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# BIPLANAR SURFACES OF ORDER THREE II

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A surface of order three $F$ in the real projective three-space $P^3$ is met by every line, not in $F$, in at most three points. $F$ is biplanar if it contains exactly one non-differentiable point $v$ and the set of tangents of $F$ at $v$ is the union of two distinct planes, say $\tau_1$ and $\tau_2$.

In [**2**], we examined the biplanar surfaces containing the line $\tau_1 \cap \tau_2$. In the present paper, we classify and describe the biplanar $F$ with the property that $\tau_1 \cap \tau_2 \cap F = \{v\}$.

We denote the planes, lines and points of $P^3$ by the letters $\alpha, \beta, \ldots, L, M, \ldots$ and $p, q, \ldots$ respectively. For a collection of flats $\alpha, L, p, \ldots, \langle \alpha, L, p, \ldots \rangle$ denotes the flat of $P^3$ spanned by them. For a set $M$ in $P^3$, $\langle \mathscr{M} \rangle$ denotes the flat of $P^3$ spanned by the points of $\mathscr{M}$.

**1. Surfaces of order three.** In this section we formally define a surface of order three, introduce some notation and list some required results.

1.1 A *surface of order three* $F$ in $P^3$ is a compact and connected set such that every intersection of $F$ with a plane is a curve of order $\leqq 3$ and there is a plane section of order three not containing any lines of $F$.

1.2 Let $\Gamma$ be a (plane) curve of order $k$, $k \leqq 3$ (see [**1**], 1.3). If $k = 1$, then $\Gamma$ is a (straight) line. If $k = 2$, then $\Gamma$ is an isolated point or a pair of lines or the image $S^1$ of a differentiable parameter curve of order two. If $k = 3$, then $\Gamma$ is (i) the union of a line and a $\Gamma'$ of order two or (ii) the image $F_*{}^1$ of a differentiable parameter curve of order three plus possibly an $S^1$ or an isolated point either disjoint from $F_*{}^1$. We denote a $\Gamma$ of order three satisfying (ii) by $F^1$.

1.3 Let $F$ be a surface of order three, $p \in F$. Let $\alpha$ be a plane through $p$. Then $p$ is *regular* in $F[\alpha \cap F]$ if there is a line $N$ in $P^3[\alpha]$ such that $p \in N$ and $|N \cap F| = 3$. Otherwise, $p$ is *irregular* in $F[\alpha \cap F]$. An $F^1$ has at most one irregular point $v$ and such a $v$ is a cusp, double point or isolated point ([**1**], 1.4).

A line $T$ is a *tangent* of $F$ at $p$ if $T$ is a tangent of some $\Gamma \subset F$ at $p$ ([**1**], 1.5). Let $\tau(p)$ be the set of tangents of $F$ at $p$. Then $p$ is *differentiable* if $p$ is regular (in $F$) and $\tau(p)$ is a plane $\pi(p)$; otherwise, $p$ is *singular*.

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We assume that every regular $p$ is differentiable and $\pi(p)$ depends continuously on $p$.

We denote by $l(p)[l(p, \alpha)]$, the number of lines of $F[\alpha \cap F]$ passing through $p$ and by $l(\alpha)$ the number of lines of $\alpha \cap F$. From 1.2, $l(\alpha) \leqq 3$.

Let $p$ be regular and $l(p) = 0$. Then $p$ is an isolated point, cusp or double point of $\pi(p) \cap F$ ([**1**], 2.3) and we call $p$ *elliptic*, *parabolic* or *hyperbolic* respectively. Let $E$, $I$ and $H$ denote the set of elliptic, parabolic and hyperbolic points of $F$ respectively.

Let $v$ be irregular in $F$. If $F$ is non-ruled, that is,

$$l(F) = |\{L \subset P^3 | L \subset F\}| < \infty,$$

then $v \in T \subset \tau(v)$ if and only if either $v \in T \subset F$ or $T \cap F = \{v\}$. Moreover, $\tau(v)$ is a plane or a union of two distinct planes or a cone of order two with vertex $v$; cf. [**5**].

1.4 Let $\mathscr{F}$ be a closed, connected subset of an $S^1$ or an $F_*{}^1$. We call $F$ a *subarc* [*subcurve*] if the end points of $\mathscr{F}$ are distinct [equal].

Let $p$ be regular. Let $\mathscr{F}(p)$ be the set of all subarcs $\mathscr{F}$ of order two in $\mathscr{F}$ such that $p \in \mathscr{F} \not\subset \pi(p)$; $\{\mathscr{F}_1, \mathscr{F}_2\} \subset \mathscr{F}(p)$. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are *p-compatible* if there is a $\beta \subset P^3 \backslash \{p\}$ and an open neighbourhood $U(p)$ of $p$ in $P^3$ such that $U(p) \cap (\mathscr{F}_1 \cup \mathscr{F}_2)$ is contained in a closed half-space of $P^3$ determined by $\beta$ and $\pi(p)$. Otherwise, $\mathscr{F}_1$ and $\mathscr{F}_2$ are *p-incompatible*.

A pair of subarcs $\mathscr{F}$ and $\mathscr{F}'$ are *compatible* [*incompatible*] if there is a $p \in \mathscr{F} \cap \mathscr{F}'$ such that $\{\mathscr{F}, \mathscr{F}'\} \subset \mathscr{F}(p)$ and $\mathscr{F}, \mathscr{F}'$ are $p$-compatible [$p$-incompatible].

We consider a subcurve of order two as an element of $\mathscr{F}(p)$ if it contains a subarc $\mathscr{F}$ such that $p \in \mathscr{F} \subset \mathscr{F}(p)$.

1.5 We describe a surface $F$ by determining the existence and the distribution of elliptic, parabolic and hyperbolic points in $F$. By way of preparation, we list the following results.

1. If $p$ is regular in $F$ and isolated in $\alpha \cap F$, then $p \in E$ and $\alpha = \pi(p)$ ([**1**], 2.3.7).

2. Let $p$ be regular, $l(p) = 0$. Then $p \in H$ if and only if there exist incompatible $\mathscr{F}$ and $\mathscr{F}'$ in $\mathscr{F}(p)$ with $p \in (\text{int } \mathscr{F}) \cap (\text{int } \mathscr{F}')$ ([**1**], 2.5.7).

3. Let $p_\lambda[\alpha_\lambda]$ be a sequence of points [planes] converging to $p[\alpha]$; $p_\lambda \in \alpha_\lambda$ for each $\lambda$.

(a) If $\alpha \cap F$ is not of order two or $\alpha \cap F$ does not contain an isolated point, then $\lim (\alpha_\lambda \cap F) = \alpha \cap F$ ([**1**], 2.4.3).

(b) If $p_\lambda$ is a cusp [isolated point] of $\alpha_\lambda \cap F$ for each $\lambda$, then $l(p) = 0$ implies that $p$ is a cusp [cusp or isolated point] and $\alpha \cap F = L \cup S^1$ implies that $L \cap S^1 = \{p\}$ ([**1**] 2.4.6 and 2.4.9).

4. Let $\gamma \cap F$ be of order two. Then $\gamma \cap F$ consists of a pair of line $L \neq L'$ and either $L' \subset \pi(L)$ ($L' \subset \pi(p)$ for each regular $p \in L$) or $L \subset \pi(L')$ ([**1**], 2.2.3).

5. Let $G$ be an open region in $F$ such that $\alpha_0 \cap \bar{G} = \emptyset$ for some $\alpha_0$, bd $(F\backslash G) =$ bd $(G)$, $\langle$bd $(G)\rangle$ is a plane and each $r \in G$ is regular. Then $G \cap E \neq \emptyset$ ([**3**], 3.7).

6. Let $F$ be non-ruled, $l(F) > 0$. Then $H \neq \emptyset$, $E$ is open and $I = \{p \in \bar{H} \cap \bar{E} | \, l(p) = 0$ and $p$ is regular$\}$ is nowhere dense in $F$ ([**3**], 3.8 and 3.9).

## 2. Biplanar surfaces.

2.0 Let $F$ be a surface of order three. A point $v \in F$ is a *binode* if $v$ is irregular in $F$ and $\tau(v)$ is the union of two distinct planes. $F$ is *biplanar* if $F$ is non-ruled and contains a binode $v$ as its only irregular point.

Henceforth, $F$ is biplanar with the binode $v$, $\tau(v)$ is the union of distinct planes $\tau_1$ and $\tau_2$, $N_0 = \tau_1 \cap \tau_2$ and $N_0 \cap F = \{v\}$. As $v \in T \subset \tau(v)$ if and only if either $v \in T \subset F$ or $T \cap F = \{v\}$, $N_0 \not\subset F$ implies that $l(v) = l(v, \tau_1) + l(v, \tau_2)$. Since each $\tau_i$ contains at least one point of $F$ distinct from $v$, we obtain that $2 \leqq l(v) \leqq 6$.

2.1 Lemma. *Let $v \in \beta$ such that $\beta \cap \tau_i$ is a line $N_i$, $i = 1, 2$.*
1. *If $N_0 = N_1 = N_2$, then $v$ is the cusp of $\beta \cap F$.*
2. *If $N_1 \neq N_0 \neq N_2$ and $l(\beta) = 0$, then $v$ is the double point of $\beta \cap F$.*
3. *If $N_j \subset F$ and $N_k \cap F = \{v\}$, then $\beta \cap F$ consists of $N_j$ and an $S^1$ such that $|N_j \cap S^1| = 2$ and $v \in N_j \cap S^1$; $\{j, k\} = \{1, 2\}$.*
4. *If $N_1 \cup N_2 \subset F$, then $\beta \cap F$ consists of three non-concurrent lines.*

*Proof.* This is immediate since $v$ is irregular in $\beta \cap F$ and $v \in L \not\subset \tau_1 \cup \tau_2$ implies that $|L \cap F| = 2$.

2.2 Lemma. *$l(\tau_i) = 1$ or $3$ for $i = 1, 2$.*

*Proof.* Since $l(\tau_i) > 0$, there is a line $M_i \subset \tau_i \cap F$ through $v$; $i = 1, 2$.
Suppose that $\tau_1 \cap F = M_1 \cup M_1'$, $M_1 \cap M_1' = \{v\}$. Then either $M_1 \subset \pi(M_1')$ or $M_1' \subset \pi(M_1)$ by 1.5.4.
Since $M_2 \not\subset \tau_1$, 2.1.4 implies that

$$\langle M_1, M_2 \rangle \cap F = M_1 \cup M_2 \cup L_{12} \text{ and}$$
$$\langle M_1', M_2 \rangle \cap F = M_1' \cup M_2 \cup L_{12}', \text{ say,}$$

where $v \notin L_{12} \cup L_{12}'$. Then $M_2 \subset \pi(M_1 \cap L_{12}) \cap \pi(M_1' \cap L_{12}')$ yields that $M_1 \not\subset \pi(M_1')$ and $M_1' \not\subset \pi(M_1)$, a contradiction.
The preceding argument is symmetric in $\tau_1$ and $\tau_2$.

2.3 THEOREM. *Let $F$ be biplanar with the binode $v$, $\tau_1 \cap \tau_2 \cap F = \{v\}$. Then $F$ is one of the following types:* (1) $l(F) = 3$ *and* $l(v) = 2$, (2) $l(F) = 7$ *and* $l(v) = 4$ *and* (3) $l(F) = 15$ *and* $l(v) = 6$.

*Proof.* By 2.2, $l(v)$ is 2, 4 or 6. Now apply 2.1.4.

We note that 2.3 and [**2**], 2.3 provide a classification of biplanar surfaces. In particular, a biplanar $F$ with the binode $v$ is identified by the ordered pair $(l(F), l(v))$ equal to (1. 1), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3), (6, 4), (7, 4), (10, 5) or (15, 6).

2.4 Before examining the surfaces listed in 2.3, we introduce the following definitions and notations.

a) Let $v \in \beta$, $l(\beta) = 0$. By 2.1, $v$ is a cusp or a double point of $\beta \cap F$. In either case, there is a unique inflection point $p_\beta \in \beta \cap F$.

If $v$ is a cusp of $\beta \cap F$, then

$$\beta \cap F = \mathscr{F} \cup \mathscr{F}'$$

where $\mathscr{F} \cap \mathscr{F}' = \{v, p_\beta\}$ and $\{\mathscr{F}, \mathscr{F}'\} \subset \mathscr{F}(p_\beta)$.

If $v$ is a double point of $\beta \cap F$, then

$$\beta \cap F = \mathscr{L} \cup \mathscr{F}_1 \cup \mathscr{F}_2$$

where $\mathscr{L} \cap (\mathscr{F}_1 \cup \mathscr{F}_2) = \{v\}$, $\mathscr{F}_1 \cap \mathscr{F}_2 = \{v, p_\beta\}$, $\{\mathscr{F}_1, \mathscr{F}_2\} \subset \mathscr{F}(p_\beta)$ and $\mathscr{L}$ is the *loop* (the subcurve of order two) of $\beta \cap F$. We note that $\beta \cap \tau_1$ and $\beta \cap \tau_2$ are the tangents of $\beta \cap F$ at $v$. We will always assume that $\lim \langle v, r\rangle = \beta \cap \tau_i$ as $r \neq v$ tends to $v$ in $\mathscr{F}_i$; $i \neq 1, 2$.

b) Let $p$, $q$, $r$ and $s$ be four mutually distinct collinear points. We say that $p$, $q$ *separates* $r$, $s$ if no segment of $\langle p, q\rangle$ bounded by $p$ and $q$ contains both $r$ and $s$. Otherwise, $p$, $q$ *does not separate* $r$, $s$.

In an obvious manner, we extend the preceding definition to coplanar lines.

c) Let $L \subset F$ and $r \in F \backslash L$ such that $\langle L, r\rangle \cap F$ consists of $L$ and an $S^1$. We denote this $S^1$ by $S^1(L, r)$.

d) Let $\mathscr{F}$ be a subarc or a subcurve, either of order two; $\alpha = \langle \mathscr{F}\rangle$. We define

$$e(\mathscr{F}) = \{p \in \alpha\backslash\mathscr{F} \mid p \text{ lies on a tangent of } \mathscr{F} \text{ at } r \text{ for some } r \in \mathscr{F}\}$$

and $i(F) = \alpha\backslash\{e(\mathscr{F}) \cup \mathscr{F}\}$.

We note that $\alpha = i(\mathscr{F}) \cup \mathscr{F} \cup e(\mathscr{F})$ and $\mathscr{F} = S^1$ implies that $i(S^1)$ is the open disk of $\langle S^1\rangle$ bounded by $S^1$.

Let $\mathscr{F}$ be a subarc of order two, $r \in \text{int}(\mathscr{F})$. Then the tangent $T$ of $\mathscr{F}$ at $r$ supports $\mathscr{F}$ at $r$ and $T \subset e(\mathscr{F}) \cup \{r\}$. Let $r \in N \subset \alpha$,

$N \neq T$. Then $N$ cuts $\mathscr{F}$ at $r$ and $N \cap i(\mathscr{F}) \neq \emptyset \neq N \cap e(F)$. Thus

$$r \in \overline{(N \cap i(F))} \cap \overline{(N \cap e(F))}.$$

2.5 THEOREM. *Let $r$ be regular such that $l(r) = 0$ and $\{\mathscr{F}, \mathscr{F}'\} \subset \mathscr{F}(r)$ with*

$$r \in (\mathrm{int}\mathscr{F}) \cap (\mathrm{int}\mathscr{F}') \text{ and } r \notin \overline{e(\mathscr{F}) \cap e(\mathscr{F}')}.$$

*Then $r$ is hyperbolic.*

*Proof.* Let $T[T']$ be the tangent of $\mathscr{F}[\mathscr{F}']$ at $r$. Since

$$r \notin \overline{e(\mathscr{F}) \cap e(\mathscr{F}')},$$

2.4 (d) yields that $\langle \mathscr{F} \rangle \cap \langle \mathscr{F}' \rangle$ is a line $N$ distinct from $T$ and $T'$. Hence $\langle T, N \rangle = \langle F \rangle$, $\langle T', N \rangle = \langle F' \rangle$, $T \neq T'$, $\pi(r) = \langle T, T' \rangle$, $N$ cuts both $\mathscr{F}$ and $\mathscr{F}'$ at $r$ and

$$r \in \overline{(N \cap e(\mathscr{F}))} \cap \overline{(N \cap e(\mathscr{F}'))}.$$

Then

$$r \notin \overline{e(\mathscr{F}) \cap e(\mathscr{F}')} = N \cap \overline{e(\mathscr{F}) \cap e(\mathscr{F}')}$$

and the preceding imply that $\overline{N \cap e(\mathscr{F})}$ and $\overline{N \cap e(\mathscr{F}')}$ are one-sided neighbourhoods of $r$ in $N$. Thus there is an open neighbourhood $U(r)$ of $r$ in $P^3$ such that

$$U(r) \cap e(\mathscr{F}) \neq \emptyset \neq U(r) \cap e(\mathscr{F}')$$

and

$$U(r) \cap e(\mathscr{F}) \cap e(\mathscr{F}') = \emptyset.$$

Then $e(\mathscr{F}) \cap e(\mathscr{F}') \subset N$ and $N \not\subset \pi(r)$ imply that for any $\beta \subset P^3 \backslash \{r\}$, $U(r) \cap (N \cap e(\mathscr{F}))$ and $U(r) \cap (N \cap e(\mathscr{F}'))$ are not contained in the same half-space of $P^3$ bounded by $\beta$ and $\pi(r)$. This is possible only if $U(r) \cap \mathscr{F}$ and $U(r) \cap \mathscr{F}'$ are also not contained in the same half-space of $P^3$ bounded by $\beta$ and $\pi(r)$. Thus $r \in H$ by 1.4 and 1.5.2.

COROLLARY. *If the line $\langle \mathscr{F} \rangle \cap \langle \mathscr{F}' \rangle$ is not the tangent of $\mathscr{F}$ or $\mathscr{F}'$ at $r$, then $r \notin \overline{e(\mathscr{F}) \cap e(\mathscr{F}')}$ if and only if $r \notin \overline{i(\mathscr{F}) \cap i(\mathscr{F}')}$.*

## 3. $F$ with three lines.

3.0 Let $F$ be biplanar with the binode $v$, $l(F) = l(v) + 1 = 3$. Let $M_i$ denote the line of $F$ through $v$ in $\tau_i$, $i = 1, 2$. By 2.1.4, $\langle M_1, M_2 \rangle \cap F$ contains a line $L$ such that $v \notin L$. Let $L \cap M_i$ be the point $m_i$.

Let $r \in F$, $l(r) = 0$. By 2.1, $v$ is the cusp of $\langle N_0, r \rangle \cap F$ and

$$\langle M_i, r \rangle \cap F = M_i \cup S^1(M_i, r)$$

where

$$v \in M_i \cap S^1(M_i, r) \text{ and } |M_i \cap S^1(M_i, r)| = 2.$$

We note that $S^1(M_i, r) \in \mathscr{F}(r)$ and if $M_i \cap S^1(M_i, r) = \{v, p_i\}$, then

$$\pi(p_i) = \langle M_i, r\rangle; i = 1, 2.$$

If $L \not\subset \pi(r)$, then $\langle L, r\rangle \cap F = L \cup S^1(L, r)$ and $S^1(L, r) \in F(r)$.

3.1 Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the closed half-spaces of $P^3$ determined by $\tau_1$ and $\tau_2$. Put $\{i, j\} = \{1, 2\}$ and let $\beta \subset \mathscr{P}_i$, $l(\beta) = 0$. Then $N_0 \subset \beta$ and $v$ is the cusp of $\beta \cap F = \mathscr{F}_1 \cup \mathscr{F}_2$.

Let $B_\lambda$ be a sequence of planes tending to $\beta$ such that $v \in \beta_\lambda \neq \beta$ and $l(\beta_\lambda) = 0$ for each $\lambda$. Then $v$ is the double point of

$$\beta_\lambda \cap F = \mathscr{L}_\lambda \cup \mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}$$

($\mathscr{L}_\lambda$, being the loop of $\beta_\lambda \cap F$) by 2.1.2. Since $\lim \beta_\lambda = \beta$, 1.5.3 implies that

$$\lim(\mathscr{L}_\lambda \cup \mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}) = \mathscr{F}_1 \cup \mathscr{F}_2.$$

We note that (for each $\lambda$) $\mathscr{L}_\lambda$ and $\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}$ are not contained in the same $\mathscr{P}_1$ or $\mathscr{P}_2$ and $\lim \mathscr{L}_\lambda$ is a curve of order $\leqq 2$. Then $\mathscr{F}_1 \cup \mathscr{F}_2 \subset \mathscr{P}_i$ and $(\mathscr{F}_1 \cup \mathscr{F}_2) \cap \mathscr{P}_j = \{v\}$ imply that

$$\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda} \subset \mathscr{P}_i \text{ and } \mathscr{L}_\lambda \subset \mathscr{P}_j$$

for $\beta_\lambda$ sufficiently close to $\beta$. Thus

$$\lim(\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}) = \mathscr{F}_1 \cup \mathscr{F}_2 \text{ and } \lim \mathscr{L}_\lambda = \{v\}.$$

Finally, $v \notin L$ implies that $\mathscr{L}_\lambda \cap L = \emptyset$ and therefore

$$(\mathscr{L}_\lambda \backslash \{v\}) \cap (M_1 \cup M_2 \cup L) = \emptyset$$

for $\mathscr{L}_\lambda$ sufficiently close to $v$.

3.2 For $i = 1, 2$, let $\mathscr{P}_{i1}$ and $\mathscr{P}_{i2}$ be the open quarter-spaces of $\mathscr{P}_i$ determined by $\langle M_1, M_2\rangle$. Put $F_{ik} = \mathscr{P}_{ik} \cap F$, $k = 1, 2$. Then

$$F_{11} \cup F_{12} \cup F_{21} \cup F_{22} = \{r \in F | \, l(r) = 0\}$$

and

$$F = F_{11} \cup F_{12} \cup F_{21} \cup F_{22} \cup M_1 \cup M_2 \cup L.$$

From 3.1, there is a sequence of loops $\mathscr{L}_\lambda$ in each $F_{i1} \cup F_{i2} \cup \{v\}$ such that $\lim \mathscr{L}_\lambda = \{v\}$. It is easy to check that there is such a sequence of loops in each $F_{ik} \cup \{v\}$. Then the continuity of the plane sections of $F$ through $v$ yields that (for each $i, k = 1, 2$) there exists a sequence $\beta_\gamma$ such that $v$ is the double point of

$$\beta_\gamma \cap F = \mathscr{L}_\gamma \cap \mathscr{F}_{1,\gamma} \cup \mathscr{F}_{2,\gamma},$$

$$\beta_\lambda \cap F_{ik} = \mathscr{L}_\gamma \backslash \{v\} \text{ and } \lim \beta_\gamma = \langle M_1, M_2 \rangle.$$

Since $\lim(\mathscr{L}_\gamma \cup \mathscr{F}_{1,\gamma} \cup \mathscr{F}_{2,\gamma}) = M_1 \cup M_2 \cup L$ and $\lim \mathscr{L}_\gamma$ is a curve of order $\leqq 2$, we readily obtain that $\lim \mathscr{L}_\gamma$ is a triangle in $M_1 \cup M_2 \cup L$ with vertices $m_1, m_2$ and $v$. Thus $F_{ik}$ is a bounded triangular region with

$$\text{bd}(F_{ik}) \subset M_1 \cup M_2 \cup L; i, k = 1, 2.$$

3.3 THEOREM. *$E \cap F_{ik}$ is a non-empty proper subset of $F_{ik}$ with $v$ in its boundary; $i, k = 1, 2$.*

*Proof.* From 3.2, there is a sequence of loops $\mathscr{L}_\gamma \subset F_{ik} \cup \{v\}$ converging to $v$. Since $F_{ik}$ is a bounded region with $\text{bd}(F_{ik}) \subset M_1 \cup M_2 \cup L$, we obtain that (for each $\gamma$) $\mathscr{L}_\gamma$ is the boundary of an open region $F_{ik}(\mathscr{L}_\gamma) \subset F_{ik}$ such that

$$\lim \text{Cl}(F_{ik}(\mathscr{L}_\gamma)) = \{v\}$$

as $\mathscr{L}_\gamma$ tends to $v$. Clearly, $F_{ik}(\mathscr{L}_\gamma)$ satisfies 1.5.5 for each $\gamma$ and thus

$$E \cap F_{ik}(\mathscr{L}_\gamma) \neq \emptyset \text{ and}$$

$$v \in \overline{E \cap F_{ik}}.$$

Let $r \in F_{ik}$. Then

$$\langle M_1, r \rangle \cap F = M_1 \cup S^1(M_1, r)$$

where

$$M_1 \cap S^1(M_1, r) = \{v, p\}, p \neq v \text{ and } \pi(p) = \langle M_1, r \rangle.$$

We note that $p \neq m_1$ and thus

$$L \cap S^1(M_1, r) = \emptyset.$$

Then $r \in F_{ik}$ implies that

$$p \in M_1 \cap \bar{F}_{ik}.$$

Since $|M_1 \cap S^1(M_1, r)| = 2$, 1.5.3 yields that $p \notin \bar{E}$ and $F_{ik} \not\subseteq E$.

3.4 We have shown that $F$ is the union of the regions $\bar{F}_{ik}$, each of which contains elliptic, parabolic and hyperbolic points. We still need to determine the "positions" of the $F_{ik}$ to obtain a representation of $F$; cf. Figure 1.

Let $r \in \mathscr{P}_i \cap F$ such that $l(r) = 0$ and $\langle L, r \rangle \cap F = L \cup S^1(L, r)$, $i \in \{1, 2\}$. Then $\langle L, r \rangle \cap \tau_i \cap F = \{m_i\}$ implies that

$$\{m_1, m_2\} \subset e(S^1(L, r)) \text{ and } S^1(L, r) \subset \text{int}(\mathscr{P}_i).$$

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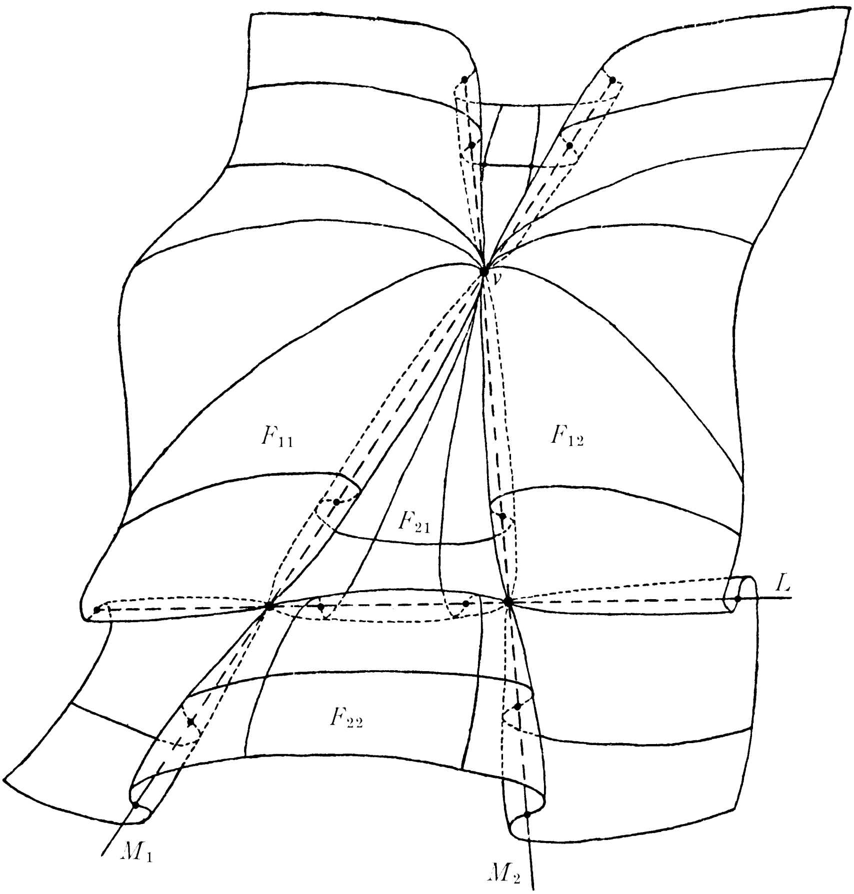


FIGURE 1

Let $\mathscr{S}$ be the space of all planes of $P^3$ through $L$ and put

$$\mathscr{S}_i = \{\alpha \in \mathscr{S} \mid \alpha \cap F = L \cup S^1 \text{ and } S^1 \subset \mathscr{P}_i\},\ i = 1, 2.$$

If $\alpha_i \in \text{bd}(\mathscr{S}_i)$, then clearly $\alpha_i = \langle M_1, M_2\rangle$ or $\alpha_i \cap F = L$ or $\alpha_i \cap F = L \cup \{r_i\}$ where $r_i$ is some point in $\mathscr{P}_i \cap F$ with $l(r_i) = 0$. Thus $\alpha \in \overline{\mathscr{S}}_1 \cap \overline{\mathscr{S}}_2$ implies that

$$\alpha = \langle M_1, M_2\rangle \text{ or } \alpha \cap F = L.$$

Since $\mathscr{S}^* = \mathscr{S}\backslash\langle M_1, M_2\rangle$ is connected and $\mathscr{S}^* \cap \overline{\mathscr{S}}_1$ and $\mathscr{S}^* \cap \overline{\mathscr{S}}_2$ are non-empty and closed in $\mathscr{S}^*$, we obtain that

$$\mathscr{S}^* \cap \overline{\mathscr{S}}_1 \cap \overline{\mathscr{S}}_2 \neq \emptyset \text{ or } \mathscr{S}\backslash(\overline{\mathscr{S}}_1 \cup \overline{\mathscr{S}}_2) \neq \emptyset.$$

In either case, there is an $\alpha_0 \in \mathscr{S}$ such that $\alpha_0 \cap F = L$.

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Let $\mathscr{Q}$ and $\mathscr{Q}'$ be the closed half-spaces of $P^3$ determined by $\langle M_1, M_2\rangle$ and $\alpha_0$ and let $v$ be the cusp of $\beta \cap F = \mathscr{F}_1 \cup \mathscr{F}_2 \subset \mathscr{P}_i$, $i \in \{1, 2\}$. Then $\mathscr{F}_1 \cup \mathscr{F}_2$ meets both $F_{i1}$ and $F_{i2}$. Since $\beta \cap \langle M_1, M_2\rangle$ supports $\beta \cap F$ at $v$, $\mathscr{F}_1 \cup \mathscr{F}_2$ is contained in either $\mathscr{Q}$ or $\mathscr{Q}'$, say $\mathscr{Q}$. As $l(r) = 0$ for $r \in F_{i1} \cup F_{i2}$, this implies that

$$F_{i1} \subset \mathscr{P}_{i1} \cap \mathscr{Q} \text{ and } F_{12} \subset \mathscr{P}_{i2} \cap \mathscr{Q}.$$

Since each $F_{ik}$ is connected, we obtain that

$$\bar{F}_{i1} \cap \bar{F}_{i2} = (\mathscr{P}_i \cap L) \cup \{v\},$$

$$\{v, m_1, m_2\} \subset \bar{F}_{1k} \cap \bar{F}_{2k} \subset M_1 \cup M_2$$

and

$$\bar{F}_{11} \cap \bar{F}_{12} \cap \bar{F}_{21} \cap \bar{F}_{22} = \{v, m_1, m_2\}; i, k = 1, 2.$$

We determine completely the positions of the $F_{ik}$ if the following result ([**4**], p. 10) about algebraic biplanar surfaces is valid for $F$: "A plane turning about its edge ($N_0$) cuts the surface in a curve with a cusp which changes direction to the opposite one whenever the turning plane has passed through one of the two real nodal planes ($\tau_i$)."

Since $\beta \cap F \subset \mathscr{P}_i \cap \mathscr{Q}$, the quote is true if there is a $\beta'$ such that $v$ is the cusp of $\beta' \cap F$ and $\beta' \cap F \subset \mathscr{P}_j \cap \mathscr{Q}'$; $\{i, j\} = \{1, 2\}$. Since $l(r) = 0$ implies that $v$ is the cusp of $\langle N_0, r\rangle \cap F$, it is sufficient to prove that

$$\operatorname{int}(\mathscr{P}_j \cap \mathscr{Q}') \cap F \neq \emptyset.$$

As in 3.1 let $\beta_\lambda$ tend to $\beta$, $v$ the double point of $\beta_\lambda \cap F = \mathscr{L}_\lambda \cup \mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}$ for each $\lambda$. Then $\mathscr{F}_1 \cup \mathscr{F}_2 \subset \mathscr{P}_i$ implies that

$$\mathscr{L}_\lambda \subset \mathscr{P}_j \text{ and } \mathscr{L}_\lambda \cap (M_1 \cup M_2 \cup L) = \{v\}$$

for $\beta_\lambda$ sufficiently close to $\beta$. Clearly, $\mathscr{F}_1 \cup \mathscr{F}_2 \subset \mathscr{Q}$ implies that $\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}$ is also contained in $\mathscr{Q}$ for $\beta_\lambda$ sufficiently close to $\beta$. Then

$$v \in \mathscr{Q} \cap \mathscr{Q}', \{v\} = \mathscr{L}_\lambda \cap (\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda})$$

and the preceding imply that $\mathscr{L}_\lambda \subset \mathscr{Q}'$ for $\mathscr{L}_\lambda$ sufficiently close to $v$. Thus

$$\operatorname{int}(\mathscr{P}_j \cap \mathscr{Q}') \neq \emptyset.$$

We observe that the surface in $P^3$ (suitably coordinatized) defined by

$$x_0^3 + x_0(x_1^2 + x_2^2) + x_1x_2x_3 = 0$$

satisfies 3.0 with

$$M_1 \equiv x_0 = x_1 = 0, M_2 \equiv x_0 = x_2 = 0 \text{ and } L \equiv x_0 = x_3 = 0.$$

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3.5 THEOREM. *Let F be a biplanar surface satisfying* 3.0. *Then*

$$F = F_{11} \cup F_{12} \cup F_{21} \cup F_{22} \cup M_1 \cup M_2 \cup L$$

*where* $F_{ik}$ *is an open region described in* 3.2 *and* 3.4 *with* $v \in \overline{E \cap F_{ik}}$, $i, k = 1, 2$.

## 4. $F$ with seven lines.

4.0 Let $F$ be biplanar with the binode $v$, $l(F) = l(v) + 3 = 7$. Let $M_i$ ($i \in \mathcal{N}_4 = \{1, 2, 3, 4\}$) denote the lines of $F$ through $v$ such that

$$\tau_1 \cap F = M_1 \cup M_2 \cup M_3 \text{ and } \tau_2 \cap F = M_4.$$

Then $\langle M_4, M_j \rangle \cap F$ contains a line $L_j$ with $v \notin L_j$, $j \in \mathcal{N}_3 = \{1, 2, 3\}$. Clearly $L_1$, $L_2$ and $L_3$ are mutually disjoint.

Let $r \in F$, $l(r) = 0$. Then $v$ is the cusp of $\langle N_0, r \rangle \cap F$ and

$$\langle M_i, r \rangle \cap F = M_i \cup S^1(M_i, r) \text{ where}$$

$$v \in M_i \cap S^1(M_i, r) \text{ and } |M_i \cap S^1(M_i, r)| = 2, i \in \mathcal{N}_4.$$

Clearly, $r$ is not an isolated point of $\langle L_j, r \rangle \cap F$ and thus

$$\langle L_j, r \rangle \cap F = L_j \cup S^1(L_j, r); j \in \mathcal{N}_3.$$

We note that

$$\{S^1(M_i, r), S^1(L_j, r)\} \subset \mathcal{F}(r).$$

For $j \in \mathcal{N}_3$, let $m_j[l_j]$ be the point of intersection of $L_j$ and $M_4[M_j]$. Finally, we assume that $M_1$, $M_3$ separate $N_0$, $M_2$.

4.1 Let $\mathcal{P}_1$ and $\mathcal{P}_2[\mathcal{Q}_0$ and $\mathcal{Q}_2]$ be the closed half-spaces of $P^3$ determined by $\tau_1$ and $\tau_2$ $[\langle M_4, M_1 \rangle$ and $\langle M_4, M_3 \rangle]$. We assume that $N_0 \subset \mathcal{Q}_0$, thus

$$M_2 \cup L_2 \cup \tau_2 \subset \mathcal{Q}_2.$$

Finally, let $\mathcal{Q}_{01}$ and $\mathcal{Q}_{03}$ be the closed quarter-spaces of $\mathcal{Q}_0$ determined by $\tau_2$ with $\langle M_4, M_k \rangle \subset \mathcal{Q}_{0k}$, $k = 1, 3$. Then

$$\begin{aligned} P^3 &= \mathcal{Q}_2 \cup \mathcal{Q}_0 \\ &= \mathcal{Q}_2 \cup \mathcal{Q}_{01} \cup \mathcal{Q}_{03} \\ &= \mathcal{Q}_2 \cup (\mathcal{P}_1 \cap \mathcal{Q}_{01}) \cup (\mathcal{P}_2 \cap \mathcal{Q}_{01}) \cup (\mathcal{P}_1 \cap \mathcal{Q}_{03}) \\ &\qquad \cup (\mathcal{P}_2 \cap \mathcal{Q}_{03}). \end{aligned}$$

Let $F_2 = \mathcal{Q}_2 \cap F$ and $F_{ik} = \operatorname{int}(\mathcal{P}_i \cap \mathcal{Q}_{0k}) \cap F$, $i \in \{1, 2\}$ and $k \in \{1, 3\}$. We observe that all the lines of $F$ are contained in $F_2$ and

$$F_{11} \cup F_{13} \cup F_{21} \cup F_{23} = \{r \in \mathcal{Q}_0 \cap F \mid l(r) = 0\}.$$

4.2 Let $\alpha \subset \mathscr{Q}_{0k}$, $\alpha \neq \tau_2$ and $l(\alpha) = 1$, $k \in \{1, 3\}$. By 2.1, there is an $r \in F$ such that

$$\alpha \cap F = M_4 \cap S^1(M_4, r) \text{ and } M_4 \cap S^1(M_4, r) = \{v, q_r\}, v \neq q_r.$$

Thus

$$\alpha = \pi(q_r), q_r \notin \{m_1, m_2, m_3\}, S^1(M_4, r) \cap \mathscr{P}_i \neq \emptyset \text{ and } F_{ik} \neq \emptyset, i = 1, 2.$$

Since $l(r') = 0$ for $r' \in S^1(M_4, r)\backslash M_4$, we obtain that

$$S^1(M_4, r) = (F_{1k} \cap S^1(M_4, r)) \cup \{v, q_r\} \cup (F_{2k} \cap S^1(M_4, r)).$$

Thus $q_r \in \bar{F}_{1k} \cap \bar{F}_{2k}$.

If $\alpha = \langle S^1(M_4, r)\rangle$ tends to $\langle M_4, M_k\rangle$ in $\mathscr{Q}_{0k}$, then $\lim (M_4 \cup S^1(M_4, r)) = M_4 \cup M_k \cup L_k$ and $\lim S^1(M_4, r)$, a curve of order two, imply that

$$\lim S^1(M_4, r) = M_k \cup L_k,$$
$$\lim (\bar{F}_{ik} \cap S^1(M_4, r)) \subset M_k \cup (\mathscr{P}_i \cap L_k)$$

and

$$\lim q_r = m_k, i = 1, 2.$$

If $a = \langle S^1(M_4, r)\rangle$ tends to $\tau_2$, then $v \in \lim S^1(M_4, r) \subseteq M_4$. If $q \neq v$ in $\lim S^1(M_4, r)$, then it is easy to check that $\pi(q) = \tau_2$ and thus $q \notin \{m_1, m_2, m_3\}$. This implies that $M_4 \cap \bar{F}_{ik}$ is the closed segment (of $M_4$) bounded by $v$ and $m_k$ such that

$$\{m_2, m_l\} \cap (M_4 \cap \bar{F}_{ik}) = \emptyset; \{k, l\} = \{1, 3\} \text{ and } i \in \{1, 2\}.$$

Therefore $m_1, m_3$ separates $v, m_2$.

By a similar consideration of $\langle M_k, r\rangle \cap F$ for $r \in F_{ik}$, we obtain that $M_k \cap \bar{F}_{ik}$ is a closed segment bounded by $v$ and $l_k$. Thus $\mathrm{bd}(F_{ik})$ is a triangle in $M_4 \cup M_k \cup L_k$ with vertices $v, m_k$ and $l_k$, $i \in \{1, 2\}$ and $k \in \{1, 3\}$.

4.3 THEOREM. *$E \cap F_{ik} \neq \emptyset$ with $v \in \overline{E \cap F_{ik}}$, $i = 1, 2$ and $k = 1, 3$.*

*Proof.* Let $v$ be the cusp of $\beta \cap F = \mathscr{F}_1 \cup \mathscr{F}_2$ and let $\beta_\lambda$ converge to $\beta$ such that $v$ is the double point of $\beta_\lambda \cap F = \mathscr{L}_\lambda \cup \mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}$ for each $\lambda$. From 2.1, we may assume that $\beta \subset \mathscr{P}_j$, $\{i, j\} = \{1, 2\}$. Then (cf. 3.1)

$$\lim(\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda}) = \mathscr{F}_1 \cup \mathscr{F}_2, \lim \mathscr{L}_\lambda = \{v\}$$

and for $\beta_\lambda$ sufficiently close to $\beta$,

$$\mathscr{F}_{1,\lambda} \cup \mathscr{F}_{2,\lambda} \subset \mathscr{P}_j, \mathscr{L}_\lambda \subset \mathscr{P}_i, l(r) = 0$$

for $r \in \mathscr{L}_\lambda\backslash\{v\}$ and thus $\mathscr{L}_\lambda$ is contained in either $\mathscr{Q}_0$ or $\mathscr{Q}_2$.

Let $\beta_\lambda$ be arbitrarily close to $\beta$. As $\beta \cap (\tau_1 \cup \tau_2) = N_0 \subset \mathcal{Q}_0$, this implies that $\beta_\lambda \cap (\tau_1 \cup \tau_2) \subset \mathcal{Q}_0$ and hence $\mathcal{L}_\lambda \subset \mathcal{P}_i \cap \mathcal{Q}_0$. Clearly, $\mathcal{L}_\lambda$ is contained in $\mathcal{P}_i \cap \mathcal{Q}_{01}$ or $\mathcal{P}_i \cap \mathcal{Q}_{03}$ and there exist $\beta_\lambda$ such that $\mathcal{L}_\lambda$ converges to $v$ in $F_{ik} \cup \{v\}$.

We may apply 3.3.

4.4 From 4.2, $M_4 \cap \bar{F}_{1k} = M_4 \cap \bar{F}_{2k}$ is the segment bounded by $v$ and $m_k$ not containing $m_2$ or $m_l$; moreover, $\pi(q) \subset \mathcal{Q}_{0k}$ for each $q \in M_4 \cap \bar{F}_{1k}$, $\{k, l\} = \{1, 3\}$. Thus $M_4 \cap F_2$ is the segment of $M_4$, bounded by $m_1$ and $m_3$, containing $m_2$.

Let $r \in F$, $l(r) = 0$. Then

$$\langle L_2, r\rangle \cap F = L_2 \cup S^1(L_2, r),$$

$$\langle L_2, r\rangle \cap \tau_2 \cap F = \{m_2\} \text{ and}$$

$$|\langle L_2, r\rangle \cap \tau_1 \cap F| = 3.$$

Hence

$$\langle m_2, \langle L_2, r\rangle \cap N_0\rangle \subset e(S^1(L_2, r)) \text{ (cf. 2.4)},$$

$$l_2 \in i(S^1(L_2, r)) \text{ and}$$

$$|L_2 \cap S^1(L_2, r| = 2.$$

4.5 Lemma. *Let* $r \in F_2$, $l(r) = 0$. *Then* $|\langle m_2, r\rangle \cap F| = 3$.

*Proof.* Since $L_1, L_2$ and $L_3$ are mutually disjoint, $S^1(L_2, r)$ meets $L_1[L_3]$ at a point $l_1^*[l_3^*]$ say. Since $l_2 \in M_2 \cap i(S^1(L_2, r))$, $S^1(L_2, r)$ meets $M_1[M_3]$ at $m_1^*[m_3^*]$ say. Clearly, $m_2$, $m_k^*$ and $l_k^*$ are collinear, $k = 1, 3$.

Let $\mathcal{H}_0$ and $\mathcal{H}_2$ be the closed half-planes of $\langle S^1(L_2, r)\rangle$ determined by $\langle m_2, m_1^*\rangle$ and $\langle m_2, m_3^*\rangle$, $L_2 \subset \mathcal{H}_2$. Then

$$\mathcal{H}_0 = \mathcal{Q}_0 \cap \langle S^1(L_2, r\rangle, \mathcal{H}_2 = \mathcal{Q}_2 \cap \langle S^1(L_2, r)\rangle$$

and $r \in \mathcal{H}_2$. Thus $\langle m_2, \langle L_2, r\rangle \cap N_0\rangle \subset \mathcal{H}_0 \cap e(S^1(L_2, r))$ implies that $|L \cap F| = 3$ for any $L \subset \mathcal{H}_2 | H_0$, and in particular, $|\langle m_2, r\rangle \cap F| = 3$.

4.6 Theorem. *Every* $r \in F_2$ *such that* $l(r) = 0$ *is hyperbolic.*

*Proof.* Let $r \in F_2$, $l(r) = 0$. Then

$$\langle M_4, r\rangle \subset \mathcal{Q}_2, \langle M_4, r\rangle \cap F_2 = M_4 \cup S^1(M_4, r) \text{ and } l(r') = 0$$

for $r' \in S^1(M_4, r)\backslash M_4$. From 4.5, $|\langle m_2, r'\rangle \cap F| = 3$ for each such $r'$ and this implies that $m_2 \in i(S^1(M_4, r))$. Hence

$$|S^1(M_4, r) \cap S^1(L_2, r)| = 2.$$

This and $m_2 \in e(S^1(L_2, r))$ imply

$$e(S^1(L_2, r)) \cap e(S^1(M_4, r)) = \emptyset.$$

Since $\{S^1(L_2, r), S^1(M_4, r)\} \subset \mathscr{F}(r)$, $r \in H$ by 2.5.

4.7 We observe that the shape of the surface (Figure 2) is simple about $M_4$ and complex near $\tau_1$. The latter reflects the degeneration of the curves $\beta \cap F = \mathscr{F}_1 \cup \mathscr{F}_2$ with $v$ is a cusp into $M_1 \cup M_2 \cup M_3$ as $\beta$ tends to $\tau_1$.

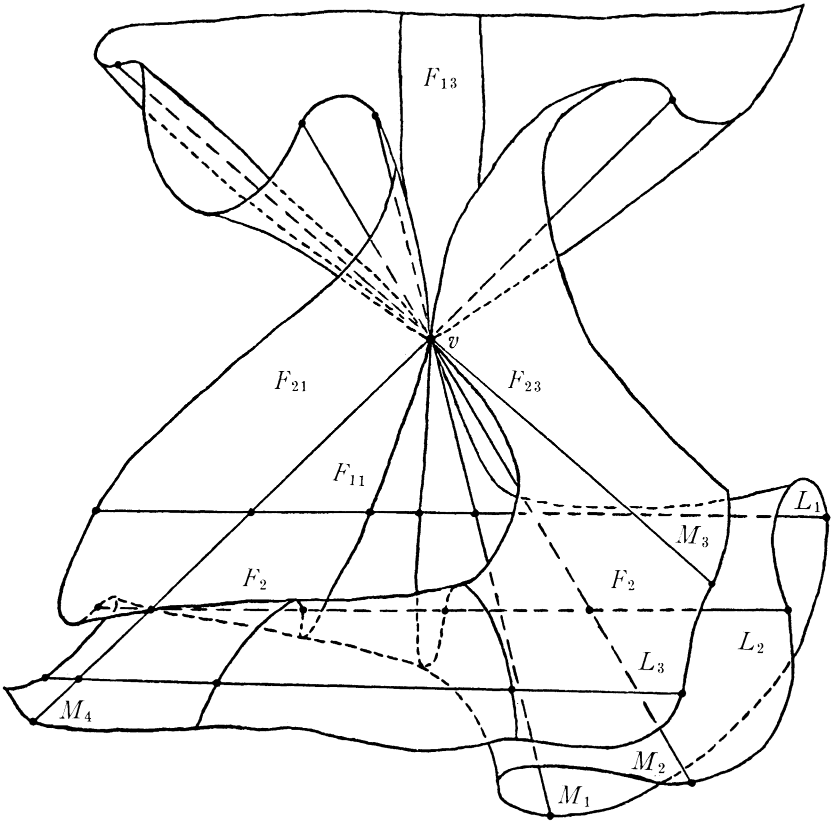


FIGURE 2

We also note that $\beta \cap F \subset \mathscr{F}_1 \cup \mathscr{F}_2$ through $v$ changes "direction" as $\beta$ passes through $\tau_1$ and $\tau_2$. (Consider $\mathscr{Q}_2$ as a plane; that is, identify $M_2$ and $M_3$, and argue as in 3.4.)

Finally, the surface in $P^3$ defined by

$$x_0^3 + x_0x_1^2 - x_0x_2^2 + x_1x_2x_3 = 0$$

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satisfies 4.0 with

$$M_1 \equiv x_1 = x_0 + x_2 = 0,\ M_2 \equiv x_1 = x_0 = 0,\ M_3 \equiv x_1 = x_0 - x_2 = 0,$$
$$M_4 \equiv x_2 = x_0 = 0,\ L_1 \equiv x_0 + x_2 = x_1 - x_3 = 0,\ L_2 \equiv x_0 = x_3 = 0$$

and

$$L_3 \equiv x_0 - x_2 = x_1 + x_3 = 0.$$

4.8 THEOREM. *Let $F$ be a biplanar surface satisfying* 4.0. *Then*

$$F = F_2 \cup F_{11} \cup F_{13} \cup F_{21} \cup F_{23}$$

*where* 1) *every $r \in F_2$ such that $l(r) = 0$ is hyperbolic and* 2) *$F_{ik}$ is an open region described in* 4.1 *to* 4.3; *$i = 1, 2$ and $k = 1, 3$.*

## 5. $F$ with fifteen lines.

5.0 Let $F$ be biplanar with the binode $v$, $l(F) = 15$ and $l(v) = 6$. Let $M_1, M_2, M_3[M_4, M_5, M_6]$ denote the lines of $F$ through $v$ in $\tau_1[\tau_2]$. We assume that $M_1, M_3$ separates $N_0, M_2$ and $M_4, M_6$ separates $N_0, M_5$. Let

$$\mathcal{N} = \{1, 2, 3\} = \{i, j, k\} \text{ and } \mathcal{N}^* = \{4, 5, 6\} = \{\lambda, \mu, \nu\}.$$

For $(i, \lambda) \in \mathcal{N} \times \mathcal{N}^*$, $\langle M_i, M_\lambda\rangle \cap F$ contains a line $L_{i\lambda}$ with $v \notin L_{i\lambda}$.

Let $r \in F$, $l(r) = 0$. For $t \in \mathcal{N} \cup \mathcal{N}^*$,

$$\langle M_t, r\rangle \cap F = M_t \cup S^1(M_t, r)$$

where

$$M_t \cap S^1(M_t, r) = \{v, q_r^t\}, v \neq q_r^t, \pi(q_r^t) = \langle M_t, r\rangle \text{ and}$$
$$S^1(M_t, r) \in \mathscr{F}(r).$$

For $(i, \lambda) \in \mathcal{N} \times \mathcal{N}^*$,

$$\langle L_{i\lambda}, r\rangle \cap F = L_{i\lambda} \cup S^1(L_{i\lambda}, r) \text{ with } S^1(L_{i\lambda}, r) \in \mathscr{F}(r).$$

5.1 Let $(i, \lambda) \in \mathcal{N} \times \mathcal{N}^*$. Since $L_{i\lambda}$ meets $M_i$ and $M_\lambda$, we obtain that

$$L_{i\lambda} \cap (M_j \cup M_k \cup M_\mu \cup M_\nu) = \emptyset.$$

It is now immediate that $L_{i\lambda}$ meets each of $L_{j\mu}$, $L_{j\nu}$, $L_{k\mu}$ and $L_{k\nu}$ and none of $L_{i\mu}$, $L_{i\nu}$, $L_{j\lambda}$ and $L_{k\lambda}$. Thus each of the following flats is a plane:

$$\langle L_{14}, L_{25}, L_{36}\rangle, \langle L_{14}, L_{26}, L_{35}\rangle, \langle L_{15}, L_{24}, L_{36}\rangle,$$
$$\langle L_{15}, L_{26}, L_{34}\rangle, \langle L_{16}, L_{24}, L_{35}\rangle, \langle L_{16}, L_{25}, L_{34}\rangle.$$

Let $L_{i\lambda} \subset \alpha$ such that $l(\alpha) = 1$, that is, $\alpha$ is distinct from $\langle M_i, M_\lambda\rangle$, $\langle L_{j\mu}, L_{k\nu}\rangle$ and $\langle L_{j\nu}, L_{k\mu}\rangle$. Then $\alpha$ meets $M_j$, $M_k$, $M_\mu$, $M_\nu$, $L_{i\mu}$, $L_{i\nu}$, $L_{j\lambda}$ and $L_{k\lambda}$ outside of $L_{i\lambda}$. Thus

$$|\alpha \cap \tau_1 \cap F| = |\alpha \cap \tau_2 \cap F| = 3,$$

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and

$$\alpha \cap F = L_{i\lambda} \cup S^1(L_{i\lambda}, r) \text{ for some } r \in F, l(r) = 0.$$

Let $\mathscr{P}_{i\lambda}$ and $\mathscr{P}_{i\lambda}^*$ be the closed half-spaces of $P^3$ determined by $\langle L_{j\mu}, L_{k\nu}\rangle$ and $\langle L_{j\nu}, L_{k\mu}\rangle$. We assume that $\langle M_i, M_\lambda\rangle \subset \mathscr{P}_{i\lambda}$ and thus $v \notin \mathscr{P}_{i\lambda}^*$. Finally, let $\mathscr{P}_{i\lambda}'$ and $\mathscr{P}_{i\lambda}''$ be the closed quarter-spaces of $\mathscr{P}_{i\lambda}$ determined by $\langle M_i, M_\lambda\rangle$.

5.2 LEMMA. *Let* $\alpha = \langle L_{i\lambda}, r\rangle \subset \mathscr{P}_{i\lambda}$, $l(r) = 0$ *and* $(i, \lambda) \in \mathscr{N} \times \mathscr{N}^*$. *Then*

1) $\alpha \cap N_0 \in e(S^1(L_{i\lambda}, r))$

*and*

2) $M_i \cap L_{i\lambda}[M_\lambda \cap L_{i\lambda}] \in i(S^1(L_{i\lambda}, r))$ *if and only if* $i = 2[\lambda = 5]$.

*Proof*. 1) Since $l(r) = 0$, $\alpha \subset \mathscr{P}_{i\lambda}'$ say. Suppose that $\alpha \cap N_0 \in i(S^1(L_{i\lambda}, r))$. Then the continuity of the plane sections of $F$ through $L_{i\lambda}$ implies that

$$\alpha' \cap N_0 \in i(S^1(L_{i\lambda}, r'))$$

for each $\alpha' = \langle S^1(L_{i\lambda}, r')\rangle \subset \mathscr{P}_{i\lambda}$.

Let $\alpha' = \langle S^1(L_{i\lambda}, r')\rangle$ tend to $\langle M_i, M_\lambda\rangle$. Then

$$\lim \alpha' \cap F = M_i \cup M_\lambda \cup L_{i\lambda} \text{ and}$$
$$\lim S^1(L_{i\lambda}, r') = M_i \cup M_\lambda.$$

As $M_i \cap L_{i\lambda} \neq M_\lambda \cap L_{i\lambda}$, this implies that $|L_{i\lambda} \cap S^1(L_{i\lambda}, r')| = 2$ for $\alpha'$ sufficiently close to $\langle M_i, M_\lambda\rangle$.

Let $p \in L_{i\lambda}\backslash(M_i \cup M_\lambda)$. Since $\lim \alpha' = \langle M_i, M_\lambda\rangle$, we obtain that

$$\alpha' \cap \langle N_0, p\rangle = \langle \alpha' \cap N_0, p\rangle \text{ tends to } \langle v, p\rangle.$$

By 2.1, $v$ is the cusp of $\langle N_0, p\rangle \cap F$ and this implies that

$$|\alpha' \cap \langle N_0, p\rangle \cap F| = k \text{ where } k \text{ is either 1 or 3}$$

for each $\alpha'$ sufficiently close to $\langle M_i, M_\lambda\rangle$.

If $k = 1$, then $\alpha' \cap \langle N_0, p\rangle \cap F = \{p\}$ and $p \in e(S^1(L_{i\lambda}, r'))$. If $k = 3$, then $\alpha' \cap \langle N_0, p\rangle \cap F = \{p, r_1, r_2\}$ (say) where $\{r_1, r_2\} \subset S^1(L_{i\lambda}, r')$ and $r_1, r_2$ separates $\alpha' \cap N_0, p$. Thus $a' \cap N_0 \in i(S^1(L_{i\lambda}, r'))$ implies that $p \in e(S^1(L_{i\lambda}, r'))$.

Since $p$ is any point of $L_{i\lambda}\backslash(M_i \cup M_\lambda)$, it is immediate that

$$L_{i\lambda} \cap S^1(L_{i\lambda}, r') = \emptyset$$

for $\langle S^1\langle L_{i\lambda}, r')\rangle$ sufficiently close to $\langle M_i, M_\lambda\rangle$; a contradiction by the preceding. Thus $\alpha \cap N_0 \in e(S^1(\mathrm{L}_{i\lambda}, r))$.

2) This is immediate since $\alpha \cap N_0 \in e(S^1(L_{i\lambda}, r))$, $|\alpha \cap \tau_j \cap F| = 3$ and $M_1, M_3[M_4, M_6]$ separates $N_0, M_2[N_0, M_5]$; $j = 1, 2$.

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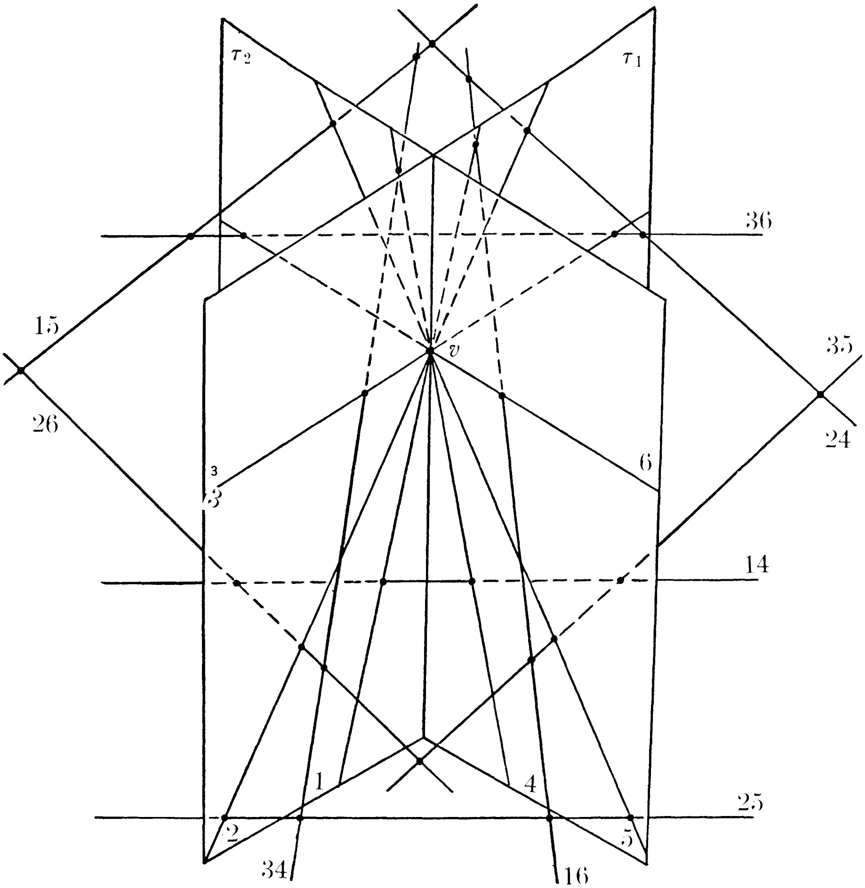



FIGURE 3

We now determine the configuration (cf. Figure 3) of the lines of $F$, or more precisely, the configuration of the lines of $F$ not containing $v$. Since $L_{i\lambda}$ intersects $M_i$, $M_\lambda$, $L_{j\mu}$, $L_{j\nu}$, $L_{k\mu}$ and $L_{k\nu}$ (cf. 5.1), it is sufficient to determine the distribution of these points of intersection in $L_{i\lambda}$, $(i, \lambda) \in \mathscr{N} \times \mathscr{N}^*$.

5.3 Let $\alpha' = \langle S^1(L_{i\lambda}, r')\rangle$ be a sequence of planes tending to $\alpha$; $l(\alpha) = 3$ and $|L_{i\lambda} \cap S^1(L_{i\lambda}, r')| = 2$ for $\alpha'$ sufficiently close to $\alpha$. Put

$$\mathscr{M}_{i\lambda}(r') = L_{i\lambda} \cap S^1(L_{i\lambda}, r').$$

Then lim $M_{i\lambda}(r')$ is the set consisting of the point(s) of intersection of $L_{i\lambda}$ with the other lines of $\alpha \cap \bar{F}$.

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the closed half-spaces of $P^3$ determined by $\tau_1$ and $\tau_2$.

i) Let $\langle S^1(L_{25}, r)\rangle \subset \mathscr{P}_{25}$. From 5.2,

$$L_{25} \cap (M_2 \cup M_5) \subset i(S^1(L_{25}, r))$$

and therefore, $|M_{25}(r)| = 2$ and either

$$\mathcal{M}_{25}(r) \subset \mathcal{P}_1 \text{ or } \mathcal{M}_{25}(r) \subset \mathcal{P}_2.$$

We assume that $\langle L_{14}, L_{36}\rangle \subset \mathcal{P}_{25}'$, $\langle L_{16}, L_{34}\rangle \subset \mathcal{P}_{25}''$ and $\mathcal{M}_{25}(r') \subset \mathcal{P}_1$ for some $\langle S^1(L_{25}, r')\rangle \subset \mathcal{P}_{25}'$. Then clearly $\mathcal{M}_{25}(r) \subset \mathcal{P}_1$ and $M_{25}(r) \cap \mathcal{P}_2 = \emptyset$ for each $\langle S^1(L_{25}, r)\rangle \subset \mathcal{P}_{25}'$. As $\mathcal{P}_{25}'$ is bounded by $\langle M_2, M_5\rangle$ and $\langle L_{14}, L_{36}\rangle$, this implies that

$$L_{25} \cap (L_{14} \cup L_{36}) \subset \mathcal{P}_1.$$

The continuity of the plane sections of $\mathcal{P}_{25} \cap F$ through $L_{25}$ and the preceding imply that $M_{25}(r) \subset \mathcal{P}_2$ and $\mathcal{M}_{25}(r) \cap \mathcal{P}_1 = \emptyset$ for each $\langle S^1(L_{25}, r)\rangle \subset \mathcal{P}_{25}''$. Thus $\langle L_{16}, L_{34}\rangle \subset \mathcal{P}_{25}''$ yields that

$$L_{25} \cap (L_{16} \cap L_{34}) \subset \mathcal{P}_2.$$

Hence $M_2 \cap L_{25}, M_5 \cap L_{25}$ separates $L_{14} \cap L_{25}, L_{16} \cap L_{25}$ but neither $L_{14} \cap L_{25}, L_{36} \cap L_{25}$ nor $L_{16} \cap L_{25}, L_{34} \cap L_{25}$.

ii) Let $\langle S^1(L_{2\lambda}, r)\rangle \subset \mathcal{P}_{2\lambda}$, $\lambda \in \mathcal{N}^*\backslash\{5\}$. From 5.2,

$$M_2 \cap L_{2\lambda} \in i(S^1(L_{2\lambda}, r)) \text{ and}$$

$$M_\lambda \cap S^1(L_{2\lambda}, r) \in e(S^1(L_{2\lambda}, r)).$$

Thus $|\mathcal{M}_{2\lambda}(r)| = 2$ and

$$|\mathcal{M}_{2\lambda}(r) \cap \mathcal{P}_1| = |\mathcal{M}_{2\lambda}(r) \cap \mathcal{P}_2| = 1.$$

As $\langle S^1(L_{2\lambda}, r)\rangle$ varies between $\langle L_{1\mu}, L_{3\nu}\rangle$ and $\langle L_{1\nu}, L_{3\mu}\rangle$ in $\mathcal{P}_{2\lambda}$, we obtain that

$$M_2 \cap L_{2\lambda}, M_\lambda \cap L_{2\lambda} \text{ separates both } L_{1\mu} \cap L_{2\lambda}, L_{3\nu} \cap L_{2\lambda} \text{ and}$$
$$L_{1\nu} \cap L_{2\lambda}, L_{3\mu} \cap L_{2\lambda}.$$

By a similar argument, we obtain that

$$M_i \cap L_{i5}, M_5 \cap L_{i5} \text{ separates both } L_{j4} \cap L_{i5}, L_{k6} \cap L_{i5} \text{ and}$$
$$L_{j6} \cap L_{i5}, L_{k4} \cap L_{i5}$$

for $i \in \mathcal{N}\backslash\{2\}$.

iii) Let $\langle S^1(L_{i\lambda}, r)\rangle \subset \mathcal{P}_{i\lambda}$, $i \in \mathcal{N}\backslash\{2\}$ and $\lambda \in \mathcal{N}^*\backslash\{5\}$. Then

$$L_{i\lambda} \cap (M_i \cup M_\lambda) \subset e(S^1(L_{i\lambda}, r))$$

by 5.2. By arguing as in the preceding cases, we obtain that

$$M_i \cap L_{i\lambda}, M_\lambda \cap L_{i\lambda} \text{ does not separate } L_{j\mu} \cup L_{i\lambda}, L_{k\nu} \cap L_{i\lambda} \text{ or}$$
$$L_{j\nu} \cap L_{i\lambda}, L_{k\mu} \cap L_{i\lambda}.$$

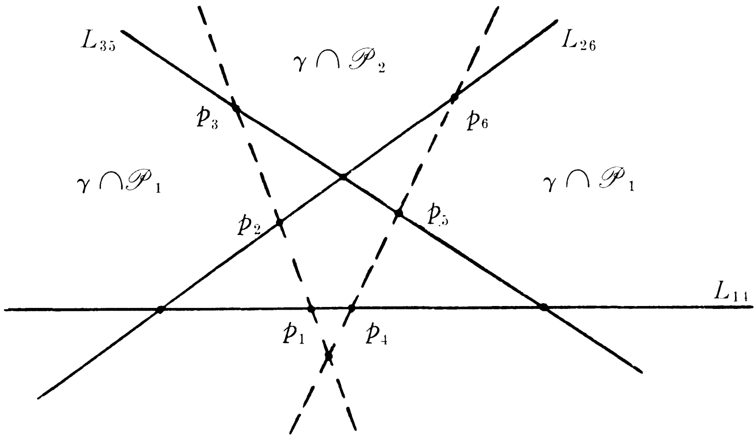
iv) We note that it must still be determined whether $M_i \cap L_{i\lambda}$, $M_\lambda \cap L_{i\lambda}$ separates $L_{j\mu} \cap L_{i\lambda}, L_{j\nu} \cap L_{i\lambda}$ for $(i, \lambda) \in \mathcal{N} \times \mathcal{N}^*\backslash\{(2, 5)\}$.

Consider $\gamma = \langle L_{14}, L_{26}, L_{35} \rangle$. Let $\gamma \cap M_t[\gamma \cap L_{i\lambda}]$ be the point $p_t[l_{i\lambda}], t \in \mathscr{N} \cup \mathscr{N}^*$ and $(i, \lambda) \in \mathscr{N} \times \mathscr{N}^* \setminus \{(1, 4), (2, 6), (3, 5)\}$. Then

$$\{p_1, p_4, l_{25}, l_{36}\} \subset L_{14}, \{p_2, p_6, l_{15}, l_{34}\} \subset L_{26}, \{p_3, p_5, l_{16}, l_{24}\} \subset L_{35}$$

and $\gamma \cap \mathscr{P}_1$ and $\gamma \cap \mathscr{P}_2$ are the closed half-planes of $\gamma$ determined by $\langle p_1, p_2, p_3 \rangle$ and $\langle p_4, p_5, p_6 \rangle$. Finally, $\gamma \cap N_0 = (p_1, p_2) \cap \langle p_4, p_5 \rangle$ and $p_1, p_3[p_4, p_6]$ separates $\gamma \cap N_0, p_2[\gamma \cap N_0, p_4]$.

Since $p_1, p_4$ does not separate $L_{26} \cap L_{14}, L_{35} \cap L_{14}$ from iii), we assume that $L_{14} \cap (L_{26} \cup L_{35}) \subset \mathscr{P}_1$ say. From ii), $p_3, p_5$ separates $L_{14} \cap L_{35}$, $L_{26} \cap L_{35}$ and therefore $L_{26} \cap L_{35} \in \mathscr{P}_2$. Since $\{l_{25}\} = \langle p_2, p_5 \rangle \cap L_{14}$ and $\{l_{36}\} = \langle p_3, p_6 \rangle \cap L_{14}$, we obtain (cf. Figure 4) that $\{l_{25}, l_{36}\} \subset \mathscr{P}_1$ and thus $p_1, p_4$ does not separate $l_{25}, L_{26} \cap L_{14}$ or $l_{36}, L_{35} \cap L_{14}$.



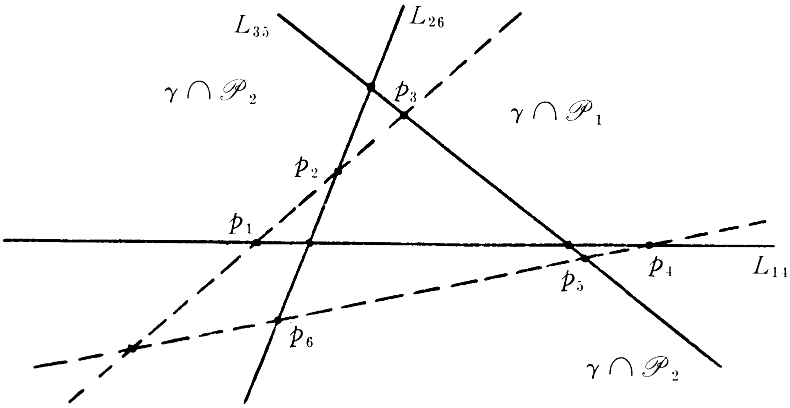




FIGURE 4

Similarly, $p_2, p_6$ does not separate $l_{15}, L_{14} \cap L_{26}$ or $l_{34}, L_{35} \cap L_{26}$ and $p_3, p_5$ does not separate $l_{24}, L_{14} \cap L_{35}$ or $l_{16}, L_{26} \cap L_{35}$. Then ii) yields that ($p_2, p_6$ separates $l_{15}, L_{35} \cap L_{26}$ and $l_{34}, L_{14} \cap L_{26}$ and) $p_3, p_5$ separates $l_{16}, L_{14} \cap L_{35}$ and $l_{24}, L_{26} \cap L_{35}$.

By arguing as in the preceding for suitable $i$ and $\lambda$, we obtain that for $(i, \lambda) \in \mathscr{N} \times \mathscr{N}^*\backslash\{(2, 5)\}$,

$$M_t \cap L_{i\lambda},\ M_\lambda \cap L_{i\lambda} \text{ separates [does not separate] } L_{j\mu} \cap L_{i\lambda},\ L_{j\nu} \cap L_{i\lambda}$$

if $\lambda = 5[\lambda \neq 5]$.

5.4 LEMMA. *Let $r \in F, l(r) = 0$. Then $\langle S^1(L_{25}, r)\rangle \cap N_0 \in e(S^1(L_{25}, r))$ if and only if $r \in \mathscr{P}_{25}$.*

*Proof.* We recall that $\mathscr{P}_{25}{}^*$ is the closed half-space of $P^3$ bounded by $\langle L_{14}, L_{36}\rangle$ and $\langle L_{16}, L_{34}\rangle$ and not containing $\langle M_2, M_5\rangle$. Hence we can choose $\langle S^1(L_{25}, r)\rangle \subset \mathscr{P}_{25}{}^*$ to vary continuously between $\langle L_{14}, L_{36}\rangle$ and $\langle L_{16}, L_{34}\rangle$. Then $\mathscr{M}_{25}(r)$ varies between $L_{25} \cap (L_{14} \cup L_{36}) \subset \mathscr{P}_1$ and $L_{25} \cap (L_{16} \cup L_{34}) \subset \mathscr{P}_2$ from 5.3 (i). As $\mathscr{M}_{25}(r) \cap (M_2 \cup M_5) = \emptyset$, this implies that there is an $\alpha^* = \langle S^1(L_{25}, r^*)\rangle \subset \mathscr{P}_{25}{}^*$ such that

$$\mathscr{M}_{25}(r^*) = L_{25} \cap S^1(L_{25}, r^*) = \emptyset.$$

Hence $L_{25} \cap (M_2 \cup M_5) \subset e(S^1(L_{25}, r^*))$ and $\alpha^* \cap N_0 \in i(S^1(L_{25}, r^*))$. It is now immediate that $\langle S^1(L_{25}, r)\rangle \cap N_0 \in i(S^1(L_{25}, r))$ for each $\langle S^1(L_{25}, r)\rangle \subset \mathscr{P}_{25}{}^*$.

The converse follows from 5.2.

5.5 From 5.3(i) and (iii), we obtain that

$$\{L_{14} \cap L_{25}, L_{36} \cap L_{25}, L_{14} \cap L_{36}\} \subset \mathscr{P}_1$$

and

$$\{L_{16} \cap L_{25}, I_{34} \cap I_{25}, L_{16} \cap L_{34}\} \subset \mathscr{P}_2.$$

By an argument similar to the one in 5.3(iv) with $\gamma = \langle L_{14}, L_{25}, L_{36}\rangle$, we obtain that either $L_{14}, L_{25}$ and $L_{36}$ are concurrent or they determine a triangle $\Delta_1$ in $\langle L_{14}, L_{36}\rangle \cap \mathscr{P}_1$ such that

a) $\Delta_1 \cap (M_t \cup L_{i\lambda}) = \emptyset$ for $t \in \mathscr{N} \cup \mathscr{N}^*$ and

$(i, \lambda) \in \mathscr{N} \times \mathscr{N}^*\backslash\{(1, 4), (2, 5), (3, 6)\}$.

Similarly, either $L_{16}, L_{25}$ and $L_{34}$ are concurrent or they determine a triangle $\Delta_2$ in $\langle L_{16}, L_{34}\rangle \cap \mathscr{P}_2$ such that

b) $\Delta_2 \cap (M_t \cup L_{i\lambda}) = \emptyset$ for $t \in \mathscr{N} \cup \mathscr{N}^*$ and

$(i, \lambda) \in \mathscr{N} \times \mathscr{N}^*\backslash\{(1, 6), (2, 5), (3, 4)\}$.

Let $\mathscr{C}_i$ denote the interior of the cone with vertex $v$ and base $\Delta_i$,

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$i = 1, 2$. It is immediate that no line of $F$ meets $\mathscr{C}_i$ and $M_t \cap \Delta_i = \emptyset$ for $t \in \mathscr{N} \cup \mathscr{N}^*$ implies that $(\tau_1 \cup \tau_2) \cap \overline{\mathscr{C}}_i = \{v\}$. Let $L_i$ be a line through $v$ in $\overline{\mathscr{C}}_i$ such that $L_i \cap \Delta_i = \emptyset$. Then $L_i \not\subset \tau_1 \cup \tau_2$ implies (cf. 1.3) that $L_i \cap F = \{v, r_i\}$ where $r_i \neq v$ and $r_i \in G_i' = \mathscr{C}_i \cap F$. It follows that $G_i'$ is an open region such that $\Delta_i \subseteq \text{bd}(G_i') \subseteq \Delta_i \cup \{v\}$ and $l(r) = 0$ for each $r \in G_i'$, $i = 1, 2$.

Let $r_\lambda$ be a sequence of points in $G_1' \cup G_2'$ converging to $v$. Let $T$ be a line of accumulation of $\langle v, r_\lambda \rangle$. Clearly $T \subseteq \tau_1 \cup \tau_2$. But $r_\lambda \in G_1' \cup G_2' \subset \overline{\mathscr{C}}_1 \cup \overline{\mathscr{C}}_2$ implies that $\langle v, r_\lambda \rangle \subset \overline{\mathscr{C}}_1 \cup \overline{\mathscr{C}}_2$ and hence $T \subset \overline{\mathscr{C}}_1 \cup \overline{\mathscr{C}}_2$. Since $(\tau_1 \cup \tau_2) \cap (\overline{\mathscr{C}}_1 \cup \overline{\mathscr{C}}_2) = \{v\}$, this is a contradiction.

Thus $\text{bd}(G_i') = \Delta_i$ and $G_i'$ satisfies the hypotheses of 1.5.5, $i = 1, 2$.

It is easy to check that a), b) and the continuity of the plane sections of $F$ through $L_{25}$ yield that $\bar{G}_1' \cup \bar{G}_2' \subset \mathscr{P}_{25}^*$. Finally, this and 5.4 readily imply that $\langle L_{14}, L_{36} \rangle \cap N_0$ and $\Delta_1[\langle L_{16}, L_{34} \rangle \cap N_0$ and $\Delta_2]$ are contained in the same closed half-plane of $\langle L_{14}, L_{36} \rangle[\langle L_{16}, L_{34} \rangle]$ determined by $L_{14}$ and $L_{36}$ $[L_{16}$ and $L_{34}]$.

If $L_{14}$, $L_{25}$ and $L_{36}[L_{16}$, $L_{25}$ and $L_{34}]$ are concurrent, let $G_1[G_2]$ denote their point of intersection. Otherwise, let $G_1 = \overline{G_1'}[G_2 = \overline{G_2'}]$. We summarize our results.

5.6 THEOREM. *If $G_i$ is not a point, then $G_i$ is a closed triangular region in $\mathscr{P}_{25}^*$ such that $l(r) = 0$ for each $r \in \text{int}(G_i)$ and $G_i \cap E \neq \emptyset$, $i = 1, 2$.*

5.7 Put $\{i, j\} = \{1, 2\}$ and let $\beta \subset \mathscr{P}_i$, $l(\beta) = 0$. Then $v$ is the cusp of $\beta \cap F = \mathscr{F}_1 \cup \mathscr{F}_2$ by 2.1. Let $\beta_\gamma$ tend to $\beta$ such that $v$ is the double point of $\beta_\gamma \cap F = \mathscr{L}_\gamma \cup \mathscr{F}_{1,\gamma} \cup \mathscr{F}_{2,\gamma}$ for each $\gamma$. Then (cf. 3.1) $\mathscr{F}_{1,\gamma} \cup \mathscr{F}_{2,\gamma}$ tends to $\mathscr{F}_1 \cup \mathscr{F}_2$ in $\mathscr{P}_i$, $\mathscr{L}_\gamma$ tends to $v$ in $\mathscr{P}_j$ and $\mathscr{L}_\gamma \cap L_{k\lambda} = \emptyset$ for $\mathscr{L}_\gamma$ sufficiently close to $v$ and $(k, \lambda) \in \mathscr{N} \times \mathscr{N}^*$.

Let $\mathscr{Q}_0$ and $\mathscr{Q}_0^*$ be the open half-spaces of $P^3$ determined by $\langle M_1, M_4 \rangle$ and $\langle M_3, M_6 \rangle$. We assume that $N_0 \subset \bar{Q}_0$ and $M_2 \cup M_5 \subset \bar{Q}_0^*$. It is immediate that $\lim \beta_\gamma = \beta$ implies that

$$\beta_\gamma \cap (\tau_1 \cup \tau_2) \subset \overline{\mathscr{Q}}_0 \text{ and } \mathscr{L}_\gamma \subset \overline{\mathscr{Q}}_0$$

for $\beta_\gamma$ sufficiently close to $\beta$.

We observe that $\mathscr{Q}_0 \cap \mathscr{P}_i$ is the union of two disjoint connected sets. Clearly, there is a sequence of loops $\mathscr{L}_\gamma$ converging to $v$ not only in $\overline{\mathscr{Q}}_0 \cap \mathscr{P}_i$ but also in the closure (in $P^3$) of each component of $\mathscr{Q}_0 \cap \mathscr{P}_i$.

From 5.5, $L_{14} \cap L_{36} \in \mathscr{P}_1$ and thus one component of $\mathscr{Q}_0 \cap \mathscr{P}_2$ is bounded by $\tau_1$, $\tau_2$ and $\langle M_1, M_4 \rangle$ and the other by $\tau_1$, $\tau_2$ and $\langle M_3, M_6 \rangle$. We denote these components by $\mathscr{Q}_{14}$ and $\mathscr{Q}_{36}$ respectively.

Let $F_{k\mu} = \mathscr{Q}_{k\mu} \cup F$; $(k, \mu) = (1, 4), (3, 6)$. It is clear that

$$\bar{F}_{k\mu} \cap (\tau_1 \cup \tau_2) \subset M_k \cup M_\mu$$

and hence

$$\text{bd}(\bar{F}_{k\mu}) \subset M_k \cup M_\mu \cup L_{k\mu}.$$

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Thus a line of $F$ passing through some point of $F_{k\mu}$ is $M_k$, $M_\mu$ or $L_{k\mu}$. Since $F_{k\mu} \subset \mathcal{Q}_{k\mu}$ and $\mathcal{Q}_{k\mu} \cap (M_k \cup M_\mu \cup L_{k\mu}) = \emptyset$ by definition, we obtain that $l(r) = 0$ for $r \in F_{k\mu}$.

As there is a sequence of loops in $F_{k\mu} \cup \{v\}$ converging to $v$, this implies that $\mathrm{bd}(F_{k\mu})$ is a triangle, determined by $M_k$, $M_\mu$ and $L_{k\mu}$, and $F_{k\mu}$ contains a sequence of elliptic points converging to $v$ (cf. 3.2 and 3.3), $(k, \mu) = (1, 4), (3, 6)$.

We wish to determine a region similar to $F_{14}$ and $F_{36}$ in each of the components of $\mathcal{Q}_0 \cap \mathcal{P}_1$. We know that there is a suitable sequence of loops in the closure of each component converging to $v$ but $L_{16} \cap L_{34} \in \mathcal{P}_2$ implies that

$$(\mathcal{Q}_0 \cap \mathcal{P}_1) \cap (L_{16} \cup L_{34}) \neq \emptyset.$$

It is easy to check that one component of $\mathcal{Q}_0 \cap \mathcal{P}_1$ meets $L_{16}$ and the other meets $L_{34}$. Hence, we consider the subsets of $\mathcal{Q}_0 \cap \mathcal{P}_1$ bounded by $\tau_1$, $\tau_2$ and either $\langle M_1, M_6 \rangle$ or $\langle M_3, M_4 \rangle$.

Let $\mathcal{Q}_{l\nu}$ be the maximal open connected subset of $\mathcal{Q}_0 \cap \mathcal{P}_1$ bounded by $\tau_1$, $\tau_2$ and $\langle M_l, M_\nu \rangle$; $(l, \nu) = (1, 6), (3, 4)$. Let $F_{l\nu} = \mathcal{Q}_{l\nu} \cap F$. By arguing as in the preceding, we obtain that $\mathrm{bd}(F_{l\nu})$ is a triangle determined by $M_l$, $M_\nu$ and $L_{l\nu}$, $l(r) = 0$ for $r \in F_{l\nu}$ and $F_{l\nu}$ contains a sequence of elliptic points converging to $v$.

Finally, we note that $F_{14}$, $F_{16}$, $F_{34}$ and $F_{36}$ are the regions $F_{11}$, $F_{12}$, $F_{21}$ and $F_{22}$ in 3.5 when we identify $M_1$, $M_2$, $M_3$ in $\tau_1$ and $M_4$, $M_5$, $M_6$ in $\tau_1$. Thus the curves $\beta \cap F = \mathcal{F}_1 \cup \mathcal{F}_2$ with the cusp $v$ change "direction" as $\beta$ passes through $\tau_1$ and $\tau_2$.

5.8 THEOREM. *There exist four open triangular regions $F_{i\lambda}$ in $F$ such that*

1) $l(r) = 0$ *for* $r \in F_{i\lambda}$,
2) $E \cap F_{i\lambda} \neq \emptyset$ *with* $v \in \overline{E \cap F_{i\lambda}}$

*and*

3) $\mathrm{bd}(F_{i\lambda}) \subset M_i \cup M_\lambda \cup L_{i\lambda}$, $(i, \lambda) \in \{1, 3\} \times \{4, 6\}$.

5.9 Let $\mathcal{Q}_2$ and $\mathcal{Q}_2^*$[$\mathcal{Q}_5$ and $\mathcal{Q}_5^*$] be the closed half-spaces of $P^3$ determined by $\langle M_2, M_4 \rangle$ and $\langle M_2, M_6 \rangle$[$\langle M_5, M_1 \rangle$ and $\langle M_5, M_3 \rangle$]. We assume that $\tau_1 \subset \mathcal{Q}_2$ and $\tau_2 \subset \mathcal{Q}_5$. Then $N_0 \subset \mathcal{Q}_2 \cap \mathcal{Q}_5$, $M_5 \cup L_{25} \subset \mathcal{Q}_2^*$ and $M_2 \cup L_{25} \subset \mathcal{Q}_5^*$.

Let $M_t \subset \alpha_t$, $l(\alpha_t) = 1$; $t = 2, 5$. Then

$$\alpha_t \cap F = M_t \cup S^1(M_t, r_t)$$

for some $r_t \in F$ and

$$M_t \cap S^1(M_t, r_t) = \{v, p_t\}$$

where $v \neq p_t$ and $\pi(p_t) = \alpha_t = \langle S^1(M_t, r_t) \rangle$.

5.10 LEMMA. *$M_t \cap L_{25} \in i(S^1(M_t, r))$ if and only if $\langle S^1(M_t, r)\rangle \subset \mathscr{Q}_t$, $t = 2, 5$.*

*Proof.* Let $\alpha = \langle S^1(M_2, r)\rangle$ converge to $\tau_1$ in $\mathscr{Q}_2$. Then

$$\lim \alpha \cap \tau_2 = N_0 \text{ and } \lim S^1(M_2, r) = M_1 \cup M_3.$$

Since $\alpha \cap \tau_2$ is a tangent of $S^1(M_2, r)$, $\alpha \cap \tau_2 \subset \overline{e(S^1(M_2, r))}$ and thus

$$N_0 \subset \lim \overline{e(S^1(M_2, r))}.$$

As $\lim S^1(M_2, r) = M_1 \cup M_3$, this implies that $\lim \overline{e(S^1(M_2, r))}$ and $\lim i(S^1(M_2, r))$ are the closed half-planes of $\tau_1$ determined by $M_1$ and $M_3$. Since $M_1$, $M_3$ separates $N_0$, $M_3$, we obtain that

$$M_2 \subset \lim i(S^1(M_2, r)).$$

Hence $M_2 \cap (L_{24} \cup I_{25} \cup I_{26}) \subset i(S^1(M_2, r))$ for $S^1(M_2, r)$ sufficiently close to $\tau_1$. The "if" condition now follows from the continuity of the plane sections of $\mathscr{Q}_2 \cap F$ through $M_2$.

We note that $\langle M_2, M_5\rangle$ is the common boundary of two quarter-spaces of $\mathscr{Q}_2^*$. If there is in each quarter-space a point $r$ such that $M_2 \cap I_{25} \in e(S^1(M_2, r))$, then the "only if" condition follows as in the preceding.

From the proof of 5.4, there are planes $\gamma$ such that $\gamma \cup F = I_{25} \cup S^1$ and $I_{25} \cap S^1 = \emptyset$. Clearly, there is a $\bar{\gamma}$ such that

$$\bar{\gamma} \cap F = L_{25} \cup S^1,\ L_{25} \cap S^1 = \emptyset$$

and $\bar{\gamma}$ does not contain the points $M_4 \cap L_{24}$ and $M_6 \cap L_{26}$.

Since $L_{25} \cap S^1 = \emptyset$, $M_2 \cap L_{25} \in e(S^1)$ and there are points $\bar{r}_1 \neq \bar{r}_2$ in $S^1$ such that

$$M_2 \cap L_{25} \in \pi(\bar{r}_1) \cap \pi(\bar{r}_2).$$

As $\{M_4 \cap L_{24}, M_6 \cap L_{26}\} \cap \bar{\gamma} = \emptyset$, this implies that

$$|\bar{\gamma} \cap \langle M_2, M_4\rangle \cap F| = |\bar{\gamma} \cap \langle M_2, M_6\rangle \cap F| = 3$$

and

$$\bar{\gamma} \cap (L_{24} \cup L_{26} \cup M_2 \cup M_4) \subset S^1.$$

Then $M_2 \cap L_{25} \in \pi(\bar{r}_1) \cap \pi(\bar{r}_2)$ umplies that the lines $\bar{\gamma} \cap \langle M_2, M_3\rangle$, $\bar{\gamma} \cap \langle M_2, M_6\rangle$ do not separate $L_{25}$, $\langle M_2 \cap L_{25}, \bar{r}_1\rangle$ and $\langle M_2 \cap L_{25}, \bar{r}_2\rangle$. Thus $\langle M_2, M_5, L_{25}\rangle \subset \mathscr{Q}_2^*$ yields that $\{\bar{r}_1, \bar{r}_2\} \subset \mathscr{Q}_2^*$. It is clear that $\bar{r}_1$ and $\bar{r}_2$ are not contained in the same quarter-space of $\mathscr{Q}_2^*$ determined by $\langle M_2, M_5\rangle$ and $M_2 \cap L_{25} \in e(S^1(M_2, \bar{r}_i))$, $i = 1, 2$.

By a similar argument, we prove the result for $t = 5$.

5.11 Let $F^* = F \backslash (G_1 \cup G_2 \cup F_{14} \cup F_{16} \cup F_{34} \cup F_{36})$. We claim that $r$ is hyperbolic for $r \in F^*$, $l(r) = 0$. By the symmetry between $\mathscr{P}_1$ and

$\mathscr{P}_2$, it is sufficient to prove the claim for $r \in \mathscr{P}_1 \cap F^*$. From 5.5 and 5.7,

$$\mathscr{P}_1 \cap (G_2 \cup F_{14} \cup F_{36}) = \emptyset \text{ and}$$
$$\mathscr{P}_1 \cap F = G_1 \cup F_{16} \cup F_{34} \cup (\mathscr{P}_1 \cap F^*).$$

Let $\bar{r} \in \mathscr{P}_1 \cap F^*$, $l(\bar{r}) = 0$. Then $v$ is the cusp of

$$\beta \cap F = \mathscr{F}_1 \cup \mathscr{F}_2, \beta = \langle N_0, \bar{r} \rangle.$$

Since $N_0$ is the tangent of both $\mathscr{F}_1$ and $\mathscr{F}_2$ at $v$, we obtain that

i) $N_0 \backslash \{v\} \subset e(\mathscr{F}_1) \cup e(\mathscr{F}_2)$.

Let $\beta \cap L_{i\lambda}$ be the point $l_{i\lambda}$, $(i, \lambda) \in \mathscr{N} \times \mathscr{N}^*$. Then $G_2 \subset \mathscr{P}_2$ implies that $l_{16}$, $l_{25}$ and $l_{34}$ are mutually distinct; cf. 5.5. Since

$$v \in \text{bd}(F_{16}) \cap \text{bd}(F_{34}) \text{ and } F_{16} \cap F_{34} = \emptyset,$$

it is clear that $\beta \cap \bar{F}_{16}$ and $\beta \cap \bar{F}_{34}$ are connected one-sided neighbourhoods of $v$ in $\beta \cap F$ bounded by $v$ and $l_{16}$ and $l_{34}$ respectively. As $l(r) = 0$ for $r \in F_{16} \cup F_{34}$, this implies that $l_{i\lambda} \notin \beta \cap (F_{16} \cup F_{34})$ and $l_{16}$, $l_{34}$ separates $l_{25}$, $\langle l_{16}, l_{34} \rangle \cap N_0$.

Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are subarcs of order two and $\langle l_{16}, l_{25}, l_{34} \rangle$ is a line, we assume that

ii) $l_{16} \in \text{int}(\mathscr{F}_1)$, $l_{34} \in \text{int}(\mathscr{F}_2)$ and $l_{25} \in \mathscr{F}_2$.

We put

iii) $U(v) = \beta \cap (\bar{F}_{16} \cup \bar{F}_{34})$.

Then $U(v)$ is a closed neighbourhood of $v$ in $\beta \cap F$ bounded by $l_{16}$ and $l_{34}$ such that $l(r) = 0$ for $r \in U(v) \backslash \{l_{16}, l_{34}, v\}$.

We note that with the possible exception that some or all of $l_{14}$, $l_{36}$ and $l_{25}$ may be coincident, all the other $l_{i\lambda}$'s are mutually distinct. Since $\beta \cap M_t = \{v\}$ for $t \in \mathscr{N} \cup \mathscr{N}^*$ and $M_1, M_3[M_4, M_6]$ separates $M_2$, $N_0[M_5, N_0]$, it is easy to check that a subarc of $\mathscr{F}_1 \cup \mathscr{F}_2$ bounded by

iv) $l_{i4}$ and $l_{i6}$ contains either $v$ or $l_{i5}$, $i \in \mathscr{N}$,

and

v) $l_{1\lambda}$ and $l_{3\lambda}$ contains either $v$ or $l_{2\lambda}$, $\lambda \in \mathscr{N}^*$.

We recall that $G_1$ is either a triangular region bounded by $L_{14}$, $L_{25}$ and $L_{36}$ or the point $l^* = l_{14} = l_{25} = l_{36}$; moreover, $G_1 \subset \mathscr{P}_{25}^*$ and $\Delta_1$ (cf. 5.5) and $\langle L_{14}, L_{36} \rangle \cap N_0$ are contained in the same half-plane bounded by $L_{14}$ and $L_{36}$.

Since $\beta \cap \mathscr{P}_{25}^*$ is the half-plane bounded by

$$\langle l_{16}, l_{25}, l_{34} \rangle \text{ and } \beta \cap \langle L_{14}, L_{25}, L_{36} \rangle \text{ with } v \notin \beta \cap \mathscr{P}_{25}^*,$$

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iii) implies that

vi) $U(v) \subseteq \beta \cap \mathscr{P}_{25} \cap F$ and $\beta \cap P_{25}^* \cap F \subseteq \beta \cap (F^* \cup G_1)$.

We claim that either $\{l_{15}, l_{35}\}$ or $\{l_{24}, l_{26}\}$ is contained in $\beta \cap \mathscr{P}_{25}^*$. Clearly, $l^* = l_{14} = l_{36}$ or $l_{14} = l_{36} \neq l_{25}$ ($l_{25} \in \pi(l_{14}) = \pi(l_{36})$) imply that

$$\beta \cap F = U(v) \cup (\beta \cap \mathscr{P}_{25}^* \cap F).$$

Hence, we may assume that $l_{14} \neq l_{36}$. Then $l_{25} \in \mathscr{F}_2$ implies $l_{14} \in \mathscr{F}_1$ and $l_{36} \in \mathscr{F}_2$ or $l_{14} \in \mathscr{F}_2$ and $l_{36} \in \mathscr{F}_1$ or $\{l_{14}, l_{36}\} \subset \mathscr{F}_1$. If $l_{14} \in \mathscr{F}_1$ and $l_{36} \in \mathscr{F}_2$, then ii) and $v \notin \mathscr{P}_{25}^*$ imply that $\mathscr{P}_{25}^* \cap \mathscr{F}_1[\mathscr{P}_{25}^* \cap \mathscr{F}_2]$ is the subarc of $\mathscr{F}_1 \cup F_2$ bounded by $l_{14}$ and $l_{16}[l_{34}$ and $l_{36}]$ not containing $v$. Hence $\{l_{15}, l_{35}\} \subset \mathscr{P}_{25}^*$ by iv). By similar arguments, we prove the claim in the other two cases.

From 5.9, $\mathscr{Q}_2^* \cap (\mathscr{F}_1 \cup \mathscr{F}_2)[\mathscr{Q}_5^* \cap (\mathscr{F}_1 \cup \mathscr{F}_2)]$ is the subarc bounded by $l_{24}$ and $l_{26}[l_{15}$ and $l_{35}]$ not meeting $N_0$ and, hence, not containing $v$. Thus

$\{l_{24}, l_{26}\}$ or $\{l_{15}, l_{35}\}$ contained in $\beta \cap \mathscr{P}_{25}^*$ and

$\beta \cap \mathscr{P}_{25} \cap F = U(v) \cup (\beta \cap \mathscr{P}_{25} \cap F^*)$

imply that

vii) $\beta \cap \mathscr{P}_{25} \cap F^* \subset \mathscr{Q}_2^* \cup \mathscr{Q}_5^*$.

Finally, we observe that $\beta \cap G_1$ determines the distribution of the $l_{i\lambda}$'s in $\beta \cap F$. Since $\Delta_1$ and $\langle L_{14}, L_{36}\rangle \cap N_0$ are contained in the same half-plane bounded by $L_{14}$ and $L_{36}$, it is easy to check the following:

a) If $\beta \cap G_1 = \emptyset$, then $|\{l_{14}, l_{25}, l_{36}\}| = 3$, $\beta \cap \mathscr{P}_{25} \cap \bar{F}^*$ is the subarc of $\mathscr{F}_1 \cup \mathscr{F}_2$, bounded by $l_{14}$ and $l_{34}$, containing $l_{25}$ but not $v$, and $l_{14}, l_{36}$ separates $l_{25}$, $\langle l_{14}, l_{36}\rangle \cap N_0$.

b) If $\beta \cap G_1$ is a point, then $\beta \cap G_1 = \{l_{25}\}$ and $l_{25} = l^*$ or $l_{25} = l_{14} \neq l_{36}$ or $l_{25} = l_{36} \neq l_{14}$. If $l_{25} = l^*$, then $\beta \cap F^* \subset \mathscr{P}_{25}^*$. If $l_{25} \neq l^*$, then $\beta \cap \mathscr{P}_{25} \cap \bar{F}^*$ is the subarc of $\mathscr{F}_1 \cup \mathscr{F}_2$, bounded by $l_{25}$ and the $l_{14}$ or $l_{36}$ distinct from $l_{25}$, not containing $v$.

c) If $\beta \cap G_1$ is neither empty nor a point, then $l_{25} \notin \{l_{14}, l_{36}\}$ and $\beta \cap G_1$ is the subarc of $\mathscr{F}_1 \cup \mathscr{F}_2$ bounded by $l_{25}$ and $l_{14}$ or $l_{36}$ and not containing $v$. If $l_{14} = l_{36}$, then $\beta \cap F^* \subset \mathscr{P}_{25}^*$. If $l_{14} \neq l_{36}$, then $\beta \cap \mathscr{P}_{25} \cap \bar{F}^*$ is the subarc of $\mathscr{F}_1 \cup \mathscr{F}_2$ bounded by $l_{14}$ and $l_{36}$ and not containing $v$ and $l_{14}, l_{36}$ does not separate $l_{25}$, $\langle l_{14}, l_{36}\rangle \cap N_0$.

From a), b) and c), we readily obtain that

viii) the inflection point of $\beta \cup F$ is contained in $\mathscr{P}_{25} \cup G_1$

and

ix) $r$, $\langle l_{25}, r\rangle \cap \mathscr{F}_1$ separates $l_{25}$, $\langle l_{25}, r\rangle \cap N_0$ for $r \in \operatorname{int}(\mathscr{P}_{25}^*) \cap \mathscr{F}_2$.

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5.12 THEOREM. *Every $r \in F^*$ such that $l(r) = 0$ is hyperbolic.*

*Proof.* As in 5.11, let $\bar{r} \in \mathscr{P}_1 \cap \mathscr{F}^*$ with $l(\bar{r}) = 0$ and $\beta = \langle N_0, \bar{r} \rangle$. Then $v$ is the cusp of

$$\begin{aligned}\beta \cap F &= \mathscr{F}_1 \cup \mathscr{F}_2 \\ &= (\beta \cap \mathscr{P}_{25} \cap F) \cup (\beta \cap \mathscr{P}_{25}^* \cap F) \\ &= U(v) \cup (\beta \cap G_1) \cup (\beta \cap \mathscr{P}_{25} \cap F^*) \cup (\beta \cap \mathscr{P}_{25}^* \cap F^*).\end{aligned}$$

If $\bar{r} \in \beta \cap \mathscr{P}_{25} \cap F^*$, then $\bar{r} \in \mathscr{P}_{25} \cap \mathscr{Q}_2^*$, say, from 5.11 vii). Hence

$$M_2 \cap L_{25} \in i(S^1(L_{25}, \bar{r})) \cap e(S^1(M_2, \bar{r}))$$

by 5.2 and 5.10 respectively. Clearly,

$$|\langle M_2 \cap L_{25}, \bar{r} \rangle \cap F| = 3, |S^1(L_{25}, \bar{r}) \cap S^1(M_2, \bar{r})| = 2 \text{ and}$$
$$e(S^1(L_{25}, \bar{r})) \cap e(S^1(M_2, \bar{r})) = \emptyset.$$

Thus $\bar{r} \in H$ by 5.0 and 2.5.

If $\bar{r} \in \beta \cap \mathscr{P}_{25}^* \cap F^*$, then

$$\{n_0\} = \langle S^1(L_{25}, \bar{r}) \rangle \cap N_0 = \langle l_{25}, \bar{r} \rangle \cap N_0 \subset i(S^1(L_{25}, \bar{r}))$$

by 5.4. We note that

$$n_0 \in e(\mathscr{F}_1) \cup e(\mathscr{F}_2)$$

from 5.11 i) and

$$\bar{r} \in \text{int}(\mathscr{F}_1) \cup \text{int}(\mathscr{F}_2)$$

from 5.11 viii).

Let $i \in \{1, 2\}$. If $\bar{r} \in \text{int}(\mathscr{F}_i)$ and $e(\mathscr{F}_i) \cap e(S^1(L_{25}, \bar{r})) = \emptyset$, then $\bar{r} \in H$ by 2.5 and the theorem is proved. Suppose that

$$\bar{r} \in \text{int}(\mathscr{F}_i) \text{ and } e(\mathscr{F}_i) \cap e(S^1(L_{25}, \bar{r})) \neq \emptyset.$$

Since $\beta \neq \langle S^1(L_{25}, \bar{r}) \rangle$, this implies that

$$i(\mathscr{F}_i) \cap i(S^1(L_{25}, \bar{r})) \neq \emptyset.$$

Then $n_0 \in e(\mathscr{F}_i) \cap i(S^1(L_{25}, \bar{r}))$ yields that

$$\mathscr{F}_i \cap i(S^1(L_{25}, \bar{r})) \neq \emptyset.$$

Since

$$\mathscr{F}_i \cap i(S^1(L_{25}, \bar{r})) \subset \beta \cap \langle S^1(L_{25}, \bar{r}) \rangle \cap F = l_{25} \cup (\beta \cap S^1(L_{25}, \bar{r})),$$

we obtain that

$$\mathscr{F}_i \cap i(S^1(L_{25}, \bar{r})) = \{l_{25}\}.$$

Thus $\beta \cap S^1(L_{25}, \bar{r}) = \{\bar{r}, r'\}$ where $\bar{r} \neq r'$ and $\bar{r}, r'$ does not separate $l_{25}, n_0$.

We note that $l_{25} \in \mathscr{F}_2 \backslash \mathscr{F}_1$ implies that $i = 2$ and $\bar{r} \in \text{int}(\mathscr{P}_{25}^*) \cap \mathscr{F}_2$ and $l_{25} \in \mathscr{F}_2 \cap \mathscr{F}_1$ implies that either $\bar{r}$ or $r'$ is contained in $\text{int}(\mathscr{P}_{25}^*)$

$\cap \mathscr{F}_2$. In either case, the preceding is a contradiction by 5.11 ix). Thus

$$e(\mathscr{F}_i) \cap (S^1(L_{25}, \bar{r})) = \emptyset$$

and $\bar{r} \in H$.

5.13 THEOREM. *Let $F$ be a biplanar surface satisfying* 5.0. *Then*

$$F = G_1 \cup G_2 \cup F_{14} \cup F_{16} \cup F_{34} \cup F_{36} \cup F^*$$

*where* 1) $G_j$ *is a point or a bounded triangular region with* $l(r) = 0$ *for* $r \in G_j$ *and* $E \cap G_j \neq \emptyset, j = 1, 2,$

2) $E \cap F_{i\lambda} \neq \emptyset$ *with* $v \in E \cap F_{i\lambda}$ *and* $l(r) = 0$ *for* $r \in F_{i\lambda}$, $(i, \lambda) \in \{1, 3\} \times \{4, 6\}$, *and*

3) *every* $r \in F^*$ *such that* $l(r) = 0$ *is hyperbolic.*

We refer to Figure 5 for a representation of $F$ with all fifteen lines

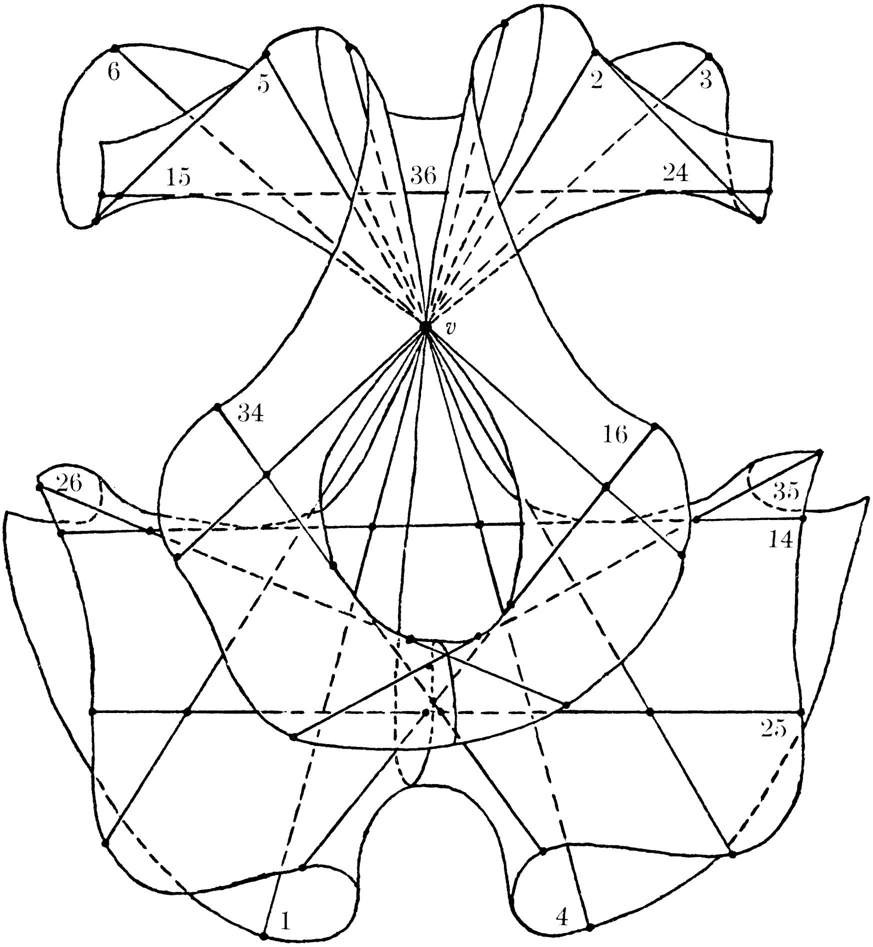


FIGURE 5

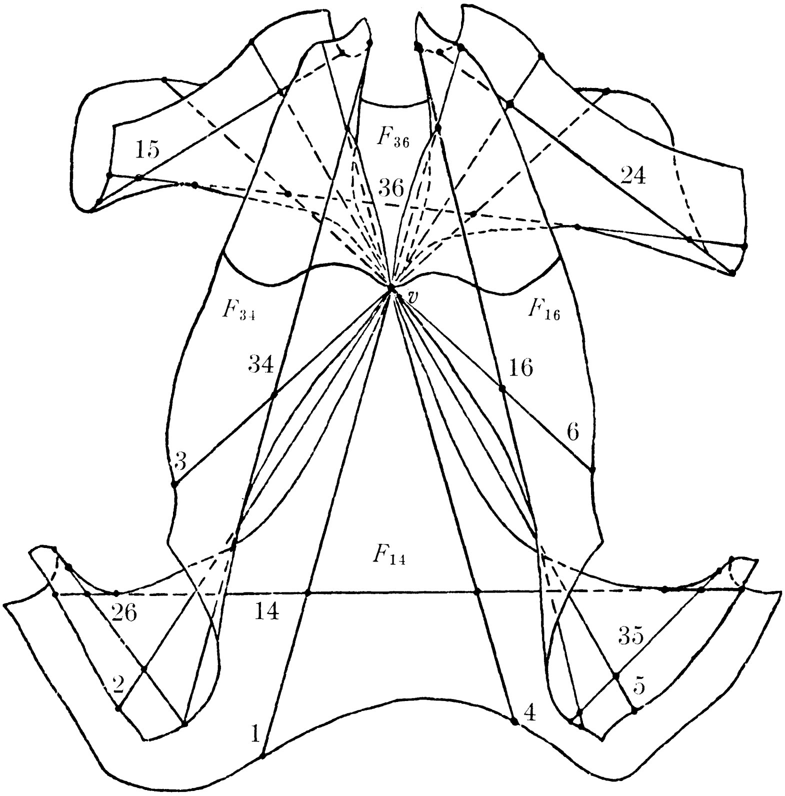


FIGURE 6

depicted. In order to indicate all lines, the collinearities in 5.1 were not accurately represented. In Figure 6, we have a truer representation of $F$ but the line $L_{25}$ is not depicted. We note that for simplicity, the lines of $F$ are labelled by their subscripts.

The surface in $P^3$ defined by $x_0^3 - x_0(x_1^2 + x_2^2) + x_1x_2x_3 = 0$ satisfies 5.0 with

$$M_1 \equiv x_2 = x_0 + x_1 = 0,\ M_2 \equiv x_2 = x_0 = 0,\ M_3 \equiv x_2 = x_0 - x_1 = 0,$$

$$M_4 \equiv x_1 = x_0 + x_2 = 0,\ M_5 \equiv x_1 = x_0 = 0,\ M_6 \equiv x_1 = x_0 - x_2 = 0,$$

$$L_{14} \equiv x_0 + x_1 + x_2 = 2x_0 + x_3 = 0,\ L_{15} \equiv x_0 + x_1 = x_2 + x_3 = 0,$$

$$L_{16} \equiv x_0 + x_1 - x_2 = x_3 - 2x_0 = 0,$$

$$L_{24} \equiv x_0 + x_2 = x_1 + x_3 = 0,\ L_{25} \equiv x_0 = x_3 = 0,$$

$$L_{26} \equiv x_0 - x_2 = x_3 - x_1 = 0,$$

$$L_{34} \equiv x_0 - x_1 + x_2 = 2x_0 - x_3 = 0,\ L_{15} \equiv x_0 + x_1 = x_2 + x_3 = 0,$$

$$L_{36} \equiv x_0 - x_1 - x_2 = 2x_0 + x_3 = 0.$$

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