AN ALTERNATIVE PROOF OF A THEOREM ON THE LEBESGUE INTEGRAL

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The theorem concerned is the following:

if f is continuous in $[a, b]$, and f' exists and is finite except at an enumerable set of points and Lebesgue integrable in $[a, b]$, then

$$\int_a^b f' = f(b) - f(a). \qquad \text{.................................(1)}$$

Proof. We assume the following theorems:

(1) the above formula is true if f is absolutely continuous (a.c.) in $[a, b]$;

(2) an indefinite integral is a.c.

We first prove the following lemma:

if f' is integrable, and f continuous but not a.c. in $[a, b]$, then given $K>0$, $l>0$, there exist two disjoint subintervals $[a', b']$ of $[a, b]$ such that

$$b' - a' < l, \; |f(b') - f(a')| > K(b' - a'), \qquad \text{........................(2)}$$

and f is not a.c. in $[a', b']$.

Proof. Since f is not a.c. in $[a, b]$, $\exists \varepsilon > 0$ such that given $\delta > 0$, there is a finite set of non-overlapping intervals $[a_r, b_r]$ such that

$$\Sigma(b_r - a_r) < \delta, \quad \Sigma\, |f(b_r) - f(a_r)| > \varepsilon. \qquad \text{........................(3)}$$

By (2), $F(x) \equiv \int_a^x |f'|$ is a.c. Hence $\exists \delta_1 > 0$ such that if $\{[a_r, b_r]\}$ is a finite set of non-overlapping intervals and $\Sigma(b_r - a_r) < \delta_1$, then

$$\Sigma\, |F(b_r) - F(a_r)| < \tfrac{1}{8}\varepsilon. \qquad \text{.................................(4)}$$

By uniform continuity, $\exists \delta_2 > 0$ such that if $|b_r - a_r| < \delta_2$, then

$$|f(b_r) - f(a_r)| < \tfrac{1}{4}\varepsilon \qquad \text{....................................(5)}$$

Now take $\delta = \min(\delta_1, \delta_2, l, \varepsilon/8K)$, and choose intervals $[a_r, b_r]$ to satisfy (3). The sum $\Sigma\, |f(b_r) - f(a_r)|$ may be divided into three parts, by putting $|f(b_r) - f(a_r)|$ into

Σ_1 if f is a.c. in $[a_r, b_r]$,

Σ_2 if f is not a.c. in $[a_r, b_r]$ and $|f(b_r) - f(a_r)| \leqq K(b_r - a_r)$,

Σ_3 if f is not a.c. in $[a_r, b_r]$ and $|f(b_r) - f(a_r)| > K(b_r - a_r)$.

Now if f is a.c. in $[a_r, b_r]$, then by (1),

$$|f(b_r) - f(a_r)| = \left| \int_{a_r}^{b_r} f' \right| \leqq \int_{a_r}^{b_r} |f'| = |F(b_r) - F(a_r)|.$$

Hence $\Sigma_1 \mid f(b_r) - f(a_r) \mid \leqq \Sigma_1 \mid F(b_r) - F(a_r) \mid < \frac{1}{8}\varepsilon$ by (4).

Also $\Sigma_2 \mid f(b_r) - f(a_r) \mid \leqq \Sigma_2 \{K(b_r - a_r)\} < K\delta \leqq \frac{1}{8}\varepsilon$.

Hence by (3), $\Sigma_3 \mid f(b_r) - f(a_r) \mid > \frac{3}{4}\varepsilon$, and so from (5) it follows that Σ_3 has at least three terms. Two of the corresponding intervals $[a_r, b_r]$ must be disjoint, and from the definition of Σ_3 satisfy the conditions of the lemma.

Now let f be a function satisfying the hypothesis of the main theorem, and let $x_1, x_2, x_3, \ldots$ be the set of points at which f' does not exist. If the theorem does not hold, then by (1), f is not a.c. in $[a, b]$. We now construct inductively a set of intervals $I_r = [a_r, b_r]$ with the following properties ($r \geqq 1$):

$$b_r - a_r < 1/r, \ \mid f(b_r) - f(a_r) \mid > r(b_r - a_r), \ I_r \subset I_{r-1}, \ x_r \notin I_r,$$

and f is not a.c. in I_r.

Let I_0 be $[a, b]$. If I_{r-1} has been constructed, the lemma may be applied to obtain two disjoint subintervals such that $b' - a' < 1/r$, $\mid f(b') - f(a') \mid > r(b' - a')$ and f is not a.c. in $[a', b']$. At least one of these does not contain x_r. Take this to be I_r.

Then $a_0 \leqq a_1 \leqq a_2 \leqq \ldots$, $b_0 \geqq b_1 \geqq b_2 \geqq \ldots$, $0 < b_r - a_r < 1/r$, and so $\{a_r\}$, $\{b_r\}$, tend to a common limit x, contained in each I_r, and hence different from each x_r, so that $f'(x)$ exists and is finite.

Now $\dfrac{f(b_r) - f(a_r)}{b_r - a_r}$ lies between $\dfrac{f(b_r) - f(x)}{b_r - x}$ and $\dfrac{f(a_r) - f(x)}{a_r - x}$, each of which tends to $f'(x)$ as $r \to \infty$. But $\left| \dfrac{f(b_r) - f(a_r)}{b_r - a_r} \right| \geqq r$, all r, and this contradiction proves the theorem.

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