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REGULARITY OF MONGE–AMPÈRE EQUATIONS IN OPTIMAL TRANSPORTATION

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We consider some recent regularity results for the Monge–Ampère equation arising in the optimal transportation problem. The Monge–Ampère equation under consideration has the following type

$$\det\{D^2u(x) - A(x, Du)\} = f(x) \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $A = \{A_{ij}\}$ is an $n \times n$ symmetric matrix defined in $\Omega \times \mathbb{R}^n$.

In optimal transportation, u is the potential function, and the matrix A and the right-hand side f are given by

$$A(x, Du) = D_x^2 c(x, T_u(x)), \tag{2}$$

$$f = |\det\{D_{xy}^2 c\}| \frac{\rho}{\rho^* \circ T_u}, \tag{3}$$

where $c(\cdot, \cdot)$ is the cost function, $T_u : x \to y$ is the optimal mapping determined by $Du(x) = D_x c(x, y)$, and ρ and ρ^* are mass distributions in the initial domain Ω and the target domain Ω^*, respectively.

We assume that the cost function $c \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies the following conditions.

(A1) For any $x, p \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $D_x c(x, y) = p$; and for any $y, q \in \mathbb{R}^n$, there is a unique $x \in \mathbb{R}^n$ such that $D_y c(x, y) = q$.

(A2) For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\det\{D_{xy}^2 c(x, y)\} \neq 0$.

(A3) For any $x, p \in \mathbb{R}^n$, and any $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$A_{ij,kl}(x, p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2, \tag{4}$$

for some constant $c_0 > 0$, where $A_{ij,kl} = D_{p_k p_l}^2 A_{ij}$ and A is given by (2).

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Under the above conditions on the cost function, the optimal mapping T_u is uniquely determined by the corresponding potential function u. Therefore, in order to study the regularity of optimal mapping it suffices to study the regularity of potential functions, that is, regularity of elliptic solutions of (1).

In the special case when the cost function is the Euclidean distance squared, equation (1) becomes the standard Monge–Ampère equation, and the various regularity results have been obtained by Caffarelli [1, 2], Delanoë [5], Urbas [12] and many other mathematicians. In particular, the interior $C^{1,\alpha}$, $C^{2,\alpha}$, $W^{2,p}$ estimates are due to Caffarelli [3]. The regularity of optimal transportation with general costs is an important open problem in the area, as pointed out by Caffarelli [4] and Villani [13]. Our goal is to establish the corresponding regularity results for general cost functions satisfying conditions (A1)–(A3), assuming the mass distributions are merely measurable or Hölder continuous, [7–9]. When the mass distributions are smooth, the C^3 smooth regularity has been obtained in [11].

The first result is the $C^{1,\alpha}$ regularity for potentials, [7]. A similar estimate was previously obtained by Loeper [10]. We give a completely different proof and our exponent is optimal when the inhomogeneous term $f \in L^\infty$.

THEOREM 1. *Let u be a potential function in the optimal transportation problem. Assume that the cost function c satisfies conditions (A1)–(A3), Ω^* is c-convex with respect to Ω, and $f \geq 0$, $f \in L^p(\Omega)$ for some $p \in (\frac{1}{2}(n+1), +\infty]$. Then $u \in C^{1,\alpha}(\overline{\Omega})$, where $\alpha = \beta(n+1)/(2n^2 + \beta(n-1))$ and $\beta = 1 - (n+1)/2p$.*

In particular, when $p = \infty$, our Hölder exponent $\alpha = 1/(2n-1)$ is optimal.

Here and below, we say that the target domain Ω^* is c-convex with respect to Ω if $D_x c(x, \Omega^*)$ is convex for any $x \in \Omega$.

The second result gives the Hölder and more general continuity estimates for second derivatives of solutions u, when the inhomogeneous term f is Hölder or Dini continuous, together with corresponding regularity results for potentials, [9].

THEOREM 2. *Assume that the cost function c satisfies (A1)–(A3) and f satisfies $C_1 \leq f \leq C_2$ for some positive constants $C_1, C_2 > 0$. Let $u \in C^2(\Omega)$ be the potential function satisfying (1). Then for all $x, y \in \Omega_\delta$, we have the estimate*

$$|D^2u(x) - D^2u(y)| \leq C\left[d + \int_0^d \frac{\omega_f(r)}{r} + d\int_d^1 \frac{\omega_f(r)}{r^2}\right], \tag{5}$$

where $d = |x-y|$, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$, $\omega_f(r) = \sup\{|f(x) - f(y)| : |x-y| < r\}$ is the oscillation of f, and the constant $C > 0$ depends only on $n, \delta, C_1, C_2, A, \sup|Du|$, and the modulus of continuity of Du.

It follows that:

(i) *if f is Dini continuous, that is, $\int_0^1 (\omega_f(r)/r)\, dr < \infty$, then the modulus of continuity of D^2u can be estimated by (5) above;*

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(ii) *if* $f \in C^{\alpha}(\Omega)$ *for some* $\alpha \in (0, 1)$*, then*

$$\|u\|_{C^{2,\alpha}(\Omega_\delta)} \le C\left[1 + \frac{\|f\|_{C^{\alpha}(\Omega)}}{\alpha(1-\alpha)}\right]; \tag{6}$$

(iii) *if* $f \in C^{0,1}(\Omega)$*, then*

$$|D^2u(x) - D^2u(y)| \le Cd[1 + \|f\|_{C^{0,1}}|\log d|] \quad \forall x, y \in \Omega_\delta. \tag{7}$$

We remark that for standard Monge–Ampère equations, these estimates were obtained by Caffarelli for Hölder continuous f [1] and by Wang for Dini continuous f [14]. In particular, the estimate (5) for standard Monge–Ampère equations was obtained by Jian and Wang [6], and previously by Wang for uniform elliptic equations [15].

The third result is the following $W^{2,p}$ estimate, which is obtained by making a more detailed study of local geometry of potential and cost functions [8].

THEOREM 3. *Assume that the cost function c satisfies (A1)–(A3),* f *is continuous,* $0 < C_1 \le f \le C_2$*, and* Ω^* *is c-convex with respect to* Ω*. Let u be the potential function satisfying (1). Then* $D^2u \in L^p(\Omega')$ *for any* $p \ge 1$*,* $\Omega' \Subset \Omega$*, and we have the estimate*

$$\|u\|_{W^{2,p}(\Omega')} \le C,$$

where C depends on $n, p, C_1, C_2, \Omega, \Omega', \Omega^*$*, and the modulus of continuity of* f*.*

In the proof of Theorem 3, we show that in a proper normalization process, the cost function is uniformly smooth and converges locally smoothly to the linear function $x \cdot y$, and the potential function converges to a quadratic function. As an application, we also obtain the following sharp $C^{1,\alpha}$ estimate for the potentials.

COROLLARY 4. *Let u be the potential function satisfying (1). Suppose that the cost function c satisfies (A1)–(A3) and* Ω^* *is c-convex with respect to* Ω*. Then if*

$$|f - 1| \le \varepsilon, \tag{8}$$

we have $u \in C^{1+\alpha}_{\mathrm{loc}}(\Omega)$ *for some* $\alpha \in (1 - C_1\varepsilon, 1]$*, and for all* $\Omega' \Subset \Omega$*,*

$$\|u\|_{C^{1+\alpha}(\Omega')} \le C,$$

where the constants $C, C_1 > 0$ *depend only on* ε, n*, and C depends also on* $\mathrm{dist}(\Omega', \partial\Omega)$.

Note that in Corollary 4 we do not assume the continuity of f and we have the linear relation

$$\alpha \ge 1 - C_1\varepsilon.$$

Moreover, our proof of Theorem 3 and Corollary 4 also implies a related result. That is, for all $p < \infty$, there exists $\varepsilon = \varepsilon(p)$ such that if f satisfies (8), then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. See [8] for more details.

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