T}^4$-linearization of the trivial bundle given by the weight $(3, 11, 2, 10) \in \mathbb{Z}^4$; in fact, by [Hau08, Corollary 4.3], we could take any weight from the relative interior of the moving cone inside the moving cone

$$\text{Mov}(X) = \bigcap_{i=1}^{6} \text{cone}(w_j; j \neq i).$$

In order to blow up the point $c_1 \in C_5$, we first embed $X_0$ in a suitable toric variety $Z_0$. Consider the polynomial ring $\mathbb{K}[T_1, \ldots, T_7]$ with the $\mathbb{Z}^4$-grading given by

$$\deg(T_i) := w_i \text{ for } 1 \leq i \leq 6, \quad \deg(T_7) := (0, 1, 0, 1),$$

and let $Q_0$ denote the matrix having these degrees as its columns. Then we have a surjection $\mathbb{K}[T_1, \ldots, T_7] \rightarrow \mathbb{K}[T_1, \ldots, T_6]$ of $\mathbb{Z}^4$-graded rings, defined by

$$T_i \rightarrow T_i \text{ for } 1 \leq i \leq 6, \quad T_7 \rightarrow T_2T_4 - T_3T_6.$$ This gives a $\mathbb{T}^4$-equivariant embedding of $\overline{X}_0 := \mathbb{K}^6$ into $\overline{Z}_0 := \mathbb{K}^7$. Note that the vanishing ideal of $\overline{X}_0$ in $\overline{Z}_0$ is generated by the polynomial

$$f_0 := T_7 - T_2T_4 + T_3T_6.$$

Consider the linearization of the trivial bundle on $\overline{Z}_0 = \mathbb{K}^7$ given by the weight $(3, 11, 2, 10) \in \mathbb{Z}^4$. Then the corresponding set of semistable points $\overline{Z}_0 \subseteq \overline{Z}_0$ is an open toric subvariety. The quotient $Z_0 := \overline{Z}_0/\mathbb{T}^4$ is a smooth projective toric variety; its fan $\Sigma_0$ has the columns $v'_1, \ldots, v'_7$ of the matrix

$$P_0 = \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & -2 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}$$

corresponding to $Q_0$ as the primitive generators of its rays, and the 10 maximal cones of the fan $\Sigma_0$ are given as

$$\text{cone}(v'_1, v'_2, v'_3), \text{cone}(v'_1, v'_2, v'_4), \text{cone}(v'_1, v'_3, v'_7), \text{cone}(v'_2, v'_3, v'_6), \text{cone}(v'_2, v'_4, v'_6),$$

$$\text{cone}(v'_2, v'_4, v'_5), \text{cone}(v'_3, v'_4, v'_6), \text{cone}(v'_4, v'_5, v'_6), \text{cone}(v'_4, v'_5, v'_7), \text{cone}(v'_5, v'_6, v'_7).$$

The closed embedding $\overline{X}_0 \subseteq \overline{Z}_0$ induces a closed embedding $X_0 \rightarrow Z_0$ of the quotient spaces, and this is a neat embedding in the sense of [Hau08]; see [Hau08, Proposition 3.14]. By our choice of the embedding, the curve $C_5$ intersects the toric orbit corresponding to $\text{cone}(v'_5, v'_7) \in \Sigma_0$ exactly in the point $c_1$.

Moreover, the polynomial $f_0$ is admissible, the $g_{k_0}$-term is just $T_2T_4 + T_3T_6$ and thus we may perform a toric ambient blow up $Z_1 \rightarrow Z_0$ at the toric orbit closure corresponding to
ON COX RINGS OF K3 SURFACES

cone(\(v'_5, v'_7\)) \in \Sigma_0. Adding the column \(v'_5 + v'_7 = (0, 0, 1)\) to the matrix \(P_0\) gives the matrix \(P_1\) describing the quotient presentation \(\hat{Z}_1 \to Z_1\).

With the proper transform \(X_1 \subseteq Z_1\) we obtain a modification \(X_1 \to X_0\) of \(X_0\) centered at the point \(c_1 \in X_0\). According to Proposition 5.2, the Cox ring of \(X_1\) is given as
\[
R(X_1) = \mathbb{K}[T_1, \ldots, T_8]/(T_7T_k - T_2T_4 + T_3T_6),
\]
where the \(\mathbb{Z}^5\)-grading assigns to the generator \(T_i\) the \(i\)th column of the matrix \(Q_1\) corresponding to \(P_1\); a direct computation shows that \(Q_1\) is the matrix given in the assertion.

We still have to check that \(X_1 = X\) holds; that means that the modification \(X_1 \to X_0\) is indeed a blow up of the point \(c_1 \in X_0\). For this, note first that, according to [Hau08, Corollary 4.13], the variety \(X_1\) inherits smoothness from its toric ambient variety \(Z_1\). Secondly, the exceptional curve over \(c_1\) is smooth and rational, and thus it must be a \((-1)\)-curve.

6. K3 surfaces with a non-symplectic involution

We now take a closer look at (complex algebraic) K3 surfaces \(X\) admitting a non-symplectic involution, i.e. an automorphism \(\sigma: X \to X\) of order two such that \(\sigma^*\omega_X = -\omega_X\) holds, where \(\omega_X\) is a non-zero holomorphic 2-form of \(X\). Since \(\text{Cl}(X) = H^2(X, \mathbb{Z}) \cap \omega_X^\perp\) holds, and \(\sigma\) is non-symplectic, one has
\[
L^\sigma := \{u \in H^2(X, \mathbb{Z}) \mid \sigma^*(u) = u\} \subseteq \text{Cl}(X)
\]
for the fixed lattice. The K3 surface \(X\) is called generic if \(\text{Cl}(X) = L^\sigma\) holds; for fixed \(\text{Cl}(X) = L^\sigma\), these K3 surfaces form a family of dimension \(20 - \text{rk}(L^\sigma)\); see [Nik83]. Our aim is to determine the Cox ring for the generic K3 surfaces with Picard number \(2 \leq \rho(X) \leq 5\), see Propositions 6.5–6.8, and for those that are generic double covers of del Pezzo surfaces, see Proposition 6.9.

For any K3 surface with a non-symplectic involution \(\sigma: X \to X\), one has a quotient surface \(Y := X/\langle \sigma \rangle\) and the quotient map \(\pi: X \to Y\). We will use the following basic facts.

**Proposition 6.1.** Let \(X\) be a generic K3 surface with a non-symplectic involution \(\sigma: X \to X\). Then the quotient map \(\pi: X \to Y\) is a double cover and:

(i) if \(\pi: X \to Y\) is unramified then the quotient surface \(Y\) is an Enriques surface;

(ii) if \(\pi: X \to Y\) is ramified, then \(Y\) is a smooth rational surface and the following statements hold:

(a) the branch divisor \(B \in \text{WDiv}(Y)\) of \(\pi\) is smooth and, denoting by \(K_Y\) the canonical divisor of \(Y\), we have
\[
\pi^*(B) = 2\pi^{-1}(B), \quad B \sim -2K_Y;
\]

(b) the pull-back \(\pi^*: \text{Cl}(Y) \to \text{Cl}(X)\) is injective and \(\pi^*(\text{Cl}(Y))\) is of index \(2^{n-1}\) in \(\text{Cl}(X)\), where \(n\) is the number of components of \(B\).

**Proof.** The fact that \(\pi: X \to Y\) is a double cover and (i) as well as (ii) up to part (b) are known; see [Zha98, Lemma 1.2]. In order to show part (b) of (ii), note first that \(\text{Cl}(Y)\) is free, because \(Y\) arises by blowing up points from \(\mathbb{P}_2\) or a Hirzebruch surface. According to [Ful98, Ex. 1.7.6], we have
\[
\pi^*(\text{Cl}_Q(Y)) = \text{Cl}_Q(X)^\sigma, \quad \pi_*\pi^*(\text{Cl}(Y)) = 2\text{Cl}(Y).
\]

989
Since $X$ is generic, the first equation tells us that $\pi^*(\text{Cl}(Y))$ is of finite index in $\text{Cl}(X)$. The second one shows that $\pi^*$ is injective. Moreover, by [BHPV04, Lemma 2.1], we have 

$$[\pi^*(\text{Cl}(Y)) : \text{Cl}(X)]^2 = \frac{\det \pi^*(\text{Cl}(Y))}{\det \text{Cl}(X)}.$$ 

Since $\text{Cl}(Y)$ is unimodular, see [BHPV04], the numerator equals $2^{g(Y)}$. By [Nik83, Theorem 4.2.2], the lattice $L^\sigma = \text{Cl}(X)$ has determinant $2^l$ and the difference $g(Y) - l$ equals $2(n - 1)$, where $n$ is the number of connected components of the branch divisor.

**Lemma 6.2.** Let $X$ be a generic K3 surface with a non-symplectic involution such that the associated double cover $\pi: X \to Y$ has branch divisor $B = C_1 + C_B$, where $C_1 \subseteq Y$ is a smooth rational curve and $C_B \subseteq Y$ is an irreducible curve.

(i) Let $w_1 \in \text{Cl}(Y)$ be the class of $C_1 \subseteq Y$. Then $(w_1, w_2, \ldots, w_r)$ is a basis of $\text{Cl}(Y)$ if and only if $(\pi^*(w_1)/2, \pi^*(w_2), \ldots, \pi^*(w_r))$ is a basis of $\text{Cl}(X)$.

(ii) With respect to bases as in (i), the homomorphism $\pi^*: \text{Cl}(Y) \to \text{Cl}(X)$ is given by the matrix

$$A := \begin{bmatrix} 2 & 0 \\ 1 & \ddots \\ 0 & \cdots & 1 \end{bmatrix}.$$ 

(iii) Let $C \subseteq X$ be any smooth rational curve and let $w \in \text{Cl}(X)$ be its class. Then precisely one of the following statements holds:

(a) $\pi(C)$ is a component of $B$ and $\pi(C)^2 = -4$;

(b) $w \in \pi^*(\text{Cl}(Y))$ and $\pi(C)^2 = -1$.

**Proof.** Since $\pi^*: \text{Cl}(Y) \to \text{Cl}(X)$ is injective, $(w_1, \ldots, w_r)$ is a basis of $\text{Cl}(Y)$ if and only if $(\pi^*(w_1), \ldots, \pi^*(w_r))$ is a basis of $\pi^*(\text{Cl}(Y))$. By Proposition 6.1(ii), we have $\pi^*(w_1) = 2u_1$ with some $u_1 \in \text{Cl}(X)$. Moreover, also by Proposition 6.1(ii), the pull-back $\pi^*(\text{Cl}(Y))$ is of index two in $\text{Cl}(X)$. This gives (i) and (ii).

To prove (iii), let $w \in \text{Cl}(X)$ denote the class of $C \subseteq X$. The adjunction formula and the Riemann–Roch theorem give $w^2 = -2$ and $h^0(w) = 1$. Since the elements of $\text{Cl}(X)$ are fixed under the involution $\sigma: X \to X$, we can conclude that $\sigma(C) = C$. If $\sigma = \text{id}$ holds on $C$, then $C$ is contained in the ramification divisor. By Proposition 6.1(ii), we have $2C = \pi^{-1}(\pi(C))$, which implies that $\pi(C)^2 = -4$. If $\sigma \neq \text{id}$ on $C$, then the restriction $\pi: C \to \pi(C)$ is a double cover. This implies that $C = \pi^{-1}(\pi(C))$ and $\pi(C)^2 = 1/2 \cdot C^2 = -1$.

We are ready to describe the quotient surfaces of generic K3 surfaces with small Picard number. In the following, we denote by $\text{Bl}_k(Z)$ the blow up of a variety $Z$ in $k$ general points. Moreover, we adopt the standard notation for integral lattices, see [BHPV04, §2, ch. 1], and $L(k)$ denotes the lattice obtained from $L$ by multiplying the intersection matrix by $k$.

**Proposition 6.3.** Let $X$ be a generic K3 surface $X$ with a non-symplectic involution and associated double cover $\pi: X \to Y$. For $2 \leq g(X) \leq 5$, the table

990
describes the intersection form of \( X \), the quotient surface \( Y \) and the branch divisor \( B \) of \( \pi \), where \( C_g \) denotes a smooth irreducible curve of genus \( g \).

Proof. According to [Nik83, §4], a lattice \( L \) of rank at most five is the fixed lattice \( L^\sigma \) of an involution \( \sigma \) on a K3 surface if and only if it is an even lattice of signature \((1, k - 1)\) which is 2-elementary, i.e. satisfies \( \text{Hom}(L, \mathbb{Z})/L = \mathbb{Z}_2^2 \), where \( 2^a = |\text{det}(L)| \). Such lattices are classified up to isometries by three invariants: the rank \( k \), the integer \( a \) and an invariant \( \delta \) defined as

\[
\delta(L) = \begin{cases} 
0 & \text{if } u^2 \in \mathbb{Z} \text{ for all } u \in \text{Hom}(L, \mathbb{Z}), \\
1 & \text{otherwise}.
\end{cases}
\]

It is easy to check that the lattices in the table are the only 2-elementary even lattices of signature \((1, k - 1)\) with \( 2 \leq k \leq 5 \), since they cover all possible triples \((k, a, \delta)\); see [Nik83, Theorem 4.3.1] and also [AN06, §2.3].

Now, suppose that the intersection form on \( \text{Cl}(X) \) is \( U \oplus A_1^{k-2} \). Then it is known that there is an elliptic fibration \( p: X \to \mathbb{P}^1 \) with a section \( E \) and \( k - 2 \) reducible fibers; see [Kon89, Lemma 3.1]. In fact, if \( e, f \) is the natural basis of \( U \) and \( v_1, \ldots, v_{k-2} \) is an orthogonal basis of \( A_1^{k-2} \), we can assume that the class of \( E \) is \( f - e \) and \( v_i \) are represented by components of the reducible fibers not intersecting \( E \).

By [Nik83, Theorem 4.2.2], the ramification divisor of \( \sigma \) is the disjoint union of a smooth irreducible curve of genus \( 12 - k \) and a smooth irreducible rational curve. This implies that \( C \) is transverse to the fibers of \( p \); hence, any fiber is preserved by \( \sigma \) and the section \( E \) is the rational curve in the ramification divisor.

A basis of \( \text{Cl}(X) \) is given by \( e, f - e, v_1, \ldots, v_{k-2} \). It follows from Lemma 6.2(ii) and (iii) that the Picard lattice of \( Y \) has intersection form

\[
\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \oplus (-1)^{k-2}.
\]

Consequently, the classification of minimal rational surfaces yields that \( Y \) is the blowing up of the Hirzebruch surface \( \mathbb{F}_4 \) at \( k - 2 \) points.

Now assume that the intersection form on \( \text{Cl}(X) \) is \( U(2) \oplus A_1^{k-2} \). Then, by [Nik83, Theorem 4.2.2], the ramification divisor has only one connected component and this is a smooth irreducible curve of genus \( 11 - k \). Thus, Proposition 6.1(ii) gives \( \text{Cl}(X) = \pi^\ast(\text{Cl}(Y)) \). It follows that the intersection form on \( \text{Cl}(Y) \) is \( U \oplus (-1)^{k-2} \). Hence, as before, we can conclude that \( Y \) is the blowing up of \( \mathbb{F}_0 \) at \( k - 2 \) points.

Similarly, if the intersection form on \( \text{Cl}(X) \) is \( (2) \oplus A_1 \), then we obtain that the intersection form on \( \text{Cl}(Y) \) is \( (1) \oplus (-1) \), the ramification divisor is a smooth irreducible curve of genus 9 and conclude that \( Y \) is the blowing up of \( \mathbb{P}^2 \) at one point.

\[\square\]
Let the Cox ring of the associated double cover $\pi \colon X \to Y$ be of the form $B = C_1 + C_B$ with $C_1, C_B \subseteq Y$ irreducible and $C_1$ rational;

the Cox ring of $Y$ is a polynomial ring $S = S'[t_1]$ with the canonical section $t_1$ of $C_1$ and a finitely generated $\mathbb{C}$-algebra $S'$.

Moreover, denote by $f \in S'$ the canonical section of $C_B$. Then the Cox ring of $X$ is given as

$$R = \pi^*(S')[T_1, T_2]/\langle T_2^2 - \pi^*(f) \rangle,$$

with the $\text{Cl}(X)$-grading defined by $\deg(\pi^*(g)) := \pi^*(\deg(g))$ for any homogeneous $g \in S'$ and $\deg(T_1) := \frac{\pi^*(w_1)}{2}, \quad \deg(T_2) := -\frac{\pi^*(2K_Y + w_1)}{2}$.

Moreover, the pull-back homomorphism $\pi^* : S \to R$ of graded rings is given on the grading groups by $\mathbb{Z}^r \to \mathbb{Z}^r, \ w \mapsto Aw$ and as a ring homomorphism by

$$t_1 \mapsto T_1^2, \quad g \mapsto \pi^*(g) \text{ for any homogeneous } g \in S'.$$

**Proof.** First note that by Proposition 4.6, the Cox ring $R$ of $X$ inherits finite generation from the Cox ring $S$ of $Y$. Consider the pull-back group of $\text{Cl}(Y)$ and the corresponding Veronese subalgebra

$$L := \pi^*(\text{Cl}(Y)) \subseteq \text{Cl}(X), \quad R_L := \bigoplus_{w \in L} R_w.$$

Write, for the moment, $B = B_1 + B_2$ and let $r$ and $b_i$ denote the canonical sections of $\pi^{-1}(B)$ and $B_i$, respectively. We claim that there is a commutative diagram of finite ring homomorphisms

\[
\begin{array}{ccc}
R & \xrightarrow{\psi} & R_L \\
\downarrow & & \downarrow \cong \\
S[T]/(T^2-b_2) & \simeq & \pi^*(S)[u_1, u_2]/\langle u_1^2 - b_1, u_2^2 - b_2, u_1u_2 - r \rangle
\end{array}
\]

where, denoting by $r_1$ and $r_2$ the canonical sections of the reduced divisors $\pi^{-1}(B_1)$ and $\pi^{-1}(B_2)$, respectively, the homomorphism $\psi$ is induced by $u_i \mapsto r_i$.

In this claim, everything is straightforward except the definition of the isomorphism $\kappa$. By Proposition 4.3, we know that $R_L$ is generated as a $\pi^*(S)$-module by 1 and a section $s \in R_L$ satisfying $s^2 = \pi^*(b)$, where $b$ denotes the canonical section of $B$. According to Lemma 6.2, we may choose $s$ to be the canonical section $r$ of the ramification divisor $\pi^{-1}(B)$. Thus, we obtain isomorphisms

$$R_L[u_1, u_2]/\langle u_1^2 - b_1, u_2^2 - b_2, u_1u_2 - r \rangle \cong \pi^*(S)[y, u_1, u_2]/\langle y^2 - b, u_1^2 - b_1, u_2^2 - b_2, u_1u_2 - y \rangle \cong \pi^*(S)[u_1, u_2]/\langle u_1^2 - b_1, u_2^2 - b_2 \rangle.$$

Now we use our assumption $S = S'[t_1]$. This enables us to define a ring homomorphism

$$\tilde{\kappa} : S[T] \to \pi^*(S)[u_1, u_2], \quad S'[g] \mapsto \pi^*(g) \in \pi^*(S'), \quad t_1 \mapsto u_1, \quad T \mapsto u_2.$$
ON COX RINGS OF K3 SURFACES

We have

\[ \text{Let } t_1^2 \to u_1^2, \text{ which defines the same element in } \pi^*(S)[u_1, u_2]/(u_1^2 - b_1, u_2^2 - b_2) \text{ as } \pi^*(t_1). \]

Consequently, \( \tilde{\kappa} \) induces the desired isomorphism

\[ \kappa: S[T]/\langle T^2 - b_2 \rangle \to \pi^*(S)[u_1, u_2]/(u_1^2 - b_1, u_2^2 - b_2). \]

The next step is to show that the homomorphism \( \psi \) of the above diagram is an isomorphism. For this, it is enough to show that \( S[T]/\langle T^2 - b_2 \rangle \) is a normal ring. Indeed, \( R_L \to R \) is of degree two,

\[ R_L \to R_L[u_1, u_2]/(u_1^2 - b_1, u_2^2 - b_2, u_1u_2 - r) \]

is of degree at least two and thus \( \psi \) is a finite morphism of degree one. If we know that \( S[T]/\langle T^2 - b_2 \rangle \) is normal, we can conclude that \( \psi \) is an isomorphism.

In order to show that \( S[T]/\langle T^2 - b_2 \rangle \) is normal, note that \( S \) can be made into a \( \mathbb{Z} \)-graded ring by assigning to each \( \mathbb{Z}^r \)-homogeneous element the \( \omega_1 \)-component of its \( \mathbb{Z}^r \)-degree. In particular, then, \( \text{deg}(b_2) \) is odd. Moreover, \( b_2 \in S \) is a prime element. Thus, we can apply the result [SS84, p. 45] and obtain that \( S[T]/\langle T^2 - b_2 \rangle \) is even factorial. In particular, it is normal.

Having verified that \( \psi \) is an isomorphism, the commutative diagram tells us that the Cox ring \( R \) of \( X \) is isomorphic to \( S[T]/\langle T^2 - b_2 \rangle \). Consequently, \( R \) is the polynomial ring \( \pi^*(S')[T_1, T_2] \) divided by the relation \( T_2^2 - \pi^*(b_2) \), where \( \pi^*(b_2) \) only depends on the first variable. The degrees of the generators \( T_i \) are easily computed using Lemma 6.2(ii).

We are ready to compute the Cox rings of generic K3 surfaces \( X \) admitting a non-symplectic involution and satisfying \( 2 \leq \varrho(X) \leq 5 \). We will work with the curves \( D_1, D_2 \subseteq \mathbb{F}_0 \) given by

\[ D_1 := \{0\} \times \mathbb{P}_1, \quad D_2 := \mathbb{P}_1 \times \{0\}, \]

the integral curves \( C_1, C_2, C_3 \subseteq \mathbb{F}_4 \) given by

\[ C_1^2 = -4, \quad C_2 := q^{-1}(0), \quad C_3 := q^{-1}(\infty), \]

where \( q: \mathbb{F}_4 \to \mathbb{P}_1 \) is the bundle projection, and the curves \( E_1, E_2 \subseteq \text{Bl}_1(\mathbb{P}^2) \) with

\[ E_1^2 = 1, \quad E_2^2 = -1. \]

Moreover, on blow ups of the surfaces \( \mathbb{F}_0 \) and \( \mathbb{F}_4 \), we denote the proper transforms of the curves \( D_1 \) and \( C_j \) again by \( D_i \) and \( C_j \).

**Proposition 6.5.** Let \( X \) be a generic K3 surface admitting a non-symplectic involution, and let \( \pi: X \to Y \) be the associated double cover. If \( \varrho(X) = 2 \) holds, then the following cases can occur.

(i) We have \( Y = \mathbb{F}_0 \). Then \( \text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \mathbb{Z} \cdot \pi^*(w_2) \) holds, where \( w_i \in \text{Cl}(Y) \) is the class of \( D_i \in \text{WDiv}(Y) \). The Cox ring of \( X \) is

\[ \mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_5]/(T_5^2 - f) \]

with a polynomial \( f \in \mathbb{C}[T_1, \ldots, T_4] \) and the degree of \( T_i \) with respect to the above basis is the \( i \)-th column of the matrix

\[ Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}. \]

(ii) We have \( Y = \mathbb{F}_4 \). Then \( \text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1)/2 \oplus \mathbb{Z} \cdot \pi^*(w_2) \) holds, where \( w_i \in \text{Cl}(Y) \) denotes the class of \( C_i \in \text{WDiv}(Y) \). The Cox ring of \( X \) is

\[ \mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_5]/(T_5^2 - f) \]

993
with a polynomial $f \in \mathbb{C}[T_1^2, T_2, T_3, T_4]$ and, with respect to the above basis, the degree of $T_i$ is the $i$th column of the matrix

$$Q = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 1 & 6 \end{bmatrix}.$$  

(iii) We have $Y = Bl_1(\mathbb{P}^2)$. Then $\text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \mathbb{Z} \cdot \pi^*(w_2)$ holds, where $w_i \in \text{Cl}(Y)$ denotes the class of $E_i \in \text{WDiv}(Y)$. The Cox ring of $X$ is

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_5]/(T_2^2 - f)$$

with a polynomial $f \in \mathbb{C}[T_1, T_2, T_3, T_4]$ and, with respect to the above basis, the degree of $T_i$ is the $i$th column of the matrix

$$Q = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$  

Proof. First note that by Proposition 6.3, the surface $Y$ is one of the three types listed in the assertion.

If $Y = \mathbb{P}_1 \times \mathbb{P}_1$ holds, then $\text{Cl}(Y) \cong \mathbb{Z}^2$ is generated by the classes $w_1, w_2$ of $D_1, D_2$ and the Cox ring of $Y$ is given by

$$\mathbb{C}[T_1, \ldots, T_4], \quad \text{deg}(T_1) = \text{deg}(T_3) = w_1, \quad \text{deg}(T_2) = \text{deg}(T_4) = w_2;$$

use e.g. Construction 5.1. Similarly, if $Y = Bl_1(\mathbb{P}^2)$, then $\text{Cl}(Y) \cong \mathbb{Z}^2$ is generated by the classes $w_1, w_2$ of $E_1, E_2$ and the Cox ring of $Y$ is given by

$$\mathbb{C}[T_1, \ldots, T_4], \quad \text{deg}(T_1) = w_1, \quad \text{deg}(T_2) = \text{deg}(T_3) = w_1 - w_2, \quad \text{deg}(T_4) = w_2.$$  

In both cases, Proposition 6.3 tells us that the branch divisor $B \subseteq Y$ is irreducible. Propositions 6.1(ii) and 4.3 thus show that the Cox ring is as claimed in (i) and (iii).

If $Y = \mathbb{F}_4$ holds, then $\text{Cl}(Y) \cong \mathbb{Z}^2$ is generated by classes $w_1, w_2$ of $C_1, C_2$ and the Cox ring of $Y$ is given by

$$\mathbb{C}[T_1, \ldots, T_4], \quad \text{deg}(T_1) = w_1, \quad \text{deg}(T_3) = w_1 + 4w_2, \quad \text{deg}(T_2) = \text{deg}(T_4) = w_2.$$  

This time, Lemma 6.2 and Proposition 6.4 show that the Cox ring of $X$ is as claimed in (ii). \qed

Proposition 6.6. Let $X$ be a generic K3 surface admitting a non-symplectic involution, and let $\pi: X \to Y$ be the associated double cover. If $g(X) = 3$ holds, then the following cases can occur.

(i) The surface $Y$ is the blow up of $\mathbb{F}_0$ at the point $(0, 0)$. If $D_3 \subseteq Y$ denotes the exceptional curve, then

$$\text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \mathbb{Z} \cdot \pi^*(w_2) \oplus \mathbb{Z} \cdot \pi^*(w_3)$$

holds, where $w_i \in \text{Cl}(Y)$ denotes the class of $D_i \in \text{WDiv}(Y)$. Moreover, the Cox ring of $X$ is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_6]/(T_6^2 - f)$$

with a polynomial $f \in \mathbb{C}[T_1, T_2, \ldots, T_5]$ and, with respect to the above basis, the degree of $T_i$ is the $i$th column of the matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$  

994
The surface $Y$ is the blow up of $\mathbb{F}_4$ at the point in $C_1 \cap C_2$, where $S_0 \in \mathbb{F}_4$ is the zero section. Then the divisor class group of $X$ is

$$\text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) / 2 \oplus \mathbb{Z} \cdot \pi^*(w_2) \oplus \mathbb{Z} \cdot \pi^*(w_3),$$

where $w_i \in \text{Cl}(Y)$ denotes the class of $C_i \in \text{WDiv}(Y)$. Moreover, the Cox ring of $X$ is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_6]/(T_6^2 - f)$$

with a polynomial $f \in \mathbb{C}[T_1^2, T_2, \ldots, T_6]$ and, with respect to the above basis, the degree of $T_i$ is the $i$th column of the matrix

$$Q = \begin{bmatrix}
1 & 0 & 2 & 0 & 3 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 3 & 1 \\
0 & 0 & 1 & -1 & -1
\end{bmatrix}.$$ 

Proof. The fact that $Y$ is either $\mathbb{P}_1 \times \mathbb{P}_1$ blown up at a point $p$ or $\mathbb{F}_4$ blown up at a point $p$ follows from Proposition 6.3. Moreover, the same proposition yields that the branch divisor has one component in the first case and two components in the second one. In both cases, applying a suitable automorphism, we may assume that the point $p$ to be blown up is as in the assertion. Then, in both cases, the surface $Y$ is toric and the computation of the Cox rings then goes in the same way as in the preceding proposition. $\square$

**Proposition 6.7.** Let $X$ be a generic K3 surface admitting a non-symplectic involution, and let $\pi : X \to Y$ be the associated double cover. If $\varrho(X) = 4$ holds, then the following cases can occur.

(i) The surface $Y$ is the blow up of $\mathbb{F}_0$ at the points $(0, 0)$ and $(\infty, \infty)$. If $D_3, D_4 \subseteq Y$ are exceptional curves corresponding to these points, then

$$\text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) \oplus \cdots \oplus \mathbb{Z} \cdot \pi^*(w_4)$$

holds, where $w_i \in \text{Cl}(Y)$ denotes the class of $D_i \in \text{WDiv}(Y)$. Moreover, the Cox ring of $X$ is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_7]/(T_7^2 - f)$$

with a polynomial $f \in \mathbb{C}[T_1^2, T_2, \ldots, T_6]$ and, with respect to the above basis, the degree of $T_i$ is the $i$th column of the matrix

$$Q = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 1 & 3 \\
0 & 0 & 1 & -1 & -1 & -1
\end{bmatrix}.$$ 

(ii) The surface $Y$ is the blow up of $\mathbb{F}_4$ at the two points $p_1 \in C_1 \cap C_2$ and $p_2 \in C_1 \cap C_3$. We have

$$\text{Cl}(X) = \mathbb{Z} \cdot \pi^*(w_1) / 2 \oplus \mathbb{Z} \cdot \pi^*(w_2) \oplus \mathbb{Z} \cdot \pi^*(w_3) \oplus \mathbb{Z} \cdot \pi^*(w_4),$$

where $w_i \in \text{Cl}(Y)$ is the class of $C_i \in \text{WDiv}(Y)$ and $C_4 \subseteq Y$ is the exceptional curve over $p_1 \in \mathbb{F}_4$. The Cox ring of $X$ is given by

$$\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_7]/(T_7^2 - f)$$
with a polynomial \( f \in \mathbb{C}[T_1^2, T_2, \ldots, T_6] \) and, with respect to the above basis, the degree of \( T_i \) is the \( i \)th column of the matrix

\[
Q = \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & 3 \\
0 & 1 & 0 & 3 & 1 & 5 \\
0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 & 4
\end{bmatrix}.
\]

**Proof.** Again, Proposition 6.3 tells us that \( Y \) is either \( \mathbb{P}_1 \times \mathbb{P}_1 \) blown up at two points \( p, q \) or \( \mathbb{F}_4 \) blown up at two points \( p, q \) and that the branch divisor has one component in the first case and two components in the second one. In both cases, we may apply a suitable automorphism, and achieve that the points \( p, q \) to be blown up are as in the assertion. Thus, again, the surface \( Y \) is toric and the computation of the Cox rings proceeds as before. \( \square \)

**Proposition 6.8.** Let \( X \) be a generic K3 surface admitting a non-symplectic involution, and let \( X \to Y \) be the associated double cover. If \( g(X) = 5 \) holds, then the following cases can occur.

(i) The surface \( Y \) is the blow up of \( \mathbb{F}_0 \) at three general points. Then the Cox ring of \( X \) is

\[
\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_{11}] / \langle f_1, \ldots, f_5, T_{11}^2 - g \rangle,
\]

where \( f_1, \ldots, f_5 \) are the Plücker relations in the variables \( T_1, \ldots, T_{10} \), i.e. we have

\[
\begin{align*}
f_1 &= T_2 T_5 - T_3 T_6 + T_4 T_7, \\
f_2 &= T_1 T_5 - T_3 T_8 + T_4 T_9, \\
f_3 &= T_7 T_6 - T_2 T_8 + T_4 T_{10}, \\
f_4 &= T_1 T_7 - T_2 T_9 + T_3 T_{10}, \\
f_5 &= T_5 T_{10} - T_6 T_9 + T_7 T_8
\end{align*}
\]

and \( g \in \mathbb{C}[T_1, \ldots, T_{10}] \) is a prime polynomial. The degree of \( T_i \in \mathcal{R}(X) \) is the \( i \)th column of

\[
Q = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\
1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 1
\end{bmatrix}.
\]

(ii) The surface \( Y \) is the blow up of \( \mathbb{F}_4 \) at three general points. Then the Cox ring of \( X \) is

\[
\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_9] / \langle T_2 T_5 + T_3 T_6 + T_7 T_8, T_9^2 - f \rangle,
\]

where \( f \in \mathbb{C}[T_1, \ldots, T_8] \) is a prime polynomial and the degree of \( T_i \in \mathcal{R}(X) \) is the \( i \)th column of

\[
Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\
0 & 1 & 0 & 0 & 0 & -2 & 3 & 4 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & 4 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1
\end{bmatrix}.
\]

**Proof.** The facts that only (i) and (ii) are possible and that the branch divisor has one component in (i) and two in (ii) follow from Proposition 6.3.

If we are in the situation (i), then \( Y \) is the blow up of \( \mathbb{P}_2 \) at four general points and hence is a del Pezzo surface. Its Cox ring is the ring of \((3 \times 3)\)-minors of a generic \((3 \times 5)\)-matrix; see [BP04, Proposition 4.1]. Moreover, by Proposition 6.1(ii), the pull-back \( \pi^* : \text{Cl}(Y) \to \text{Cl}(X) \) is an isomorphism. Thus, taking the same basis of \( \text{Cl}(Y) \) as in the proof of [BP04, Proposition 4.1], the assertion follows from Propositions 4.3 and 6.1(ii).
If we are in situation (ii), then the assertion is a direct consequence of Propositions 5.3 and 6.4 and Lemma 6.2. Note that the canonical class of $Y$ can be determined according to [BH07, Proposition 8.5] as the degree of the relation minus the sum of the degrees of the generators of the Cox ring of $Y$.

If $X$ is a generic K3 surface with a non-symplectic involution such that the associated double cover has an irreducible branch divisor, then we can proceed with the computation of Cox rings as follows.

**Proposition 6.9.** Let $X$ be a generic K3 surface with a non-symplectic involution, associated double cover $\pi: X \to Y$ and intersection form $U(2) \oplus A^{k-2}_1$, where $5 \leq k \leq 9$. Then $Y$ is a del Pezzo surface of Picard number $k$ and:

(i) the Cox ring $\mathcal{R}(X)$ is generated by the pull-backs of the $(-1)$-curves of $Y$, the section $T$ defining the ramification divisor and, for $k = 9$, the pull-back of an irreducible section of $H^0(Y, -K_Y)$;

(ii) the ideal of relations of $\mathcal{R}(X)$ is generated by quadratic relations of degree $\pi^*(D)$, where $D^2 = 0$ and $D \cdot K_Y = -2$, and the relation $T^2 - f$ in degree $-2\pi^*(K_Y)$, where $f$ is the pull-back of the canonical section of the branch divisor.

**Proof.** As in the proof of Proposition 6.3, we use [Nik83, Theorem 4.2.2] to see that the ramification divisor of $\pi: X \to Y$ is irreducible. Then Proposition 6.1(ii) yields $\text{Cl}(X) = \pi^*(\text{Cl}(Y))$. It follows that the intersection form on $\text{Cl}(Y)$ is $U \oplus (-1)^{k-2}$. Consequently, $Y$ is the blow up of $\mathbb{F}_0$ at $k - 2$ general points and hence is a del Pezzo surface.

It is known that $\mathcal{R}(Y)$ is generated by all the $(-1)$-curves of $Y$ plus, if $k = 9$, an irreducible section of $H^0(Y, -K_Y)$; see [BP04, Theorem 3.2]. The ideal of relations of $\mathcal{R}(Y)$ is generated by quadratic relations of degree $D$, where $D$ is a conic bundle, i.e. we have $D^2 = 0$ and $D \cdot K_Y = -2$; see [LV09, Lemmas 2.2, 2.3 and 2.4]. Thus, the statement follows from Proposition 4.3.

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