A NOTE ON ALMOST BALANCED BIPARTITIONS OF A GRAPH

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Abstract

Let $G$ be a graph of order $n \geq 6$ with minimum degree $\delta(G) \geq 4$. Arkin and Hassin ['Graph partitions with minimum degree constraints', Discrete Math. 190 (1998), 55–65] conjectured that there exists a bipartition $S, T$ of $V(G)$ such that $\lfloor n/2 \rfloor - 2 \leq |S|, |T| \leq \lceil n/2 \rceil + 2$ and the minimum degrees in the subgraphs induced by $S$ and $T$ are at least two. In this paper, we first show that $G$ has a bipartition such that the minimum degree in each part is at least two, and then prove that the conjecture is true if the complement of $G$ contains no complete bipartite graph $K_{3,r}$, where $r = \lfloor n/2 \rfloor - 3$.

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1. Introduction

All graphs considered here are finite, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph. The complement of $G$ is denoted by $\overline{G}$. For $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraphs induced by $S$ and $V(G) - S$, respectively. When $S = \{v\}$, we simplify $G - \{v\}$ to $G - v$. Let $N_S(v)$ be the set of the neighbours of a vertex $v$ contained in $S$, $N_S[v] = N_S(v) \cup \{v\}$ and $d_S(v) = |N_S(v)|$. A $k$-vertex is a vertex of degree $k$. We call $k$-vertices adjacent to $v$ $k$-neighbours of $v$. The minimum degree of $G$ is denoted by $\delta(G)$. Simply, we write $\delta(G[S])$ as $\delta(S)$. A complete bipartite graph of order $s + t$ is denoted by $K_{s,t}$. For $X, Y \subseteq V(G)$, define $(X, Y)_G = \{uv \in E(G) \mid u \in X, v \in Y\}$ and let $G[X, Y]$ be a graph with vertex set $X \cup Y$ and edge set $(X, Y)_G$. Let $P$ be a path. We denote by $\overrightarrow{P}$ the path $P$ with a given orientation and by $\overleftarrow{P}$ the path $P$ with the reverse orientation. If $u, v \in V(P)$, then $u\overrightarrow{P}v$ ($u\overleftarrow{P}v$, respectively) denotes the consecutive vertices of $P$ from $u$ to $v$ in the direction specified by $\overrightarrow{P}$ ($\overleftarrow{P}$, respectively). To contract an edge $e = uv$ of a graph $G$ is to delete $e$, and then identify its ends and delete possible parallel edges. The resulting graph is

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denoted by \( G/e \). A bipartition \( S, T \) of \( V(G) \) is said to be \emph{balanced} or \emph{almost balanced} if 
\[ \lfloor n/2 \rfloor \leq |S|, |T| \leq \lceil n/2 \rceil \] 
or 
\[ \lfloor n/2 \rfloor - 2 \leq |S|, |T| \leq \lceil n/2 \rceil + 2. \] 
A bipartition \( S, T \) of \( V(G) \) is said to be an \( (s, t) \)-\emph{bipartition} if \( \delta(S) \geq s \) and \( \delta(T) \geq t \), where \( s, t \) are nonnegative integers.

In [6], Stiebitz showed that every graph with minimum degree at least \( s + t + 1 \) admits an \( (s, t) \)-bipartition. Kaneko [4] and Diwan [2] strengthened this result, proving that it suffices to assume that the minimum degree is at least \( s + t \) or \( s + t - 1 \) \((s, t \geq 2)\) if \( G \) contains no cycles shorter than four or five, respectively.

It is natural to ask an analogous question on balanced or almost balanced bipartitions. Let \( s \) and \( t \) be two nonnegative integers. Is there an integer \( k \) such that every graph with minimum degree at least \( k \) admits a balanced or an almost balanced \( (s, t) \)-bipartition? In [5], Maurer proved an interesting result, from which it is easy to see that every connected graph with minimum degree at least two admits a balanced \((1, 1)\)-bipartition.

**Theorem 1.1.** Let \( G \) be a connected graph of order \( n \) with \( \delta(G) \geq 2 \). Then, for any positive integer \( l \) with \( 2 \leq l \leq n - 2 \), \( G \) admits a \((1, 1)\)-bipartition \( S, T \) such that \( |S| = l \) and \( |T| = n - l \).

Arkin and Hassin [1] have given the following conjecture for graphs with minimum degree at least four.

**Conjecture 1.2.** Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq 4 \). Then \( G \) admits an almost balanced \((2, 2)\)-bipartition.

Note that if a graph \( G \) has a \((2, 2)\)-bipartition, then clearly \(|G| \geq 6\), and so we need only consider whether Conjecture 1.2 is true for \( n \geq 6 \). In [3], El-Zahar established the following theorem.

**Theorem 1.3** (El-Zahar [3]). Let \( G \) be a graph of order \( n = n_1 + n_2 \) with \( \delta(G) \geq \lfloor n_1/2 \rfloor + \lfloor n_2/2 \rfloor \); then \( G \) contains two vertex disjoint cycles of lengths \( n_1 \) and \( n_2 \), where \( n_1, n_2 \geq 3 \) are two integers.

By Theorem 1.3, if \( \delta(G) \geq 4 \), then \( G \) contains two vertex disjoint cycles of lengths \( n_1 = n_2 = 3 \) if \( n = 6 \), \( n_1 = 3 \) and \( n_2 = 4 \) if \( n = 7 \) and \( n_1 = n_2 = 4 \) if \( n = 8 \). This is to say that Conjecture 1.2 is true for \( 6 \leq n \leq 8 \). Up to now, no results were obtained on Conjecture 1.2 and so it is still open. In this paper, we first improve a result on \((2, 2)\)-bipartitions due to Stiebitz [6] by showing the following theorem.

**Theorem 1.4.** Let \( G \) be a graph of order \( n \geq 6 \) with \( \delta(G) \geq 4 \). Then \( G \) admits a \((2, 2)\)-bipartition.

By Theorem 1.4 and the definition of almost balanced bipartitions, it is easy to obtain the following corollary.

**Corollary 1.5.** Let \( G \) be a graph of order \( n \geq 6 \) with \( \delta(G) \geq 4 \). If \( n \leq 11 \), then \( G \) admits an almost balanced \((2, 2)\)-bipartition.
Next we show that Conjecture 1.2 is true under some additional constraint. This is the main result of this paper.

**Theorem 1.6.** Let $G$ be a graph of order $n \geq 6$ with $\delta(G) \geq 4$. If $\overline{G}$ contains no $K_{3,r}$, then $G$ admits an almost balanced $(2,2)$-bipartition, where $r = \lfloor n/2 \rfloor - 3$.

Obviously, our result supports the truth of Conjecture 1.2.

### 2. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemma.

**Lemma 2.1.** Let $G$ be a graph of order $n \geq 6$ with $\delta(G) \geq 4$. Then $G$ contains two vertex disjoint cycles.

**Proof.** If $G$ is disconnected, then the result holds trivially. So, we may assume that $G$ is connected. Let $P = v_1v_2 \cdots v_p$ be any longest path of $G$. By the maximality of $P$, we have $N_G(v_1) \subseteq V(P)$ and $N_G(v_p) \subseteq V(P)$. Since $\delta_G(v_1) \geq 4$, we may assume that $v_i, v_j \in N_G(v_1)$, where $2 < i < j < p$. If $v_1v_p \notin E(G)$, then, since $\delta_G(v_p) \geq 4$, there is either some $k$ with $i + 1 \leq k < p - 1$ such that $v_k \in N_G(v_p)$ or some $l, m$ with $1 < l < m \leq i - 1$ such that $v_l, v_m \in N_G(v_p)$.

Thus, $v_1 \overrightarrow{P} v_i v_j v_1$ and $v_p v_l \overrightarrow{P} v_m v_p$ in the former case and $v_1 v_i \overrightarrow{P} v_j v_1$ and $v_p v_l \overrightarrow{P} v_m v_p$ in the latter case are two vertex disjoint cycles of $G$. Therefore, we have $v_1 v_p \in E(G)$. In this case, $p = n$. Noting that $v_{i-1} \overrightarrow{P} v_i v_j \overrightarrow{P} v_p$ and $v_{j-1} \overrightarrow{P} v_1 v_j \overrightarrow{P} v_p$ are also longest paths of $G$, by similar arguments as before, we may assume that $v_{i-1}, v_{j-1} \in N(v_p)$. If $j > i + 1$, then $v_j \overrightarrow{P} v_i v_1$ and $v_p v_{j-1} \overrightarrow{P} v_p$ are two vertex disjoint cycles and hence $j = i + 1$. If $i > 3$, then $v_2 \overrightarrow{P} v_{i-1} v_j \overrightarrow{P} v_p$ is a longest path and so $v_2 v_p \in E(G)$. Thus, $v_1 v_j v_1$ and $v_2 \overrightarrow{P} v_{i-1} v_j v_1$ are two vertex disjoint cycles. If $i = 3$, then, since $p = n \geq 6$, we have $4 = j < p - 1$, whence $v_{p-1} \overrightarrow{P} v_j v_1 v_{i-1} v_1$ is a longest path and so $v_1 v_{p-1} \in E(G)$. Hence, $v_p v_{i-1} v_j v_1$ and $v_i v_j \overrightarrow{P} v_{p-1} v_1$ are two vertex disjoint cycles. □

**Proof of Theorem 1.4.** By Lemma 2.1, $G$ has two vertex disjoint subgraphs $H_1$ and $H_2$ such that $\delta(H_i) \geq 2$ for $i = 1, 2$. Choose $H_1$ and $H_2$ such that $|H_1| + |H_2|$ is as large as possible. If $|H_1| + |H_2| < n$, set $H_3 = G - V(H_1) - V(H_2)$. By the choice of $H_1$ and $H_2$, we have $d_H(h) \leq 1$ for any $h \in V(H_3)$ and $i = 1, 2$, which implies that $\delta(H_3) \geq 2$ since $\delta(G) \geq 4$. In this case, $H_1$ and $H_2 \cup H_3$ satisfy $\delta(H_1) \geq 2$ and $\delta(H_2 \cup H_3) \geq 2$, which contradicts the choice of $H_1$ and $H_2$. □

### 3. Proof of Theorem 1.6

**Proof of Theorem 1.6.** We will use induction on $n$. By Corollary 1.5, Theorem 1.6 holds for $6 \leq n \leq 11$. Now we assume that $n \geq 12$ and that the result holds for all small $n$. In the following, we let $r = \lfloor n/2 \rfloor - 3$. 


Firstly, we show that $G$ admits a $(2, 2)$-bipartition $S, T$ such that $|\lvert S\rvert - \lvert T\rvert| \leq \lceil (n - 1)/2\rceil - \lfloor (n - 1)/2\rfloor + 5$.

If there exists $x \in V(G)$ such that $\delta(G - x) \geq 4$, then $\delta(G') \geq 4$ and $\overline{G'}$ contains no $K_{3,r}$, as $\overline{G'} \subseteq \overline{G}$, where $G' = G - x$. Thus, $G'$ admits an almost balanced $(2, 2)$-bipartition $S', T'$ by induction. Since $d_G(x) \geq 4$, we have $d_{S'}(x) \geq 2$ or $d_{T'}(x) \geq 2$, say $d_{S'}(x) \geq 2$. Let $S = S' \cup \{x\}$, $T = T'$; then $S, T$ is a $(2, 2)$-bipartition of $G$ and $|S| - |T| = |S'| + 1 - |T'| \leq \lceil (n - 1)/2\rceil - \lfloor (n - 1)/2\rfloor + 5$. So, we assume that $\delta(G - x) = 3$ for any $x \in V(G)$.

By $(\ast)$, we have $\delta(G) = 4$ and, if $d_G(x) = 4$, then $x$ has a 4-neighbor.

Let $u, v$ be two adjacent 4-vertices.

**Claim 1.** $u, v$ have at most two common neighbours.

To prove our claim, suppose that $u, v$ have three common neighbours $w_1, w_2, w_3$. Denote $X = V(G) - \{u, v, w_1, w_2, w_3\}$; then $N_G(u) = N_G(v) = X$. Since $\overline{G}$ contains no $K_{3,r}$, we have $|N_G(w) \cap X| \leq r - 1$ for any $w \in V(G) - \{u, v\}$. Then $|N_G(w_i) \cap X| \geq n - r - 4$ for $1 \leq i \leq 3$ and $|N_G(x) \cap X| \geq n - r - 5$ for any $x \in X$. Noting that $d_G(x) \geq 4$, we have $|N_G(x) \cap (V(G) - \{u, v, w_1, w_2\})| \geq 2$. If $n = 12$, then $|N_G(w_3) \cap (V(G) - \{u, v, w_1, w_2\})| \geq n - r - 4 = 5$; thus, $\{u, v, w_1, w_2\}, V(G) - \{u, v, w_1, w_2\}$ is an almost balanced $(2, 2)$-bipartition of $G$. If $n \geq 13$, let $x_0 \in X$; then $x_0 u, x_0 v \in E(\overline{G})$. Since $|N_G(x) \cap (X - x_0)| \geq n - r - 6 = n - \lfloor n/2\rfloor - 2 \geq 5$ for any $x \in X - x_0$ and $|N_G(w_i) \cap (X - x_0)| \geq n - r - 5 = n - \lfloor n/2\rfloor - 2 \geq 5$ for $1 \leq i \leq 3$, we have $\delta(G - x_0) \geq 4$, which contradicts $(\ast)$. This proves our claim.

**Claim 2.** If $u, v$ have a common neighbour $w$, then $d_G(w) \geq 5$.

To prove our claim, suppose to the contrary that $d_G(w) = 4$. Then $|V(G) - N_G[u] - N_G[v] - N_G[w]| \geq n - 9 \geq \lfloor n/2\rfloor - 3 = r$ that is, there exist $x_1, x_2, \ldots, x_r \in V(G) - N_G[u] - N_G[v] - N_G[w]$ and $\overline{G}[\{u, v, w\}, \{x_1, x_2, \ldots, x_r\}]$ is a $K_{3,r}$, which is a contradiction. This proves our claim.

Let $G' = G/uv$ and denote by $w$ the vertex resulting from the contraction of $e$. By Claim 1, we have $|N_G(u) \cup N_G(v)| \geq 4$ and $d_{G'}(w) \geq 4$. For any $x \in V(G) - \{u, v\}$, if $x$ is not a common neighbour of $u, v$, we have $d_{G'}(x) = d_G(x) \geq 4$; if $x$ is a common neighbour of $u, v$, we have $d_{G'}(x) = d_G(x) - 1 \geq 4$ by Claim 2. Therefore, $\delta(G') \geq 4$. Since $\overline{G'}$ contains no $K_{3,r}$, by induction, $G'$ admits an almost balanced $(2, 2)$-bipartition $S', T'$, Assume without loss of generality that $w \in S'$.

If $d_{S' - w}(u) = 0$, then we must have $d_{T'}(u) \geq 3$, as $d_G(u) \geq 4$. Since $d_{S'}(w) \geq 2$, we have $d_{S' - w}(v) \geq 2$. Let $S = (S' - w) \cup \{v\}$, $T = T' \cup \{u\}$, then $S, T$ is a $(2, 2)$-bipartition of $G$ and $|S| - |T| = |S'| - |T'| - 1 \leq \lceil (n - 1)/2\rceil - \lfloor (n - 1)/2\rfloor + 5$. Thus, by symmetry of $u$ and $v$, we may assume that $d_{S' - w}(u) \geq 1$ and $d_{S' - w}(v) \geq 1$. Let $X = (S' - w) \cup \{u, v\}$, $T = T'$; then $S, T$ is a $(2, 2)$-bipartition of $G$ and $|S| - |T| = |S'| + 1 - |T'| \leq \lceil (n - 1)/2\rceil - \lfloor (n - 1)/2\rfloor + 5$. 


Next, we show that $G$ admits an almost balanced $(2, 2)$-bipartition.

By the argument above, $G$ admits a $(2, 2)$-bipartition $S, T$ with $|S| - |T| \leq [(n-1)/2] - [(n-1)/2] + 5$. Assume without loss of generality that $|S| \geq |T| \geq 3$. If $|S| - |T| \leq [n/2] - [n/2] + 4$, then we see that $S, T$ is an almost balanced $(2, 2)$-bipartition. Therefore, we have $[n/2] - [n/2] + 5 \leq |S| - |T| \leq [(n-1)/2] - [(n-1)/2] + 5$; thus, $n$ is even and $|S| = [(n-1)/2] + 3 = [n/2] + 3$ and $|T| = [(n-1)/2] - 2 = [n/2] - 3$. In the following, we will show that $G$ admits an almost balanced $(2, 2)$-bipartition under these conditions.

If $δ(S) = 2$, we assume that $H$ is any component of the subgraph induced by the 2-vertices of $G[S]$ and $V(H) = \{x_1, \ldots, x_p\}$. Clearly, $H$ is a path or a cycle. Let $P = x_1x_2 \cdots x_p$ be a path in $H$. Since $d_S(x_i) = 2$ and $d_G(x_i) \geq 4$, we have $d_T(x_i) \geq 2$ for $1 \leq i \leq p$. If $p \geq 3$, then $|S - \bigcup_{i=1}^3 N_S(x_i)| \geq |S| - 5 = [n/2] - 2 \geq r$. Let $y_1, y_2, \ldots, y_r \in S - \bigcup_{i=1}^3 N_S(x_i)$. Then $\overline{G}(\{x_1, x_2, x_3\}, \{y_1, y_2, \ldots, y_r\}) = K_{3,r}$, which is a contradiction. Therefore, $p \leq 2$. If $p = 2$ and $x_1, x_2$ have a common neighbour $x$ in $S$, then $d_S(x) \geq 4$. Otherwise, $|S - N_S[x] - N_S[x_1] - N_S[x_2]| \geq |S| - 4 = [n/2] - 1 > r$, that is, there exist $y_1, y_2, \ldots, y_r \in S - N_S[x] - N_S[x_1] - N_S[x_2]$; then $\overline{G}(\{x, x_1, x_2\}, \{y_1, y_2, \ldots, y_r\})$ is a $K_{3,r}$, which is a contradiction. Thus, $δ(S - \{v_1, \ldots, v_p\}) \geq 2$. Let $S^* = S - V(P)$, $T^* = T \cup V(P)$. Noting that $δ(T) \geq 2$, $d_T(x_i) \geq 2$, we have $δ(T \cup V(P)) \geq 2$. Since $|S'| - |T'| = |S| - |T| - 2p \leq 4$, we see that $S^*, T^*$ is an almost balanced $(2, 2)$-bipartition of $G$.

Now we assume that $δ(S) \geq 3$. Denote $X = \{x \mid x \in S$ and $d_T(x) \geq 1\}$. Then $X$ contains all 3-vertices of $G[S]$. Since $|T| = [n/2] - 3 = r$ and $\overline{G}$ contains no $K_{3,r}$, we have $(S, T)_G \not\equiv 0$, that is, $X \not\equiv 0$. Choose $x \in X$ such that $d_S(x)$ is as small as possible and let $x_1$ be a neighbour of $x$ in $T$. Since $\overline{G}$ contains no $K_{3,r}$, we have $(N_S(x), T)_G \not\equiv 0$. Choose $y \in N_S(x) \cap X$ such that $d_S(y)$ is as small as possible and let $y_1$ be a neighbour of $y$ in $T$. If $δ(S - \{x, y\}) = 1$, then $δ(S) = 3$ and $x, y$ have a common neighbour $z$ of degree three in $S$. Noting that $z \in X$, by the choice of $x, y$, we have $d_S(x) = d_S(y) = 3$. Since $|S - N_S[x] - N_S[y] - N_S[z]| \geq |S| - 6 = [n/2] - 3 = r$, there exist $w_1, w_2, \ldots, w_r \in S - N_S[x] - N_S[y] - N_S[z]$. But then $\overline{G}(\{x, y, z\}, \{w_1, w_2, \ldots, w_r\})$ is a $K_{3,r}$, which is a contradiction. Therefore, we have $δ(S - \{x, y\}) \geq 2$. Noting that $δ(T) \geq 2$, $\{x_1, y\} \subseteq N_{T\cup\{x_1\}}(x)$ and $\{x, y\} \subseteq N_{T\cup\{x_1\}}(y)$, we have $δ(T \cup \{x, y\}) \geq 2$. Let $S^* = S - \{x, y\}$, $T^* = T \cup \{x, y\}$. Since $|S'| - |T'| = |S| - |T| - 4 = 2$, we see that $S^*, T^*$ is an almost balanced $(2, 2)$-bipartition of $G$.

Therefore, the proof of Theorem 1.6 is complete. □

References


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