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REAL HYPERSURFACES OF NON-FLAT COMPLEX SPACE FORMS WITH GENERALIZED ξ- PARALLEL JACOBI STRUCTURE OPERATOR

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Abstract. The aim of the present paper is the classification of real hypersurfaces M equipped with the condition $Al = lA$, $l = R(., \xi)\xi$, restricted in a subspace of the tangent space T_pM of M at a point p. This class is large and difficult to classify, therefore a second condition is imposed: $(\nabla_\xi l)X = \omega(X)\xi + \psi(X)lX$, where $\omega(X)$, $\psi(X)$ are 1-forms. The last condition is studied for the first time and is much weaker than $\nabla_\xi l = 0$ which has been studied so far. The Jacobi Structure Operator satisfying this weaker condition can be called generalized ξ-parallel Jacobi Structure Operator.

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1. Introduction. An n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g).

Real hypersurfaces in $\mathbb{C}P^n$ which are homogeneous, were classified by R. Takagi [**12**]. The same author classified real hypersurfaces in $\mathbb{C}P^n$, with constant principal curvatures in [**13**]. Berndt gave the equivalent result for Hopf hypersurfaces in $\mathbb{C}H^n$ [**1**] where he divided real hypersurfaces into four model spaces, named A_0, A_1, A_2 and B. Analytic lists of constant principal curvatures can be found in the previously mentioned references as well as in [**7**, **9**]. Real hypersurfaces of type A_1 and A_2 in $\mathbb{C}P^n$ and of type A_0, A_1 and A_2 in $\mathbb{C}H^n$ are said to be hypersurfaces of *type A* for simplicity and appear quite often in classification theorems. Real hypersurfaces of type A_1 in $\mathbb{C}H^n$ are divided into types $A_{1,0}$ and $A_{1,1}$ [**7**]. Finally we mention that real hypersurfaces satisfying $\phi A = A\phi$, in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ were classified by Okumura [**10**], and Montiel and Romero [**8**] respectively. For more information and examples on real hypersurfaces, we refer to [**9**].

A Jacobi field along geodesics of a given Riemannian manifold (M, g) plays an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. For any vector field X, the Jacobi operator is defined by R_X: $R_X(Y) = R(Y, X)X$, where R denotes the curvature tensor and Y is a vector field on M. R_X is a self-adjoint endomorphism in the tangent space of M, and is related to the Jacobi differential equation, which is given by $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}} Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ on M, where $\dot{\gamma}$ denotes the velocity vector along γ on M. In a real hypersurface M of a complex space form $M_n(c)$, $c \neq 0$, the Jacobi operator on M with respect to the structure vector field ξ, is called the structure Jacobi operator and is denoted by $lX = R_\xi(X) = R(X, \xi)\xi$. Conditions including this operator, generate larger classes than the conditions including the Riemannian tensor $R(X, Y)Z$. So, operator l has been studied by quite a few authors and under several conditions.

In 2007, Ki, Pérez, Santos and Suh [**6**] classified real hypersurfaces in complex space forms with ξ-parallel Ricci tensor and structure Jacobi operator. Cho and Ki in [**3**] classified the real hypersurfaces whose structure Jacobi operator is symmetric along the Reeb flow ξ and commutes with the shape operator A.

In the present paper we classify real hypersurfaces M satisfying the condition $lA = Al$, restricted in the subspace $\mathbb{D} = ker(\eta)$ of T_pM for every point $p \in M$, where $\ker(\eta)$ consists of all vectors fields orthogonal to the Reeb flow ξ. This class is quite large and rather difficult to be classified, therefore a second condition had to be imposed: $(\nabla_\xi l)X = \omega(X)\xi + \psi(X)lX$, where $\omega(X)$, $\psi(X)$ are 1-forms. This condition is much weaker than the condition $\nabla_\xi l = 0$ that has been used so far [**3**, **4**, **5**, **6**]. Therefore a larger class is produced. In particular, the following theorem is proved:

Theorem 1.1. *Let M be a real hypersurface of a complex space form $M_n(c)$, $n > 2$ $(c \neq 0)$, satisfying*

$$lAX = AlX, \quad \forall X \in \mathbb{D}, \tag{1.1}$$

and

$$(\nabla_\xi l)X = \omega(X)\xi + \psi(X)lX, \tag{1.2}$$

for every vector field $X \in TM$, where $\omega(X)$, $\psi(X)$ are 1-forms. Then M is a Hopf hypersurface. Furthermore, if $\eta(A\xi) \neq 0$ then M is of type A.

The Jacobi Structure Operator satisfying (1.2) will be called *generalized ξ-parallel Jacobi Structure Operator*.

2. Preliminaries. In this section, we explain explicitly the notions that were mentioned in Section 1, as well as the notions that will appear in the paper. We also give a series of equations that will be our basic tools in our calculations and conclusions.

Let M_n be a Kaehlerian manifold of real dimension 2n, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold: $J^2X = -X$, $G(JX, JY) = G(X, Y)$, $\widetilde{\nabla}J = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighbourhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and $\xi = -JN$, $\eta(X) = g(X, \xi)$, $g = G|_M$.

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By properties of the almost complex structure J and the definitions of η and g, the following relations hold **[2]**:

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y). \tag{2.2}$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then, A is symmetric and the following equations are satisfied:

$$\nabla_X \xi = \phi A X, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.3}$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}[&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \tag{2.5}$$

The tangent space T_pM, for every point $p \in M$, is decomposed as following:

$$T_pM = \mathbb{D}^\perp \oplus \mathbb{D},$$

where $\mathbb{D} = ker(\eta) = \{X \in T_pM : \eta(X) = 0\}$.

The subspace $ker(\eta)$ is more usually referred as $\mathbb{D}$ and called holomorphic distribution of M at p. Based on the decomposition of T_pM, by virtue of (2.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{2.6}$$

where $\beta = |\phi\nabla_\xi \xi|$, α is a smooth function on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in ker(\eta)$, provided that $\beta \neq 0$.

If the vector field $A\xi$ is expressed as $A\xi = \alpha\xi$, then ξ is called *principal vector field*.

Finally, differentiation will be denoted by ( ). All manifolds, vector fields, etc., of this paper are assumed to be connected and of class C^∞.

3. Auxiliary lemmas and relations. In this section, we will be working in the set $\mathcal{N} = \{p \in M : \beta \neq 0$ in a neighbourhood around $p\}$. By putting $X = \xi$ in (1.2), combined with (2.3) and (2.6), we obtain $\beta l\phi U = -\omega(\xi)\xi$. The inner product of the last equation with ξ yields $l\phi U = 0$ which is analysed from (2.4) and (2.6) giving $(4\alpha A + c)\phi U = 0$. From the last equation, it follows that $\alpha \neq 0$ in $\mathcal{N}$.

LEMMA 3.1. *Let M be a real hypersurface of a complex space form $M_n(c)$, $n > 2$ $(c \neq 0)$, satisfying (1.1) and (1.2). Then, the following relations hold in $\mathcal{N}$:*

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$$AU = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right) U + \beta\xi, \qquad A\phi U = -\frac{c}{4\alpha}\phi U. \tag{3.1}$$

$$\nabla_\xi \xi = \beta\phi U, \qquad \nabla_U \xi = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\phi U, \qquad \nabla_{\phi U}\xi = \frac{c}{4\alpha}U. \tag{3.2}$$

$$\nabla_\xi U = W_1, \qquad \nabla_U U = W_2, \qquad \nabla_{\phi U} U + \frac{c}{4\alpha}\xi = W_3. \tag{3.3}$$

$$\nabla_\xi \phi U = \phi W_1 - \beta\xi, \quad \nabla_U \phi U = \phi W_2 + \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\xi, \quad \nabla_{\phi U}\phi U = \phi W_3. \tag{3.4}$$

where W_1, W_2, W_3 are vector fields orthogonal to U, ξ.

Proof. From (2.4), we get

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - g(AX, \xi)A\xi, \tag{3.5}$$

which, for $X = U$ yields

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi. \tag{3.6}$$

The scalar product of (3.6) with U yields

$$g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \tag{3.7}$$

where $\gamma = g(lU, U)$. We have already shown in the beginning of this section that $l\phi U = 0 \Rightarrow A\phi U = -\frac{c}{4\alpha}\phi U$ holds. Therefore

$$g(AU, \phi U) = g(A\phi U, U) = 0. \tag{3.8}$$

From (3.7), (3.8) and $g(AU, \xi) = g(A\xi, U) = \beta$ we obtain $AU = (\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha} - \frac{c}{4\alpha})U + \beta\xi + \lambda W$, where W is a vector field satisfying $W \perp \{U, \phi U, \xi\}$. Combining the decomposition of AU with (2.6) and (3.6), we obtain $lU = \gamma U + \alpha\lambda W$.

Summarizing the results so far, we have proved the following:

$$lU = \gamma U + \alpha\lambda W, \quad l\phi U = 0, \tag{3.9}$$

$$AU = \left(\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right) U + \beta\xi + \lambda W, \quad A\phi U = -\frac{c}{4\alpha}\phi U. \tag{3.10}$$

Condition (1.1) yields the next calculations: $AlU = lAU \Rightarrow g(AlU, \xi) = g(lAU, \xi) \Rightarrow g(lU, A\xi) = 0$, since l is symmetric and $l\xi = 0$. Expanding $g(lU, A\xi) = 0$ with the aid of (2.6) and (3.9), we obtain $\gamma = 0$. Now, we expand $AlU = lAU$ with the aid of $\gamma = 0$ and (3.5), obtaining $\lambda = 0$. So, from the conclusions of this paragraph and (3.10), we have proved (3.1).

From equation (3.1) and relation (2.3) for $X = \xi$, $X = U$, $X = \phi U$, we obtain (3.2). Next, we remind of the rule

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{3.11}$$

We define $W_1 = \nabla_\xi U$. By virtue of (3.11) for $X = Z = \xi$, $Y = U$ and for $X = \xi$, $Y = Z = U$, it is shown respectively $\nabla_\xi U \perp \xi$ and $\nabla_\xi U \perp U$. In a similar way, we

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define $W_2 = \nabla_U U$. Equation (3.11) for $X = Y = Z = U$ and $X = Z = U,\ Y = \xi$ yields respectively $\nabla_U U \perp U$ and $\nabla_U U \perp \xi$. Finally, (3.11) for $X = \phi U,\ Y = Z = U$ and $X = \phi U,\ Y = U, Z = \xi$ (with the aid of (3.2)) yields respectively $\nabla_{\phi U} U \perp U$ and $g(\nabla_{\phi U} U, \xi) = -\frac{c}{4\alpha}$. Therefore, we define $W_3 = \nabla_{\phi U} U + \frac{c}{4\alpha}\xi$ and (3.3) has been proved. In order to prove (3.4), we use the second of (1.3) with the following combinations: (i) $X = \xi,\ Y = U$, (ii) $X = Y = U$, (iii) $X = \phi U,\ Y = U$, and make use of (2.6), (3.1), (3.3). □

In order to proceed with the rest of the paper, the following functions are defined:

$$\kappa_1 = g(W_1, \phi U),\quad \kappa_2 = g(W_2, \phi U),\quad \kappa_3 = g(W_3, \phi U). \tag{3.12}$$

Lemma 3.2. *Let M be a real hypersurface of a complex space form $M_n(c)$, $n > 2$ $(c \neq 0)$, satisfying (1.1) and (1.2). Then, the following relations hold in $\mathcal{N}$:*

$$AW_1 = -\frac{c}{4\alpha}W_1, \qquad A\phi W_1 = -\frac{c}{4\alpha}\phi W_1 - \frac{\kappa_1\beta}{\alpha}A\xi.$$

Proof. From (1.2), we obtain $(\nabla_\xi l)U = \omega(U)\xi + \psi(U)lU$. The previous relation is analysed by virtue of (3.9), $\gamma = \delta = \lambda = 0$ and Lemma 3.1, giving $lW_1 = -\omega(U)\xi$. The inner product of the last equation with ξ yields $\omega(U) = 0$ which means $lW_1 = 0$, which is expanded from (3.5) giving $AW_1 = -\frac{c}{4\alpha}W_1$.

In a similar way, (1.2) yields $(\nabla_\xi l)\phi U = \omega(\phi U)\xi + \psi(\phi U)l\phi U$. The last equation is developed by virtue of (3.9), $\epsilon = \delta = \mu = 0$ and Lemma 3.1, giving $l\phi W_1 = -\omega(\phi U)\xi$, whose inner product with ξ yields $\omega(\phi U) = 0$. This means $l\phi W_1 = 0$, which is expanded from (3.5) giving $A\phi W_1 = -\frac{c}{4\alpha}\phi W_1 - \frac{\kappa_1\beta}{\alpha}A\xi$. □

Lemma 3.3. *Let M be a real hypersurface of a complex space form $M_n(c)$, $n > 2$ $(c \neq 0)$, satisfying (1.1) and (1.2). Then, in $\mathcal{N}$ we have $\kappa_1 = -4\alpha$ and $\kappa_2 = -4\beta + \frac{c}{4\alpha\beta}(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha})$.*

Proof. Putting $X = U,\ Y = \xi$ in (2.5), we obtain

$$(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U.$$

Combining the last equation with (2.6) and Lemmas 3.1, 3.2 it follows :

$$\begin{aligned}(U\alpha)\xi + (U\beta)U + \beta W_2 + \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\frac{c}{4\alpha}\phi U \\ -\xi\left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U - \frac{\beta^2}{\alpha}W_1 - (\xi\beta)\xi = 0.\end{aligned}$$

Taking the scalar products of the last relation with ξ and U respectively, we obtain

$$(U\alpha) = (\xi\beta), \tag{3.13}$$

and

$$(U\beta) = \left(\xi\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\right). \tag{3.14}$$

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Combining the last three equations, we have

$$\frac{c}{4\alpha}\left(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha}\right)\phi U+\frac{\beta^2}{\alpha}W_1-\beta W_2=0. \tag{3.15}$$

Putting $X=\phi U,\ Y=\xi$ in (2.5), we obtain

$$(\nabla_{\phi U}A)\xi-(\nabla_\xi A)\phi U=\frac{c}{4}U,$$

which is expanded with the aid of Lemmas 3.1, 3.2 and (2.6), resulting to

$$\left[\frac{3\beta c}{4\alpha}+\alpha\beta+\kappa_1\beta-(\phi U\alpha)\right]\xi \tag{3.16}$$

$$-\left[(\phi U\beta)+\frac{c}{4\alpha}\left(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha}\right)-\beta^2-\kappa_1\frac{\beta^2}{\alpha}\right]U+\frac{c}{4\alpha^2}(\xi\alpha)\phi U-\beta W_3=0.$$

By taking the scalar products of (3.16) with ξ, U, ϕU and making use of (3.12), we acquire respectively

$$(\phi U\alpha)=\frac{3\beta c}{4\alpha}+\alpha\beta+\kappa_1\beta, \tag{3.17}$$

$$(\phi U\beta)=\frac{c}{4\alpha}\left(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha}\right)+\beta^2+\kappa_1\frac{\beta^2}{\alpha}. \tag{3.18}$$

$$\beta\kappa_3=\frac{c}{4\alpha^2}(\xi\alpha). \tag{3.19}$$

Relation $(\nabla_U A)\phi U-(\nabla_{\phi U}A)U=-\frac{c}{2}\xi$ holds due to (2.5). It is further analysed using Lemmas 3.1, 3.2 leading to

$$\begin{aligned}&\frac{c}{4\alpha^2}(U\alpha)\phi U+\left[\frac{c}{2\alpha}\left(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha}\right)+\beta^2-(\phi U\beta)\right]\xi\\&\quad+\left[-\frac{3\beta c}{4\alpha}+\frac{\beta^3}{\alpha}+\left(\phi U\left(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha}\right)\right)\right]U-\frac{c}{4\alpha}\phi W_2-A\phi W_2\\&\quad+AW_3+\left(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha}\right)W_3=0.\end{aligned} \tag{3.20}$$

The scalar product of (3.20) with U, combined with Lemma 3.1, (3.12) and the symmetry of A, yields

$$\frac{\kappa_2\beta^2}{\alpha}-\frac{3\beta c}{4\alpha}+\frac{\beta^3}{\alpha}+\phi U\left(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha}\right)=0.$$

The above equation is modified, first, by expanding the term $\phi U(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha})$ and then, by replacing the terms $(\phi U\alpha)$, $(\phi U\beta)$ from (3.17), (3.18). The result is

$$\kappa_2\beta-\kappa_1\frac{\beta^2}{\alpha}-\frac{c}{4\alpha}\left(\frac{c}{4\alpha}-\frac{\beta^2}{\alpha}\right)-c-\kappa_1\frac{c}{4\alpha}=0. \tag{3.21}$$

On the other hand, the scalar product of (3.15) with ϕU, because of (3.12), yields

$$\kappa_2\beta - \kappa_1\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) = 0. \tag{3.22}$$

From (3.21) and (3.22), we obtain $\kappa_1 = -4\alpha$, $\kappa_2 = -4\beta + \frac{c}{4\alpha\beta}(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha})$. □

The scalar product of (3.15) with ϕW_1 yields:

$$g(\phi W_1, W_2) = g(W_1, \phi W_2) = 0. \tag{3.23}$$

4. The set $\mathcal{N}$ is the empty set. In this section, we prove that $\mathcal{N} = \emptyset$. In order to do that, we need the following lemma:

LEMMA 4.1. *Let M be a real hypersurface of a complex space form $M_n(c)$, $n > 2$ $(c \neq 0)$ satisfying (1.1) and (1.2) in $\mathcal{N}$. Then, $\kappa_3 = 0$ holds in $\mathcal{N}$.*

Proof. Taking the scalar product of (3.20) with ϕU, because of Lemma 3.1, (3.12) and the symmetry of A, we get $(U\alpha) = \frac{4\alpha\beta^2}{c}\kappa_3$. Combining the last equation with (3.13) and (3.14), we have

$$(U\alpha) = (\xi\beta) = \frac{4\alpha\beta^2}{c}\kappa_3, \quad (\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3, \quad (U\beta) = \beta\left(\frac{4\beta^2}{c} + 1\right)\kappa_3. \tag{4.1}$$

By making use of (2.5) for $X = \phi W_2$, $Y = \xi$ and using (2.3), (2.6), we obtain

$$(\phi W_2\alpha)\xi + \alpha\phi A\phi W_2 + (\phi W_2\beta)U + \beta\nabla_{\phi W_2}U$$
$$- A\phi A\phi W_2 - \nabla_\xi A\phi W_2 + A\nabla_\xi\phi W_2 = \frac{c}{4}W_2.$$

The scalar product with ξ, due to (2.2), (2.3), (2.6), (3.11), (3.12), (3.23), (3.1) the symmetry of A and Lemmas 3.1, 3.3, implies

$$(\phi W_2\alpha) = \beta\kappa_3\Big(\frac{16\alpha\beta^2}{c} + \frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\Big). \tag{4.2}$$

From (3.16)–(3.18) and (4.1), it is shown $W_3 = \kappa_3\phi U$, which is combined with (3.17) and Lemma (3.3) giving

$$(W_3\alpha) = 3\beta\left(\frac{c}{4\alpha} - \alpha\right)\kappa_3. \tag{4.3}$$

In a similar way, equation (2.5) yields $(\nabla_{\phi W_1}A)U - (\nabla_U A)\phi W_1 = 0$,which by virtue of Lemma 3.1 is further developed as

$$\left(\phi W_1\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\right)U + \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\nabla_{\phi W_1}U$$
$$+ (\phi W_1\beta)\xi + \beta\phi A\phi W_1 - A\nabla_{\phi W_1}U - \nabla_U A\phi W_1 + A\nabla_U\phi W_1 = 0.$$

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The scalar product of the last equation with ξ by using Lemmas 3.1, 3.3, (2.3), (2.6), (3.11), (3.12) leads to

$$(\phi W_1\beta) = 4\alpha\beta\kappa_3\left(1+\frac{4\beta^2}{c}\right). \tag{4.4}$$

By virtue of Lemma 3.3 and (3.17). (4.1), we have

$$[\phi U, U]\alpha = \big(\phi U(U\alpha)\big) + \frac{3\alpha\beta}{c}\left[8\beta^2 + c - \frac{c^2}{4\alpha^2}\right]\kappa_3.$$

On the other hand from Lemmas 3.1, 3.3 and (4.1)–(4.3), we obtain:

$$[\phi U, U]\alpha = (\nabla_{\phi U}U - \nabla_U\phi U)\alpha = \frac{\alpha\beta}{c}\left(-12\beta^2 + \frac{c^2}{\alpha^2} - 5c - \frac{\beta^2}{\alpha^2}c\right)\kappa_3.$$

The last two equations imply

$$\big(\phi U(U\alpha)\big) = 2\beta\left(\frac{7c}{8\alpha} - \frac{18\alpha\beta^2}{c} - 4\alpha - \frac{\beta^2}{2\alpha}\right)\kappa_3. \tag{4.5}$$

Following a similar way, from the action of $[\phi U, \xi]$ on β we calculate $\big(\phi U(\xi\beta)\big)$. In particular, we expand the derivative $[\phi U, \xi]\beta = \phi U(\xi\beta) - \xi(\phi U\beta)$ by virtue of Lemma 3.3 and (3.18), (4.1), and then calculate the same derivative, from relation $[\phi U, \xi]\beta = (\nabla_{\phi U}\xi - \nabla_\xi\phi U)\beta$, with the aid of Lemmas 3.1, 3.3, (4.1), (4.4). The final result is

$$\big(\phi U(\xi\beta)\big) = 2\beta\left(\frac{3c}{8\alpha} - \frac{18\alpha\beta^2}{c} - 2\alpha + \frac{\beta^2}{2\alpha}\right)\kappa_3. \tag{4.6}$$

Because of (3.13), the relations (4.5) and (4.6) yield

$$\left(\frac{c}{2} - 2\alpha^2 - \beta^2\right)\kappa_3 = 0. \tag{4.7}$$

Let us assume there exists a point $p \in \mathcal{N}$ at which $\kappa_3 \neq 0$. Then, there exists a neighbourhood V_1 of p such that $\kappa_3 \neq 0$ in V_1. Therefore, (4.7) yields $2\alpha^2 + \beta^2 = \frac{c}{2}$, which is differentiated along ξ, with the aid of (4.1) and $\kappa_3 \neq 0$, giving $2\alpha^2 + \beta^2 = 0$ which is a contradiction. This means there are no points of $\mathcal{N}$ where $\kappa_3 \neq 0$ and so $\kappa_3 = 0$ holds in $\mathcal{N}$. □

From Lemma 4.1 and (4.1) we have $(U\alpha) = (\xi\alpha) = 0 \Rightarrow [U, \xi]\alpha = 0$. But the last equation, because of Lemmas 3.1, 3.3 yields

$$\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)(\phi U\alpha) - (W_1\alpha) = 0. \tag{4.8}$$

Using the same Lemmas and relations, we prove $[U, \xi]\beta = 0$ and

$$\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)(\phi U\beta) - (W_1\beta) = 0. \tag{4.9}$$

Furthermore, from (2.5) for $X = W_1$, $Y = \xi$, taking the scalar product with ξ and U, using the Lemmas 3.1, 3.3 we have respectively

$$(W_1\alpha) = \beta|W_1|^2 - 3\beta c - 4\alpha^2\beta. \tag{4.10}$$

$$(W_1\beta) = c\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) + \frac{\beta^2}{\alpha}|W_1|^2 - 4\alpha\beta^2. \tag{4.11}$$

Relations (3.17), (4.8), (4.10) and Lemma 3.3 lead to

$$\frac{\beta}{\alpha}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\left(\frac{3\beta c}{4\alpha} - 3\alpha\beta\right) + \frac{3\beta^2 c}{\alpha} = \frac{\beta^2}{\alpha}|W_1|^2 - 4\alpha\beta^2, \tag{4.12}$$

while relations (3.18), (4.9), (4.11) and Lemma 3.3 lead to

$$\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\left[\frac{c}{4\alpha}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right) - 3\beta^2\right] - c\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) = \frac{\beta^2}{\alpha}|W_1|^2 - 4\alpha\beta^2. \tag{4.13}$$

We equate the left sides of (4.12) and (4.13) and then modify this new equation by subtracting $-3\beta^2\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)$ from both sides. The result is

$$\frac{c}{4\alpha}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\left[\frac{2\beta^2}{\alpha} + \frac{c}{4\alpha})\right] + \frac{3\beta^2 c}{\alpha} + c\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) = 0,$$

which is multiplied by $\frac{4\alpha^3}{c}$, giving

$$\left(\beta^2 - \frac{c}{4}\right)\left(\beta^2 + \frac{c}{8}\right) = -4\alpha^2\beta^2 - \frac{\alpha^2 c}{2} = -4\alpha^2\left(\beta^2 + \frac{c}{8}\right). \tag{4.14}$$

If we had $\frac{c}{8} = -\beta^2 < 0$ at some point of M, then the same would hold in a neighbourhood of this point. In this case, by differentiation of $\frac{c}{8} = -\beta^2$ along ϕU in this neighbourhood and by virtue of (3.18), Lemma 3.3, we would get $c = 4\alpha^2 > 0$ which is a contradiction. Therefore, $\frac{c}{8} \neq -\beta^2$ and (4.14) yields

$$\beta^2 + 4\alpha^2 = \frac{c}{4}. \tag{4.15}$$

We differentiate (4.15) along ϕU and make use of (3.17), (3.18), (4.15), Lemma 3.3, obtaining $\beta^2 + 4\alpha^2 = \frac{2c}{3}$ which contradicts (4.15). Therefore, we have a contradiction in $\mathcal{N}$ and $\mathcal{N}$ is the empty set. Thus, M is a Hopf hypersurface. □

5. Proof of Theorem 1.1. Since M is Hopf, we have $A\xi = \alpha\xi$ and α is constant [**9**]. The scalar product of condition (1.2) with ξ, due to the symmetry of l, $l\xi = 0$, (3.11) and $\nabla_\xi\xi = \phi A\xi = \alpha\phi\xi = 0$, yields $\omega(X) = 0$ $\forall X \in TM$ and so condition (1.2) becomes

$$(\nabla_\xi l)X = \psi(X)lX.$$

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In addition, replacing the vector field X with $2X$ in the above relation and due to the linearity of ψ, l, we have

$$(\nabla_\xi l)X = 2\psi(X)lX.$$

The above two equations hold for every $X \in \mathbb{D}$ and therefore we obtain $\psi(X)lX = 0$, $\forall X \in \mathbb{D}$. However l cannot be locally zero [**11**], which means $\psi(X) = 0\, \forall X \in \mathbb{D}$. The last equation and $(\nabla_\xi l)\xi = 0$, reform (1.2) as $(\nabla_\xi l)X = 0$, which is further analysed leading to $\alpha(\nabla_\xi A)X = 0$.

Next, we recall the following equation which holds in every Hopf hypersurface [**9**]:

$$A\phi AX - \frac{\alpha}{2}(A\phi + \phi A)X - \frac{c}{4}\phi X = 0. \tag{5.1}$$

Relation $\alpha(\nabla_X A)\xi - \alpha(\nabla_\xi A)X = -\alpha\frac{c}{4}\phi X$ holds $\forall X \in \mathbb{D}$ due to (2.5). It is combined with $\alpha(\nabla_\xi A)X = 0$ and further developed, giving $\alpha^2\phi AX = \alpha A\phi AX - \alpha\frac{c}{4}\phi X$. The right term of this equality is replaced from (5.1) resulting to

$$\alpha(\phi A - A\phi)X = 0, \quad \forall X \in D. \tag{5.2}$$

From (5.2) and $(\phi A - A\phi)\xi = 0$, we obtain $\alpha = 0$ or M is of type A [**8**, **10**] and the theorem is proved.

Finally, we give two propositions.

Proposition 5.1. *Every Hopf hypersurface satisfies (1.1).*

Proof. If M is Hopf, then (2.4) yields $lX = (\alpha A + \frac{c}{4})X$, $\forall X \in D$. By virtue of the last equation, we have $lAX = AlX$. □

Proposition 5.2. *Every real hypersurface of type A satisfies (1.2) with* $\omega(X) = 0$ *and* $\psi(X) = 0$. *Every Hopf hypersurface with* $\alpha = 0$ *satisfies the same condition.*

Proof. Let M be of type A and $X \in D$ a principal vector field with principal curvature λ, and α the principal curvature of ξ. (2.4) yields $lX = (\alpha A + \frac{c}{4})X$, $\forall X \in D$. Furthermore, in a real hypersurface of type A we have $\lambda^2 = \alpha\lambda + \frac{c}{4}$, thus from the last two equations we have $lX = \lambda^2 X$, which is used to show $(\nabla_\xi l)X = 0$. The last equation and $(\nabla_\xi l)\xi = \nabla_\xi l\xi - l\nabla_\xi\xi = 0$ show that real hypersurfaces of type A satisfy (1.2) with $\omega = \psi = 0$.

If M is Hopf with $\alpha = 0$ then (2.4) yields $lX = \frac{c}{4}X$ for every $X \in D$. Therefore, $(\nabla_\xi l)X = 0$ holds. In addition we have $(\nabla_\xi l)\xi = 0$, thus $(\nabla_\xi l)X = 0$ holds for every X. □

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