Canad. J. Math. Vol. **61** (6), 2009 pp. 1407–1436

Traces, Cross-Ratios and 2-Generator Subgroups of SU(2, 1)

Pierre Will

Abstract. In this work, we investigate how to decompose a pair (A, B) of loxodromic isometries of the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ under the form $A = I_1I_2$ and $B = I_3I_2$, where the I_k's are involutions. The main result is a decomposability criterion, which is expressed in terms of traces of elements of the group $\langle A, B\rangle$.

1 Introduction

Let G be the fundamental group of an arbitrary hyperbolic Riemann surface, and let $n > 1$ be an integer. Although some important rigidity results have been obtained (see [15,31]), the moduli space of discrete and faithful representations $\rho\colon G \to \mathrm{PU}(n, 1)$ has not been given an explicit description, even when $n = 2$. There exist only two examples of non-trivial Fuchsian groups for which such a description has been carried out (both for $n = 2$). In the first case, G admits the presentation $\langle i_1, i_2, i_3 \mid i_k^2, (i_j i_k)^\infty\rangle$ (these are the so-called complex hyperbolic ideal triangle groups, as in [16, 28, 29]). In the second case, G is the modular group $\mathrm{PSL}(2, \mathbb{Z})$ (see [5]).

We are especially interested in representations of non-compact Riemann surfaces of finite area, that is, with a finite number of cusps. In this case, in addition to discreteness and faithfulness, the representation is often required to be *type-preserving*: the homotopy classes of loops around cusps should be mapped to parabolic isometries. Several examples of families of such representations have been described, concerning the 3-punctured sphere (see [4,16,28]), the 1-punctured torus (see [35]), and surface groups of finite index in the modular group (see [5, 17]). The 3-punctured sphere and the 1-punctured torus have a common feature: their fundamental groups are both isomorphic to the free group of rank 2 with generators $\mathtt{m}$ and $\mathtt{n}$, $F_2 = \langle \mathtt{m}, \mathtt{n}\rangle$. This is our motivation to study 2-generator subgroups of $\mathrm{PU}(2, 1)$.

In the case of the complex hyperbolic line, $\mathrm{Isom}(\mathbf{H}^1_{\mathbb{C}}) = \langle \mathrm{PU}(1, 1), z \mapsto \bar{z}\rangle$. It is a classical result that any non-elementary 2-generator subgroup G of $\mathrm{PU}(1, 1)$ is contained with index 2 in a triangle group $G^\star$ (see for instance [1, 8, 11]): if A and B are the generators, then there exist three involutions σ_1, σ_2, and σ_3 such that $A = \sigma_1\sigma_2$ and $B = \sigma_3\sigma_2$. This remark is the first step in the classification of discrete 2

Received by the editors January 17, 2007; revised October 8, 2007.
This work was revised during a stay at the MPIM in Bonn, supported by the Max Planck Gesellschaft
AMS subject classification: Primary: 14L24; secondary: 22E40, 32M15, 51M10.

generator subgroups of $\mathrm{PU}(1,1)$ (see [11]). As examples, note that if $\mathbf{H}^1_{\mathbb{C}}/G$ is a pair of pants, $G^\star$ is generated by three antiholomorphic involutions (reflections in geodesics) whereas if $\mathbf{H}^1_{\mathbb{C}}/G$ is a 1-punctured torus, the three involutions are (holomorphic) half-turns.

In the setting of $\mathbf{H}^2_{\mathbb{C}}$, many of the known examples of discrete groups are related to *triangle groups*, that is, groups generated by three involutions (see for instance [16, 35]). It also turns out that knowing that G is contained with finite index in a triangle group can lead to considerable simplification in the study of the discreteness of G. As an example, Deraux, Falbel and Paupert have shown [2] that Mostow's lattices [21] are contained with finite index in a triangle group, and this simplifies their construction of a fundamental domain. For these reasons, we wish to investigate the possibility for a 2-generator subgroup of $\mathrm{PU}(2,1)$ to be contained with index 2 in a triangle group. More precisely, we examine the following question. Let $\rho \in \mathrm{Hom}(F_2, \mathrm{PU}(2,1))$ be a representation. Does there exist a triple of involutions (I_1, I_2, I_3) such that $\rho(\mathtt{m}) = I_1 I_2$ and $\rho(\mathtt{n}) = I_3 I_2$? In the complex hyperbolic 2-space, there two types of maximal totally geodesic subspaces: the complex lines and the $\mathbb{R}$-planes (see Section 2.3). Each of these subspaces is the fixed point set of an isometric involution: complex lines are fixed pointwise by complex symmetries, and $\mathbb{R}$-planes by Lagrangian reflections. In view of this, we make the following definition.

Definition 1.1
- A pair of isometries (A, B) of PU(2,1) is said to be $\mathbb{R}$-*decomposable* (resp. $\mathbb{C}$-*decomposable*) if there exist three Lagrangian reflections (resp. three complex symmetries) I_1, I_2, and I_3 such that $A = I_1 I_2$ and $B = I_3 I_2$ holds.
- We will say that a representation ρ of F_2 in PU(2,1) is $\mathbb{R}$- or $\mathbb{C}$-decomposable if the pair $(\rho(\mathtt{m}), \rho(\mathtt{n}))$ is.

We will describe necessary and sufficient conditions for $\mathbb{R}$- and $\mathbb{C}$-decomposability of a pair (A, B), written in terms of traces of elements of the group generated by A and B. Note that an element of $\mathrm{PU}(2,1)$ admits 3 lifts to $\mathrm{SU}(2,1)$, which are obtained one from another by multiplication by a cube root of 1. The trace of an isometry is thus well defined up to this indetermination. We will say that an isometry has real trace if and only if it admits a lift to $\mathrm{SU}(2,1)$ which has real trace. If the five isometries A, B, AB, $A^{-1}B$, and $[A, B]$ have real trace, then the group generated by A and B preserves a totally geodesic subspace (see Remark 20).

The main result of this work is the following.

Theorem 1.2 *Let A and B be two loxodromic isometries of $\mathbf{H}^2_{\mathbb{C}}$ and $G = \langle A, B\rangle$. Assume that G does not preserve a totally geodesic subspace.*

(i) *The following two propositions are equivalent.*
  - (a) *The isometry $[A, B]$ has real trace.*
  - (b) *The pair (A, B) is $\mathbb{R}$-decomposable.*

(ii) *The following two propositions are equivalent.*
  - (a) *The isometries A, B, AB and $A^{-1}B$ all have real traces.*
  - (b) *Either the pair (A, B) is $\mathbb{C}$-decomposable, or the pair (A^2, B^2) is $\mathbb{C}$-decomposable.*

We denote the PU(2, 1)-representation variety of F_2 by

$$\mathcal{M} = \operatorname{Hom}(F_2, \mathrm{PU}(2,1)) / \mathrm{PU}(2,1).$$

Let $\mathcal{R}^{\mathrm{lox}}$ be the subset of $\operatorname{Hom}(F_2)\ \mathrm{PU}(2,1))$ defined by

$$\{\rho \colon F_2 \to \mathrm{PU}(2,1) \mid \rho(\mathtt{m}) \text{ and } \rho(\mathtt{n}) \text{ are loxodromic }\}.$$

Theorem 1.2 provides a decomposability criterion for those classes of representations belonging to $\mathcal{M}^{\mathrm{lox}}$, the open subset of $\mathcal{M}$ defined by $\mathcal{M}^{\mathrm{lox}} = \mathcal{R}^{\mathrm{lox}}/PU(2,1)$.

Our approach to this problem is based on the interplay between two different coordinate system on $\mathcal{M}^{\mathrm{lox}}$.

- The first one is described in Section 3. It is the restriction to $\mathcal{M}^{\mathrm{lox}}$ of what we will call *trace coordinates* on $\mathcal{M}$.
- The second system of coordinates, which we will refer to as *KR coordinates*, is based on the classification of the ideal tetrahedra of $\mathbf{H}^2_{\mathbb{C}}$ by the Koranyi–Reimann complex cross-ratio on the Heisenberg group (see [14, 19]). It is described in Section 4.

First, we characterize the decomposability of a pair of loxodromic isometries using KR coordinates. Second, we translate the result in terms of traces. The transition from KR coordinates to trace coordinates is done in Section 5.2.

As we will see in Section 3, the trace coordinates on $\mathcal{M}$ arise from trace coordinates on the categorical quotient of $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$ by the diagonal action of $\mathrm{SL}(3,\mathbb{C})$ by conjugation (see Section 3.1). Two irreducible representations ρ_1 and ρ_2 of F_k in $\mathrm{SL}(n,\mathbb{C})$ are conjugate if and only if they have the same character, that is, if $\operatorname{tr}\rho_1(\mathtt{w}) = \operatorname{tr}\rho_2(\mathtt{w})$ for any word $\mathtt{w} \in F_k$. Since the ring of invariants of $\mathrm{SL}(n,\mathbb{C})$ is noetherian (see [24]), there exists a finite family of words on which the above equality should be tested to guarantee the equality of characters. Such a family provides an effective criterion to determine whether or not two representations of F_k are conjugate. In the case of F_2 and $\mathrm{SL}(3,\mathbb{C})$, an explicit and minimal such family of words is known (see for instance [20, 33, 34] among others). Using this fact, we will see that two Zariski dense representations of F_2 in SU(2, 1) are conjugate in SU(2, 1) if and only if their characters coincide on the five words $\mathtt{m}$, $\mathtt{n}$, $\mathtt{mn}$, $\mathtt{m}^{-1}\mathtt{n}$, $[\mathtt{m},\mathtt{n}]$ (see Proposition 3.7). In the case of $\mathrm{SL}(2,\mathbb{C})$, the analogous result has been known since the end of the nineteenth century with the work of Fricke [10] or Vogt [32]. A modern and self-contained treatment of this material may be found in [12, 13], where it is shown that the $\mathrm{SL}(2,\mathbb{C})$-character variety of F_2 is $\mathbb{C}^3$ (see also [8]).

On the other hand, KR coordinates are specially fit for $\mathcal{M}^{\mathrm{lox}}/\mathrm{PU}(2,1)$. A pair of loxodromic isometries (A, B) is associated with an ideal tetrahedron whose vertices are the fixed points of A and B. The ideal tetrahedra of $\mathbf{H}^2_{\mathbb{C}}$ are classified by three Koranyi–Reimann complex cross-ratios, which we call ω_a, ω_b and ω_c (this is Proposition 4.7, due to Falbel [3]). The KR coordinates of a representation corresponding to two matrices A and B are $(\operatorname{tr} A, \operatorname{tr} B, \omega_a, \omega_b, \omega_c)$. We will see (Propositions 4.18 and 4.19) that the $\mathbb{R}$- or $\mathbb{C}$-decomposability of a loxodromic pair (A, B) is equivalent to the existence of a symmetry of the associated ideal tetrahedron, realized either by a antiholomorphic involution or a complex symmetry. These two kinds of symmetries of tetrahedra are easy to detect using cross ratios (see Proposition 4.17). These complex cross-ratios have been used in several works in the last few years. Falbel [3] used them to study CR structures on the complement of the figure eight knot. Parker and Plattis [22] used the same parametrisation of $\mathcal{M}^{\mathrm{lox}}/\operatorname{PU}(2,1)$ as we are using here to describe a complex hyperbolic equivalent of the Fenchel–Nielsen coordinates.

Note that there exist relations between the traces of the five words $\mathtt{m}, \mathtt{n}, \mathtt{mn}, \mathtt{m}^{-1}\mathtt{n}$, and $[\mathtt{m}, \mathtt{n}]$. More precisely, $\operatorname{tr}[\mathtt{m}, \mathtt{n}]$ is the solution of a quadratic equation which coefficients are polynomials in the other four traces. Similarly, the three complex cross ratios are linked by two real relations, and any of the three is determined by the two others up to complex conjugation. Hence, once $\operatorname{tr} A$ and $\operatorname{tr} B$ are fixed, and, according to the choice of the coordinate system, once either ω_b and ω_c or $\operatorname{tr} AB$ and $\operatorname{tr} A^{-1}B$ are fixed, there is an order 2 ambiguity about the conjugacy class of the pair (A, B). The relation is made clear in Section 5.2, where we show (Proposition 5.2) that the pair $(\operatorname{tr} AB, \operatorname{tr} A^{-1}B)$ is the image of $(\omega_b^{-1}, \omega_c)$ under a real affine bijection, of which coefficients depend only on the conjugacy classes of A and B.

As a direct consequence of Theorem 1.2, we will see in Proposition 5.3 that the classes of $\mathbb{R}$-decomposable representation in $\mathcal{M}^{\mathrm{lox}}$ appear as the fixed points of an involution defined on $\mathcal{M}$. This result is of the same nature as those obtained by Schaffhauser [27] in a different frame. Next, Theorem 5.4, obtained as a corollary of Theorem 1.2, is a rigidity result asserting that a representation $\rho\colon F_2 \to \operatorname{PU}(2,1)$ such that $\rho([\mathtt{m}, \mathtt{n}])$ is unipotent is either reducible or $\mathbb{R}$-decomposable.

Our work is organized as follows. In Section 2, we provide some necessary background about the complex hyperbolic 2-space and its isometries. The trace coordinates on $\mathcal{M}$ are described in Section 3. We review the case of $\operatorname{SL}(3,\mathbb{C})$ in Section 3.1, before going to $\operatorname{SU}(2,1)$ in Section 3.2. In Section 3.3, we show that the coordinate system on the set of $\operatorname{PU}(2,1)$-conjugacy classes of complex triangle groups described by Pratoussevitch [23] is obtained from the trace coordinates on $\mathcal{M}$. In Section 4, we define the complex cross-ratio, which we use to define the KR coordinates on $\mathcal{M}^{\mathrm{lox}}$. We study the link between decomposability and symmetry of tetrahedra in terms of complex cross-ratios. Next, we bring together traces and cross ratios in Section 5, and show how to pass from one system to the other. Theorem 1.2 and its corollary, Theorem 5.4, are proved at this point. The last section is devoted to the study of two examples: the representations of the fundamental groups of the sphere with three holes and of the torus with one hole.

2 Preliminary Material

Throughout this paper, we will use the following notation: $F_2 = \langle \mathtt{m}, \mathtt{n} \rangle$ is the free group of rank 2 with generators $\mathtt{m}$ and $\mathtt{n}$, M^T is the transposed matrix of M, and $\mathbf{P}$ is the projection map of $\mathbb{C}^3 \setminus \{0\}$ onto the projective plane $\mathbb{C}\mathbf{P}^2$.

2.1 The Complex Hyperbolic Plane

We denote by $\mathbb{C}^{2,1}$ the vector space $\mathbb{C}^3$ equipped with a hermitian form of signature $(2,1)$. In this work, we will only use the hermitian form given by the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We will denote by V^- (resp. V^0, V^+) the set of negative (resp. null, positive) vectors for the hermitian form associated with J.

Definition 2.1 The complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is the image of V^- under $\mathbf{P}$, equipped with the distance fonction d given by

$$\cosh^2\left(\frac{\mathrm{d}(p,q)}{2}\right) = \frac{\langle \mathbf{p},\mathbf{q}\rangle\langle \mathbf{q},\mathbf{p}\rangle}{\langle \mathbf{p},\mathbf{p}\rangle\langle \mathbf{q},\mathbf{q}\rangle}, \tag{2.1}$$

where p and q are two points of $\mathbf{P}(V^-)$, and $\mathbf{p}$ and $\mathbf{q}$ are lifts of p and q to $\mathbb{C}^{2,1}$.

Note that any point p of $\mathbf{H}^2_{\mathbb{C}}$ may be lifted to $\mathbb{C}^{2,1}$ as

$$\mathbf{p} = \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix} \text{ with } z \in \mathbb{C}, u > 0, \text{ and } t \in \mathbb{R}. \tag{2.2}$$

The triple (z,t,u) is called the *horospherical coordinates* of $\mathbf{H}^2_{\mathbb{C}}$. The boundary of $\mathbf{H}^2_{\mathbb{C}}$ contains the projections of those vectors as in relation (2.2) for which $u = 0$, together with the point ∞ associated with the vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. For any $k > 0$, the hypersurface $\{u = k\}$ is a horosphere of $\mathbf{H}^2_{\mathbb{C}}$ centered at ∞.

2.2 The Isometries of $\mathbf{H}^2_{\mathbb{C}}$.

The isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is generated by $\mathrm{PU}(2,1)$ and the complex conjugation. For later use, we note that since U(2,1) is the set of matrices A satisfying the relation

$$\overline{A}^T JA = J, \tag{2.3}$$

any matrix of U(2,1) satisfies

$$\operatorname{tr} A^{-1} = \overline{\operatorname{tr} A}. \tag{2.4}$$

The usual trichotomy of isometries of $\mathbf{H}^1_{\mathbb{C}}$ also holds in the case of $\mathbf{H}^2_{\mathbb{C}}$.

Definition 2.2 An isometry of $\mathbf{H}^2_{\mathbb{C}}$ $\varphi \in \mathrm{PU}(2,1)$ is *elliptic* if it has a fixed point inside $\mathbf{H}^2_{\mathbb{C}}$, *parabolic* if it has exactly one fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$, *loxodromic* if it has exactly two fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$. A *regular elliptic* isometry is an elliptic isometry which has a lift having three distinct eigenvalues.

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

Remark 1 An isometry of $\mathrm{PU}(2,1)$ has exactly three lifts to $\mathrm{SU}(2,1)$, which are obtained one from the other by multiplication by a cube root of 1. Therefore the trace of an isometry is defined up to multiplication by a cube root of 1. Thus if A_1 and A_2 are two isometries of $\mathbf{H}^2_{\mathbb{C}}$, the assertion $\operatorname{tr} A_1 = \operatorname{tr} A_2$ should be understood as "A_1 and A_2 admit lifts to $\mathrm{SU}(2,1)$ of which traces are equal up to multiplication by a cube root of 1." Similarly, we will say that an isometry has real trace if and only if it admits a lift to $\mathrm{SU}(2,1)$ having real trace. We will freely identify a holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ with one of its lifts to $\mathrm{SU}(2,1)$ without further mention.

As in the case of PU(1,1), the trace of an isometry provides information about its type.

Proposition 2.3 *Let* $f(z) = |z|^4 - 8\mathrm{Re}\,(z^3) + 18|z|^2 - 27$. *An isometry* $A \in \mathrm{SU}(2,1)$ *is loxodromic if and only if* $f(\operatorname{tr} A) > 0$, *and regular elliptic if and only if* $f(\operatorname{tr} A) < 0$.

A proof of this proposition may be found in [14, Ch. 6]. Note that f is just the resultant of χ and χ', where χ is the characteristic polynomial of a generic matrix of $\mathrm{SU}(2,1)$ (see relation (2.6) in the proof of Proposition 2.6).

Remark 2 If $f(\operatorname{tr} A) = 0$, the isometry A may either be a complex reflection or a parabolic isometry. There are two main types of parabolic isometries: unipotent (or pure) parabolic and screw parabolic (which are also called *ellipto-parabolic* in [14]). A parabolic isometry is pure if and only if it admits a lift of trace 3. The set of pure parabolic isometries fixing a given boundary point is a copy of the Heisenberg group. There are two $\mathrm{PU}(2,1)$-conjugacy classes of pure parabolics.

- The first one contains those unipotent isometries A such that $A - \mathrm{Id}$ is nilpotent of order 2. In this case, A preserves a complex line, and is sometimes called a *vertical translation*.
- The second one contains those unipotent isometries A such that $A - \mathrm{Id}$ is nilpotent of order 3 if $A - \mathrm{Id}$ is nilpotent of order 3. In this case, A preserves an $\mathbb{R}$-plane, and is sometimes called a *horizontal translation*.

Note that complex lines and $\mathbb{R}$-planes are the two kinds of maximal totally geodesic subspaces in $\mathbf{H}^2_{\mathbb{C}}$. They are defined in the Section 2.3 above. Further information may be found in [14].

Remark 3 Antiholomorphic isometries may also be lifted to $\mathrm{SU}(2,1)$ in the following way: if α is an antiholomorphic isometry, there exists a matrix $A \in \mathrm{SU}(2,1)$ such that for any point m in $\mathbf{H}^2_{\mathbb{C}}$, $\alpha(m) = \mathbf{P}(A\overline{\mathbf{m}})$, where $\mathbf{m}$ is a lift of m to $\mathbb{C}^{2,1}$.

2.3 Involutions and Totally Geodesic Subspaces of $\mathbf{H}^2_{\mathbb{C}}$

Since the metric associated with the distance function given by relation (2.1) has non-constant curvature (see [14, Ch. 3]), there are no totally geodesic real hypersurface in $\mathbf{H}^2_{\mathbb{C}}$ and the maximal totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$ have real dimension two. There are two types of such subspaces: the complex lines and the $\mathbb{R}$-planes.

Definition 2.4 An $\mathbb{R}$-*plane* is the intersection with $\mathbf{H}^2_{\mathbb{C}}$ of the projectivization of a Lagrangian vector subspace of $\mathbb{C}^{2,1}$. A *complex line* of $\mathbf{H}^2_{\mathbb{C}}$ is the intersection of a complex projective line of $\mathbb{C}\mathbf{P}^2$ with $\mathbf{H}^2_{\mathbb{C}}$, whenever this intersection is non-empty.

The $\mathbb{R}$-planes are all the images of the subset of $\mathbf{H}^2_{\mathbb{C}}$ defined by

$$\mathbf{H}^2_{\mathbb{R}} = \{(x, 0, u), x \in \mathbb{R},\ u > 0\}$$

under PU(2, 1). The reference $\mathbb{R}$-plane $\mathbf{H}^2_{\mathbb{R}}$ is fixed by the complex conjugation. As a consequence, any $\mathbb{R}$-plane P is fixed pointwise by a unique antiholomorphic isometry of order 2: the *Lagrangian reflection about* P. See [14, 34] for more details.

The complex lines are all the images of the subset of $\mathbf{H}^2_{\mathbb{C}}$ defined by

$$\{(0, t, u), t \in \mathbb{R},\ u > 0\}$$

under PU(2, 1). The latter subspace is the intersection of $\mathbf{H}^2_{\mathbb{C}}$ with $\mathbf{P}(\mathbf{c}_0^{\perp})$, where $\mathbf{c}_0^{\perp}$ is the subspace of $\mathbb{C}^{2,1}$ hermitian orthogonal to the positive vector $\mathbf{c}_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. Hence, there is a bijection between the set of complex lines of $\mathbf{H}^2_{\mathbb{C}}$ and the subset $\mathbf{P}(V^+)$ of $\mathbb{C}\mathbf{P}^2$. If C is a complex line, a vector $\mathbf{c}$ of $\mathbb{C}^{2,1}$ such that $C = \mathbf{P}(\mathbf{c}^{\perp}) \cap \mathbf{H}^2_{\mathbb{C}}$ is said to be *polar to* C. A complex line C is fixed by a unique involutive holomorphic isometry, associated to the transformation of SU(2, 1) given by

$$Z \longmapsto -Z + 2\frac{\langle Z, \mathbf{c}\rangle}{\langle \mathbf{c}, \mathbf{c}\rangle}\mathbf{c}, \tag{2.5}$$

where $\mathbf{c}$ is polar to C. We will call this isometry the *complex symmetry about* C.

Remark 4 Let I_1 and I_2 be two Lagrangian reflections with lifts to SU(2, 1) the matrices M_1 and M_2. Since the I_k's are anti holomorphic, a lift of the product $I_1 \circ I_2$ is given by $M_1\overline{M}_2$. Similarly, the fact that the I_k's are involutions is written in terms of the M_k's through the relation $M_k\overline{M}_k = 1$.

Remark 5 Pairs of complex lines (C_1, C_2) are classified up to PU(2, 1) by the invariant $\Phi = |\langle \mathbf{c}_1, \mathbf{c}_2\rangle|^2/\langle \mathbf{c}_1, \mathbf{c}_1\rangle\langle \mathbf{c}_2, \mathbf{c}_2\rangle$. This invariant describes the relative position of C_1 and C_2, and Φ is greater than 1 if and only if C_1 and C_2 are disjoint. We refer the reader to [14] for more information.

2.4 More about Loxodromic Isometries

In this paragraph, $A \in SU(2, 1)$ is a loxodromic isometry of $\mathbf{H}^2_{\mathbb{C}}$ (see [14, pp. 204–209]).

Proposition 2.5 *There exists $\lambda \in \mathbb{C} \setminus 0$ such that $|\lambda| \neq 1$ and the eigenvalues of A are $\lambda, \bar{\lambda}/\lambda$, and $1/\bar{\lambda}$.*

Remark 6 The fixed point p_A (resp. q_A) of A associated with the eigenvalue of modulus greater (resp. smaller) than 1 is attractive (resp. repulsive). The third fixed point of A belongs to $\mathbf{P}(V^+)$, and any lift of it is polar to the complex line of $\mathbf{H}^2_{\mathbb{C}}$ spanned by p_A and q_A.

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

Remark 7 It follows from Proposition 2.5 that A is conjugate in $\mathrm{SU}(2,1)$ to the diagonal matrix $\mathrm{diag}(\lambda, \bar{\lambda}/\lambda, 1/\bar{\lambda})$.

Proposition 2.6 *Two loxodromic isometries are conjugate in* $\mathrm{SU}(2,1)$ *if and only if they have the same trace (up to multiplication by a cubic root of* 1*).*

Proof The characteristic polynomial of A is

$$\chi_A = X^3 - \operatorname{tr} A \cdot X^2 + \operatorname{tr} A^{-1} \cdot X - 1 = X^3 - \operatorname{tr} A \cdot X^2 + \overline{\operatorname{tr} A} \cdot X - 1. \tag{2.6}$$

The spectrum of A is thus determined by $\operatorname{tr} A$ (and does not change if one multiplies A by a cube root of 1). ∎

Remark 8 Note also that the conjugacy class of A is also fully determined by one of its eigenvalues of modulus different from 1.

The following proposition will be needed in Section 4.4.

Proposition 2.7 (i) *If E_1 and E_2 are two isometric involutions such that $A = E_1 \circ E_2$, both E_1 and E_2 permute the fixed points of A.*

(ii) *Let ι_1 be a Lagrangian reflection. There exists ι_2 such that $A = \iota_1 \circ \iota_2$ if and only if ι_1 swaps the fixed points of A.*

(iii) *The following two conditions are equivalent.*

  (a) $\operatorname{tr} A$ *is real.*

  (b) *Either A or A^2 may be decomposed in the form $I_1 \circ I_2$, with I_1 and I_2 two complex symmetries.*

Proof Both (i) and (ii) are classical results (see for instance [7,34]). Let us prove (iii). It is a simple computation to check that if A is loxodromic with real trace, then (2.6) has three roots $(1, t, t^{-1})$ with $t \in \mathbb{R}$. Hence, λ is real. Using the fact that for a real number x, the function f given in Proposition 2.3 factors to $f(x) = (x+1)(x-3)^3$; we see that one of two options occurs: $\lambda > 1$ and $\operatorname{tr} A > 3$, or $\lambda < -1$ and $\operatorname{tr} A < -1$. In the first case, A is conjugate to

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & -1 & 0 \\ 1/\lambda & 0 & 0 \end{bmatrix}.$$

The first (resp. second) matrix in the right-hand side product corresponds to the symmetry about the complex line polar to $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ (resp. $\begin{bmatrix} \lambda & 0 & 1 \end{bmatrix}^T$). In the second case, λ^2 is greater than 1, and thus A^2 may be decomposed in the same way.

Assume now that A can be decomposed into $I_1 \circ I_2$. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be polar vectors to the mirrors of I_1 and I_2, such that $\langle \mathbf{c}_k, \mathbf{c}_k \rangle = 1$. Using relation (2.5), it is seen that $\operatorname{tr} A = -1 + 4|\langle \mathbf{c}_1, \mathbf{c}_2 \rangle|^2$ is real. Since A is loxodromic, the mirrors of I_1 and I_2 are disjoints, which yields in terms of the $\mathbf{c}_i$'s that $|\langle \mathbf{c}_1, \mathbf{c}_2 \rangle|$ is greater than 1 (see Remark 5, and also Section 3.3). If A^2 can be decomposed as a product of two complex symmetries, then $\operatorname{tr}(A^2) > 3$. Now, using the Cayley–Hamilton theorem,

$$\operatorname{tr}(A^2) = (\operatorname{tr} A)^2 - 2\overline{\operatorname{tr} A}. \tag{2.7}$$

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

As a consequence, either $\operatorname{tr} A$ is real or $\operatorname{Re}(\operatorname{tr} A) = -1$. The latter case leads to $\operatorname{tr}(A^2) \leqslant 3$, which is absurd. ■

2.5 The Cartan Invariant

Definition 2.8 Let p_1, p_2, and p_3 be three points of $\partial \mathbf{H}^2_{\mathbb{C}}$. The quantity defined by $\mathbb{A}(p_1, p_2, p_3) = -\arg\big(\langle \mathbf{p}_1, \mathbf{p}_2\rangle\langle \mathbf{p}_2, \mathbf{p}_3\rangle\langle \mathbf{p}_3, \mathbf{p}_1\rangle\big)$ does not depend on the choice of the lifts $\mathbf{p}_i$'s of the p_i's, and is called the Cartan invariant of the triple (p_1, p_2, p_3).

The following proposition sums up the main properties of the Cartan invariant (see [14, Ch. 7] for a proof).

Proposition 2.9 (i) *Two triples of boundary points are identified by an element of* PU(2, 1) *if and only if they have the same Cartan invariant.*

(ii) *The Cartan invariant satisfies the cocycle relation*

$$\mathbb{A}(p_1, p_2, p_3) - \mathbb{A}(p_1, p_2, p_4) + \mathbb{A}(p_1, p_3, p_4) - \mathbb{A}(p_2, p_3, p_4) = 0, \tag{2.8}$$

for any 4-tuple of boundary points.

(iii) *Let (p_1, p_2, p_3) be a triple of boundary points of $\mathbf{H}^2_{\mathbb{C}}$.*
   - (a) $\mathbb{A}(p_1, p_2, p_3)$ *belongs to the interval* $[-\pi/2, \pi/2]$.
   - (b) $\mathbb{A}(p_1, p_2, p_3) = 0$ *if and only if the p_i's lie in the boundary of an $\mathbb{R}$-plane.*
   - (c) $|\mathbb{A}(p_1, p_2, p_3)| = \pi/2$ *if and only if the p_i's lie in the boundary of a complex line.*

We will need the following lemma.

Lemma 2.10 *Let p_1, p_2, p_3, and p_4 be four points of $\partial \mathbf{H}^2_{\mathbb{C}}$, such that $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p_1, p_2, p_4) = \mathbb{A}(p_1, p_3, p_4) = 0$. Then the four points are contained in the boundary of an $\mathbb{R}$-plane.*

Proof Under this assumption, there exist two $\mathbb{R}$-planes, L_3 and L_4, respectively containing (p_1, p_2, p_3) and (p_1, p_2, p_4). Applying, if necessary, a holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$, we may assume that the four points admit the lift to $\mathbb{C}^3$ given by

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{p}_3 = \begin{bmatrix} -1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \text{ and } \mathbf{p}_4 = \begin{bmatrix} -|z|^2 \\ z\sqrt{2} \\ 1 \end{bmatrix} \text{ with } z \in \mathbb{C}.$$

Using these lifts, we compute the hermitian triple product, and obtain

$$\langle \mathbf{p}_1, \mathbf{p}_3\rangle\langle \mathbf{p}_3, \mathbf{p}_4\rangle\langle \mathbf{p}_4, \mathbf{p}_1\rangle = -(1 + |z|^2) + 2\bar{z}.$$

The latter quantity is real if and only if z is real, that is, if the four points are in the $\mathbb{R}$-plane $\mathbf{H}^2_{\mathbb{R}}$. ■

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

3 Trace Coordinates on $\mathcal{M}$

3.1 The Case of $\mathrm{SL}(3,\mathbb{C})$

Definition 3.1 For any pair $(M,N) \in \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$, we call $t_{M,N}$ the list of traces

$$(\operatorname{tr} M, \operatorname{tr} N, \operatorname{tr} MN, \operatorname{tr} M^{-1}N, \operatorname{tr} M^{-1}, \operatorname{tr} N^{-1}, \operatorname{tr}(MN)^{-1}, \operatorname{tr}(M^{-1}N)^{-1}).$$

We will need the following result.

Theorem 3.2 *Two irreducible representations ρ_1 and ρ_2 of F_2 in $\mathrm{SL}(3,\mathbb{C})$ are conjugate in $\mathrm{SL}(3,\mathbb{C})$ if and only if*

(i) $t_{\rho_1(\mathtt{m}),\rho_1(\mathtt{n})} = t_{\rho_2(\mathtt{m}),\rho_2(\mathtt{n})}$,
(ii) $\operatorname{tr}(\rho_1([\mathtt{m},\mathtt{n}])) = \operatorname{tr}(\rho_2([\mathtt{m},\mathtt{n}]))$.

Theorem 3.2 has been proved independently in several works [20, 30, 33]. See also [9, Ch. 10]. It is a consequence of Propositions 3.3 and 3.4 above.

Proposition 3.3 *There exist two polynomials S and P in $\mathbb{Z}[x_1,\dots,x_8,T]$ such that for any pair $(M,N) \in \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$ the two traces $\operatorname{tr}([M,N])$ and $\operatorname{tr}([M^{-1},N])$ are the roots of the following polynomial in T: $T^2 - S(t_{M,N})T + P(t_{M,N})$.*

The proof of this proposition is done by showing that both the sum and product of $\operatorname{tr}([M,N])$ and $\operatorname{tr}([M^{-1},N])$ are polynomials in the $t_{M,N}$. The main technique is to make repeated use of trace identities obtained from the Cayley–Hamilton theorem (see [9, 20, 34]).

Proposition 3.4 *Let $\mathtt{w}$ be a element of F_2. There exists a polynomial*

$$P_{\mathtt{w}} \in \mathbb{C}[x_1,\dots,x_8,T]$$

such that for any representation $\rho \in \mathrm{Hom}(F_2, \mathrm{SL}(3,\mathbb{C}))$,

$$\operatorname{tr}(\rho(\mathtt{w})) = P_{\mathtt{w}}\big(t_{\rho(\mathtt{m}),\rho(\mathtt{n})}, \operatorname{tr}\rho([\mathtt{m},\mathtt{n}])\big)$$

Proof See [9, 20, 34] for a proof. ∎

Proposition 3.4 means that the ring of invariants of $\mathrm{SL}(3,\mathbb{C})\times\mathrm{SL}(3,\mathbb{C})$ is generated by the polynomials $t_{M,N}$ and $\operatorname{tr}([M,N])$.

Remark 9 The categorical quotient of $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$ by the diagonal action of $\mathrm{SL}(3,\mathbb{C})$ is thus

$$\mathbb{C}[x_1,\dots,x_8,T]/(T^2 - ST + P).$$

3.2 Passing from $\mathrm{SL}(3,\mathbb{C})$ to $\mathrm{SU}(2,1)$

Definition 3.5 For any representation ρ of F_2 in $\mathrm{SU}(2,1)$, let $t_\rho^{\mathrm{SU}(2,1)}$ be the list

$$\left(\operatorname{tr}\rho(\mathtt{m}),\operatorname{tr}\rho(\mathtt{n}),\operatorname{tr}\rho(\mathtt{mn}),\operatorname{tr}\rho(\mathtt{m}^{-1}\mathtt{n}),\operatorname{tr}\rho([\mathtt{m},\mathtt{n}])\right).$$

In the case of a representation of F_2 in $\mathrm{SU}(2,1)$, we can reduce the number of traces involved in Proposition 3.4 using relation (2.4). This yields the following.

Proposition 3.6 *Let* $\mathtt{w}$ *be an element of* F_2. *There exists a polynomial* $Q_{\mathtt{w}} \in \mathbb{C}[z,\bar{z}]$ *(where* $z=(z_1,z_2,z_3,z_4,z_5)$*) such that for any representation* $\rho \in \operatorname{Hom}(F_2,\mathrm{SU}(2,1))$,

$$\operatorname{tr}(\rho(\mathtt{w})) = Q_{\mathtt{w}}\left(t_\rho^{\mathrm{SU}(2,1)},\overline{t_\rho^{\mathrm{SU}(2,1)}}\right).$$

The following proposition was found independently by V. T. Khoi [18].

Proposition 3.7 *Let* ρ_1 *and* ρ_2 *be two representations of* F_2 *in* $\mathrm{SU}(2,1)$ *such that* $\rho_1(F_2)$ *and* $\rho_2(F_2)$ *are Zariski-dense in* $\mathrm{SU}(2,1)$. *The two representations are conjugate in* $\mathrm{SU}(2,1)$ *if and only if* $t_{\rho_1}^{\mathrm{SU}(2,1)} = t_{\rho_2}^{\mathrm{SU}(2,1)}$.

Remark 10 A subgroup of $\mathrm{PU}(2,1)$ is Zariski-dense if and only if it acts on $\mathbb{C}\mathbf{P}^2$ without global fixed point.

Proof Since the traces are conjugacy invariants, it suffices to show that ρ_1 and ρ_2 are conjugate in $\mathrm{SU}(2,1)$ whenever $t_{\rho_1}^{\mathrm{SU}(2,1)} = t_{\rho_2}^{\mathrm{SU}(2,1)}$.

Under the latter assumption, it follows from Proposition 3.2 that ρ_1 and ρ_2 are conjugate in $\mathrm{SL}(3,\mathbb{C})$. Thus, there exists a matrix $A \in \mathrm{SL}(3,\mathbb{C})$ such that

$$\rho_2(\mathtt{w}) = A\rho_1(\mathtt{w})A^{-1} \text{ for any word } \mathtt{w} \in F_2.$$

Denote by $\|Z\|^2$ the hermitian product $\langle Z,Z\rangle$ associated with the hermitian form. Consider the hermitian form on $\mathbb{C}^3$ defined by $N(Z)=\|AZ\|$. The image of ρ_1 is contained in the unitary group associated with N: $N(\rho_1(\mathtt{w})x) = \|\rho_2(\mathtt{w})Ax\|$, and since $\rho_2(\mathtt{w})$ is in $\mathrm{SU}(2,1)$, $N(\rho_1(\mathtt{w})\cdot x) = N(x)$ for any $\mathtt{w} \in F_2$. This equality extends to all of $\mathrm{SU}(2,1)$ because the image of ρ_1 is Zariski-dense: $N(Mx)=N(x)$ for any $M \in \mathrm{SU}(2,1)$. In other words, N is $\mathrm{SU}(2,1)$-invariant. As a consequence, N is proportional to $\|\cdot\|$, and there exists a complex number λ such that $\lambda A \in SU(2,1)$. The result follows. ■

Remark 11 If M and N are two matrices of $\mathrm{SU}(2,1)$, then

$$\operatorname{tr}[M^{-1},N] = \operatorname{tr} M^{-1}NMN^{-1} = \operatorname{tr} NMN^{-1}M^{-1} = \operatorname{tr}[M,N]^{-1}.$$

Hence, $\operatorname{tr}([M,N])$ and $\operatorname{tr}([M^{-1},N])$ are conjugate. In consequence of Propositions 3.3 and 3.7, once the conjugacy classes of M, N, MN, and $M^{-1}N$ are fixed, there are two possible conjugacy classes for the group $\langle M,N\rangle$, corresponding to the two possible (complex conjugate) values for $\operatorname{tr}([M,N])$.

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

Remark 12 Let $\mathcal{V}$ be the real algebraic subvariety of $\mathbb{C}^5$ associated with the polynomial $T^2 - S(z,\bar{z})T + P(z,\bar{z})$, where $z = (z_1, z_2, z_3, z_4)$. For any $\rho \in \mathcal{M}$, the point $(\operatorname{tr}\rho(\mathtt{m}), \operatorname{tr}\rho(\mathtt{n}), \operatorname{tr}\rho(\mathtt{mn}), \operatorname{tr}\rho(\mathtt{m}^{-1}\mathtt{n}))$ lies on $\mathcal{V}$. Note that not all points of $\mathcal{V}$ correspond to representations of F_2 into $\mathrm{SU}(2,1)$. We provide in Section 4 a condition on a point of $\mathcal{V}$ to represent an element of $\mathcal{M}^{\mathrm{lox}}$, which is simply expressed in terms of cross-ratios (see Proposition 4.5 and Corollary 4.6).

3.3 Trace in Complex Triangle Groups

Definition 3.8 A *complex triangle group* is a subgroup of $\mathrm{PU}(2,1)$ generated by three complex symmetries.

Pratoussevitch gave a criterion to decide whether two complex triangle groups are conjugate or not [23]. As we will see in this section, this criterion follows from Proposition 3.7. We first recall the definition of the coordinate system on complex triangle groups described by Pratoussevitch.

Let $\mathcal{G}$ be the set of triples of complex symmetries. The duality between the complex lines of $\mathbf{H}^2_{\mathbb{C}}$ and their polar vectors associates with an element of $\mathcal{G}$ a triple $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ of unit vectors of $\mathbb{C}^{2,1}$ (*i.e.,* such that $\langle \mathbf{c}_k, \mathbf{c}_k\rangle = 1$). The complex symmetry I_k is then given by $I_k(Z) = -Z + 2\langle Z, \mathbf{c}_k\rangle \mathbf{c}_k$.

We will use the following notation.

- We define z_1, z_2, and z_3 by $z_k = \langle n_{k+1}, n_{k+2}\rangle = r_k e^{i\theta_k}$, with $r_k > 0$ and $\theta_k \in [0, 2\pi[$ (indices are taken modulo 3).
- The real number $\alpha = \arg\bigl(\prod_{k=1}^{3} \langle n_{k+1}, n_{k+2}\rangle\bigr)$ is called the *angular invariant* of the triple (I_1, I_2, I_3). It equals $\theta_1 + \theta_2 + \theta_3$ modulo 2π.

The four quantities r_1, r_2, r_3, and α are independent of the chosen unit lifts (note that the θ_k's are not invariant, but their sum is). Define the following mapping:

$$\varphi\colon \mathcal{G} \longrightarrow \mathbb{R}^3 \times S^1$$
$$(I_1, I_2, I_3) \longrightarrow (r_1, r_2, r_3, \alpha).$$

As we will see, φ is not onto $\mathbb{R}^3 \times S^1$. Precisely, the following lemma is due to Pratoussevitch [23]. We give a proof of it for completeness.

Lemma 3.9 *The image of φ is the set $\{(r_1, r_2, r_3, \alpha), 2r_1r_2r_3\cos\alpha < r_1^2 + r_2^2 + r_3^2 - 1\}$.*

Proof The existence of a triple of complex symmetries satisfying

$$\varphi(I_1, I_2, I_3) = (r_1, r_2, r_3, \alpha)$$

is equivalent to the existence of a triple of unit vectors (n_1, n_2, n_3) such that $|z_k| = r_k$ $(k = 1, 2, 3)$ and $\arg(z_1z_2z_3) = \alpha$. These values are realized if and only if the Gram matrix associated with these three vectors, $Q = (\langle n_i, n_j\rangle)_{(i,j)}$ has signature $(2,1)$. Since Q has trace 3, it has exactly one negative eigenvalue if and only if its determinant is negative. By a computation, we obtain:

$$\det Q = 2r_1r_2r_3\cos\alpha - (r_1^2 + r_2^2 + r_3^2) + 1. \tag{3.1}$$ ■

Proposition 3.10 *Consider* (I_1, I_2, I_3) *and* (I_1', I_2', I_3')*, two triples of complex symmetries, and call* G *and* G' *the associated triangle groups. Assume that* G *and* G' *are Zariski dense. The following conditions are equivalent.*

(i) $\varphi(I_1, I_2, I_3) = \varphi(I_1', I_2', I_3')$.

(ii) *There exists a holomorphic isometry* $g \in \mathrm{PU}(2,1)$ *such that* $I_k' = gI_kg^{-1}$, $k = 1, 2, 3$.

Proof The quantities r_1, r_2, r_3, and α are conjugacy invariants. Thus (ii) implies (i).

To prove the second implication, we set $A = I_1I_2$ and $B = I_3I_2$, and compute the traces of A, B, AB, $A^{-1}B$, and $[A,B]$ from r_1, r_2, r_3, and α. The result follows from Proposition 3.7. The computation of these traces is based on the following two remarks:

- Let $\mathbf{c}$ be a vector of $\mathbb{C}^{2,1}$, polar to a complex line $C \subset \mathbf{H}^2_{\mathbb{C}}$, such that $\langle \mathbf{c}, \mathbf{c}\rangle = 1$. The complex symmetry with respect to C may be written $-\mathrm{Id} + 2\mathbf{c}\mathbf{c}^*$, where $\mathbf{c}^*$ is the linear form dual to $\mathbf{c}$.
- If $\mathbf{c}_1$, $\mathbf{c}_2, \dots, \mathbf{c}_n$ are vectors in $\mathbb{C}^{2,1}$, then it holds $\mathrm{tr}\big((\mathbf{c}_1\mathbf{c}_1^*)(\mathbf{c}_2\mathbf{c}_2^*)\cdots(\mathbf{c}_n\mathbf{c}_n^*)\big) = \langle \mathbf{c}_2, \mathbf{c}_1\rangle\langle \mathbf{c}_3, \mathbf{c}_2\rangle \cdots \langle \mathbf{c}_1, \mathbf{c}_n\rangle$ (see [23] for details).

Using these facts, we compute the traces, and obtain

$$\mathrm{tr}\,A = 4r_3^2 - 1, \quad \mathrm{tr}\,B = 4r_1^2 - 1, \quad \mathrm{tr}\,A^{-1}B = \mathrm{tr}\,I_1I_3 = 4r_2^2 - 1, \tag{3.3}$$

$$\mathrm{tr}\,AB = \mathrm{tr}\,I_1I_2I_3I_2 = 16r_1r_3(r_1r_3 - r_2\cos\alpha) + 4r_2^2 - 1. \tag{3.4}$$

We know from relation (2.7) that $\mathrm{tr}[A,B] = \mathrm{tr}(I_1I_2I_3)^2 = (\mathrm{tr}\,I_1I_2I_3)^2 - 2\overline{\mathrm{tr}\,I_1I_2I_3}$. Hence, it suffices to show that $\mathrm{tr}\,I_1I_2I_3$ is determined by the r_i's and α. A direct computation shows that $\mathrm{tr}\,I_1I_2I_3 = 8r_1r_2r_3e^{i\alpha} - 4(r_1^2 + r_2^2 + r_3^2) + 3$. The result follows. ∎

Remark 13 Sandler [25] proved a combinatorial formula allowing recursive calculation of traces in an ideal triangle group, which Pratoussevitch [23] generalized to the case of groups generated by three arbitrary complex reflections.

Remark 14 Let us express the invariants r_1, r_2, r_3, and $\cos(\alpha)$ in in terms of traces. Keeping $A = I_1I_2$ and $B = I_3I_2$,

$$r_1^2 = \frac{1 + \mathrm{tr}\,B}{4}, \quad r_2^2 = \frac{1 + \mathrm{tr}\,A^{-1}B}{4}, \quad \text{and} \quad r_3^2 = \frac{1 + \mathrm{tr}\,A}{4}.$$

Plugging these values into (3.4), we obtain

$$\mathrm{tr}\,AB = (1 + \mathrm{tr}\,A)(1 + \mathrm{tr}\,B) - 16r_1r_2r_3\cos\alpha + \mathrm{tr}\,A^{-1}B.$$

This yields finally

$$\cos\alpha = \frac{1}{2}\frac{(1 + \mathrm{tr}\,A)(1 + \mathrm{tr}\,B) + \mathrm{tr}\,A^{-1}B - \mathrm{tr}\,AB}{\sqrt{(1 + \mathrm{tr}\,A)(1 + \mathrm{tr}\,B)(1 + \mathrm{tr}\,A^{-1}B)}}.$$

Again, the values of $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr} AB$, and $\operatorname{tr} A^{-1}B$ determine the conjugacy class of the triple (I_1, I_2, I_3) only up to an order two indetermination, which corresponds to the two possible values of $\sin(\alpha)$.

The condition (3.1) guaranteeing the existence of a triangle group for given parameters r_1, r_2, r_3, α may now be rewritten in terms of traces:

$$\operatorname{tr} A \operatorname{tr} B - (\operatorname{tr} A + \operatorname{tr} B + \operatorname{tr} AB + \operatorname{tr} A^{-1}B) + 3 < 0.$$

We finish this section with the following remark which will be needed in Section 6.2.

Remark 15 We will say that a complex triangle group $\langle I_1, I_2, I_3\rangle$ is *symmetric* if there exists an isometry E of order 3 such that $I_2 = EI_1E^{-1}$ and $I_3 = E^{-1}I_1E$. A complex triangle group is symmetric if and only if $z_1 = z_2 = z_3 = z$. The condition of existence of a symmetric triangle group with parameter z is obtained from (3.1):

$$2\mathrm{Re}\,(z^3) < 3|z|^2 - 1.$$

Two symmetric triangle groups are conjugate in $\mathrm{PU}(2,1)$ if and only if their parameters z are equal modulo multiplication by a cube root of 1. As a consequence, any symmetric complex triangle group is represented by a unique z in the domain:

$$\mathcal{D} = \{lx + iy, x < -1/2, y > 1 + \sqrt{3}x, y > 1 - \sqrt{3}x\}. \tag{3.5}$$

4 KR Coordinates on $\mathcal{M}^{\mathrm{lox}}$

4.1 The Complex Cross-Ratio: Definitions.

We will call an *ideal tetrahedron* of $\mathbf{H}^2_{\mathbb{C}}$ any ordered 4-tuple of boundary points of $\mathbf{H}^2_{\mathbb{C}}$. There is a slight abuse here, since a tetrahedron is usually defined to be a simplex. Here, we will not deal with faces or edges.

Definition 4.1 [19] Let $(p_1, p_2, p_3, p_4) \subset (\mathbf{H}^2_{\mathbb{C}})^4$ be an ideal tetrahedron, and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ be lifts of the $\mathbf{p}_i$'s to $\mathbb{C}^{2,1}$. The complex number

$$\mathbf{X}(p_1, p_2, p_3, p_4) = \frac{\langle \mathbf{p}_3, \mathbf{p}_1\rangle\langle \mathbf{p}_4, \mathbf{p}_2\rangle}{\langle \mathbf{p}_4, \mathbf{p}_1\rangle\langle \mathbf{p}_3, \mathbf{p}_2\rangle} \in \mathbb{C} \setminus \{0\}$$

is independent of the choice of the lifts, and is called the *complex cross-ratio* of the ideal tetrahedron (p_1, p_2, p_3, p_4).

As a direct consequence of the definition, it is seen that the complex cross-ratio satisfies the following identities

$$\begin{aligned} \mathbf{X}(p_1, p_2, p_3, p_4) &= \mathbf{X}(p_2, p_1, p_3, p_4)^{-1} = \mathbf{X}(p_1, p_2, p_4, p_3)^{-1} \\ &= \overline{\mathbf{X}(p_3, p_4, p_1, p_2)}. \end{aligned} \tag{4.1}$$

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

Remark 16 The complex cross-ratio and the hermitian triple product are linked by the identity

$$\mathbf{X}(p_1,p_2,p_3,p_4)=\frac{\langle\mathbf{p}_1,\mathbf{p}_2\rangle\langle\mathbf{p}_2,\mathbf{p}_3\rangle\langle\mathbf{p}_3,\mathbf{p}_1\rangle}{\langle\mathbf{p}_1,\mathbf{p}_2\rangle\langle\mathbf{p}_2,\mathbf{p}_4\rangle\langle\mathbf{p}_4,\mathbf{p}_1\rangle}\cdot\frac{|\langle\mathbf{p}_2,\mathbf{p}_4\rangle|^2}{|\langle\mathbf{p}_2,\mathbf{p}_3\rangle|^2}.$$

As a consequence, the complex cross-ratio is related to the Cartan invariant by

$$\begin{aligned}\arg(\mathbf{X}(p_1,p_2,p_3,p_4)) &= \arg(\langle\mathbf{p}_1,\mathbf{p}_2\rangle\langle\mathbf{p}_2,\mathbf{p}_3\rangle\langle\mathbf{p}_3,\mathbf{p}_1\rangle)\\ &\qquad -\arg(\langle\mathbf{p}_1,\mathbf{p}_2\rangle\langle\mathbf{p}_2,\mathbf{p}_4\rangle\langle\mathbf{p}_4,\mathbf{p}_1\rangle)\\ &\equiv \mathbb{A}(p_1,p_2,p_4)-\mathbb{A}(p_1,p_2,p_3).\end{aligned}\tag{4.2}$$

Now using relation (4.1), it is seen that the product $\mathbf{X}(p_1,p_2,p_3,p_4)\mathbf{X}(p_3,p_4,p_1,p_2)$ is real and positive. Using relation (4.2), it is a direct calculation to obtain the cocyle relation (2.8).

The following proposition provides a geometric interpretation of the complex cross-ratio.

Proposition 4.2 **(Goldman [14])** *Let (p_1,p_2,p_3,p_4) be an ideal tetrahedron of $\mathbf{H}^2_{\mathbb{C}}$, and C the complex line generated by p_1 and p_2. Let z_{12} be a coordinate on C, such that $z_{12}(C)$ is the right half-plane of $\mathbb{C}$, with $z_{12}(p_1)=0$ and $z_{12}(p_2)=\infty$. Then*

$$\mathbf{X}(p_1,p_2,p_3,p_4)=\frac{z_{12}(\Pi(p_4))}{z_{12}(\Pi(p_3))}.$$

Proof See [14, p. 227]. ∎

Corollary 4.3 *Let p_1, p_2, p_3, and p_4 be boundary points of $\mathbf{H}^n_{\mathbb{C}}$, such that*

$$\mathbf{X}(p_1,p_2,p_3,p_4)$$

is real and negative. Then the four points are contained in a common complex line, and the geodesic (p_1p_2) separates p_3 and p_4.

Proof Using Proposition 4.2, we see that the only possibility for $\mathbf{X}(x_1,x_2,x_3,x_4)$ to be real and negative is that $z_{12}(\Pi(x_3))$ and $z_{12}(\Pi(x_4))$ are both on the imaginary axis of $\mathbb{C}$ with opposite argument. The result follows. ∎

4.2 Definition of the KR Coordinates on $\mathcal{M}^{\mathrm{lox}}$

4.2.1 Classification of Ideal Tetrahedra

Definition 4.4 Let p_1, p_2, p_3, and p_4 be four points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Define ω_a, ω_b, and ω_c to be the three cross-ratios given by $\omega_a=\mathbf{X}(p_1,p_2,p_3,p_4)$, $\omega_b=\mathbf{X}(p_1,p_4,p_2,p_3)$, and $\omega_c=\mathbf{X}(p_1,p_3,p_4,p_2)$. We denote by $[[p_1,p_2,p_3,p_4]]$ the vector $\begin{bmatrix}\omega_a & \omega_b & \omega_c\end{bmatrix}^T$.

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

Not all vectors of $\mathbb{C}^3$ can be seen as $[[\tau]]$ for some ideal tetrahedron τ. More precisely, the following proposition is due to Falbel [3].

Proposition 4.5 **(Falbel)** *Let z_a, z_b, and z_c be three complex numbers. The following two conditions are equivalent.*

(i) *There exists an ideal tetrahedron τ such that $[[\tau]] = \begin{bmatrix} z_a & z_b & z_c \end{bmatrix}^T$;*
(ii) *z_a, z_b and z_c satisfy the two relations*

$$|z_a z_b z_c| = 1, \tag{4.3}$$

$$2\mathrm{Re}\ (z_a) = \frac{1}{|z_c|^2}\left(\left|1 - \frac{1}{z_b}\right|^2 - 1\right) + \left|1 - \frac{1}{z_c}\right|^2. \tag{4.4}$$

Now, even two arbitrary complex numbers cannot be two of the three cross ratios of a tetrahedron, (see [22]).

Corollary 4.6 *Let z_b and z_c be two complex numbers. There exist $z_a \in \mathbb{C}$ and an ideal tetrahedron τ such that $[[\tau]] = \begin{bmatrix} z_a & z_b & z_c \end{bmatrix}^T$ if and only if z_b and z_c satisfy the following inequality:*

$$\left|1 - |\omega_c - |^2 - \left|\frac{1}{\omega_b} - 1\right|^2\right| \leqslant 2\frac{|\omega_c|}{|\omega_b|}. \tag{4.5}$$

Note that Parker and Platis [22] used a different convention in the choice of the three cross-ratios classifying ideal tetrahedra. Their three cross-ratios are called $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$, and are related to ours by a permutation of p_1, p_2, p_3, and p_4. The following proposition is due to Falbel[3].

Proposition 4.7 **(Falbel)** *Let τ_1 and τ_2 be two ideal tetrahedra of $\mathbf{H}^2_{\mathbb{C}}$. There exists an isometry g such that $g(\tau_1) = \tau_2$ if and only if $[[\tau_1]] = [[\tau_2]]$.*

Falbel's original proof uses a normalized form for ideal tetrahedra. A proof using only linear algebra may be found in [34, Ch. 4].

Corollary 4.8 *Let τ_1 and τ_2 be two ideal tetrahedra of $\mathbf{H}^2_{\mathbb{C}}$. There exists an antiholomorphic isometry g such that $g(\tau_1) = \tau_2$ if and only if $[[\tau_1]] = \overline{[[\tau_2]]}$.*

Proof The forward implication is clear from the definition of the complex cross-ratio. To prove the reverse, let I be any Lagrangian reflection. Then $[[I(\tau_2)]] = [[\tau_1]]$. Applying Proposition 4.7, we obtain a holomorphic isometry h mapping τ_1 to $I(\tau_2)$. The isometry $I \circ h$ is an antiholomorphic isometry mapping τ_1 to τ_2. ∎

Definition 4.9 An ideal tetrahedron is said to be *flat* if it is contained in a totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$.

Lemma 4.10 *An ideal tetrahedron $\tau = (p_1, p_2, p_3, p_4)$ is flat if and only if ω_a, ω_b, and ω_c are all real. Moreover,*

(i) *τ is contained in a complex line if and only if one of ω_a, ω_b, and ω_c is negative;*

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

(ii) *τ is contained in an $\mathbb{R}$-plane if and only if ω_a, ω_b, and ω_c are all positive.*

Proof (i) This case is a consequence of Corollary 4.3.

(ii) Assume that the three numbers ω_a, ω_b, and ω_c are real and positive. The relation (4.2) yields the following equalities

$$\mathbb{A}(p_1, p_2, p_4) = \mathbb{A}(p_1, p_2, p_3),$$
$$\mathbb{A}(p_1, p_4, p_3) = \mathbb{A}(p_1, p_4, p_2),$$
$$\mathbb{A}(p_1, p_3, p_2) = \mathbb{A}(p_1, p_3, p_4).$$

Since $\mathbb{A}(p_1, p_3, p_2) = -\mathbb{A}(p_1, p_2, p_4)$, we obtain

$$-\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p_1, p_2, p_4) = \mathbb{A}(p_1, p_2, p_4) = \mathbb{A}(p_1, p_2, p_3) = 0$$

Hence, Lemma 2.10 shows that p_1, p_2, p_3, and p_4 are contained in an $\mathbb{R}$-plane.

Conversely, if the four points are contained in an $\mathbb{R}$-plane,

$$\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p_1, p_2, p_4) = 0.$$

Thus $\omega_a = \mathbf{X}(p_1, p_2, p_3, p_4)$ is real and positive, as shown by relation (4.2). In a similar way, it is checked that ω_b and ω_c are real and positive. ∎

4.2.2 Classification of Pairs of Loxodromic Isometries

Recall that if A is a loxodromic isometry, we denote by p_A and q_A, respectively, its attractive and repulsive fixed points. We denote by $\tau_{A,B}$ the ideal tetrahedron (p_A, q_A, p_B, q_B) associated with a pair of loxodromic isometries (A, B), and by τ_ρ the ideal tetrahedron $\tau(\rho(\mathtt{m}), \rho(\mathtt{n}))$ associated with a representation $\rho \in \mathfrak{R}^{\text{lox}}$.

Lemma 4.11 *Let (A_1, B_1) and (A_2, B_2) be two pairs of loxodromic elements. The following two conditions are equivalent.*

(i) *There exists $C \in \mathrm{PU}(2,1)$ such that $A_2 = CA_1C^{-1}$ and $B_2 = CB_1C^{-1}$.*

(ii) *The following two relations hold.*

  (a) $[[\tau_{A_1,B_1}]] = [[\tau_{A_2,B_2}]]$,

  (b) $\operatorname{tr} A_1 = \operatorname{tr} A_2$ *and* $\operatorname{tr} B_1 = \operatorname{tr} B_2$.

Proof All the quantities involved in (ii) are invariant under conjugation in PU(2, 1). Thus (i) implies (ii).

Assume that (a)and (b) are satisfied. According to Proposition 4.7, (a) implies the existence of some $g \in \mathrm{SU}(2,1)$ mapping τ_{A_1,B_1} onto τ_{A_2,B_2}. The pair (A_2, B_2) is thus conjugate to a pair having the same fixed points as (A_1, B_1). Next, (b) shows that A_1 and A_2 are conjugate, and B_1 and B_2 as well (see Proposition 2.6). Since a loxodromic isometry is fully determined by its conjugacy class and its fixed points, the result is shown. ∎

Proposition 4.12 *The mapping*

$$\Psi\colon \mathcal{M}^{\mathrm{lox}} / \,\mathrm{PU}(2,1) \longrightarrow \mathbb{C}^5$$
$$[\rho] \longmapsto (\mathrm{tr}\,\rho(\mathtt{m}), \mathrm{tr}\;\rho(\mathtt{n}), [\![\tau_\rho]\!])$$

is one-to-one. Its image is the subset of $\mathbb{C}^5$

$$\Big\{ (z_{\mathtt{m}}, z_{\mathtt{n}}, \omega_a, \omega_b, \omega_c,) \;\Big|\; |\omega_a\omega_b\omega_c| = 1,\ 2\mathrm{Re}\,(\omega_c) = \tfrac{1}{|\omega_b|^2}(|1 - \tfrac{1}{\omega_a}|^2 - 1) + |1 - \tfrac{1}{\omega_b}|^2,\ f(z_{\mathtt{m}}) > 0,\ \textit{and}\ f(z_{\mathtt{n}}) > 0 \Big\},$$

where f is the function defined in Proposition 2.3.

Proof A point of $\mathcal{M}^{\mathrm{lox}}$ is a coset for the diagonal action by conjugation of $\mathrm{PU}(2,1)$ on $\mathfrak{R}^{\mathrm{lox}}$. The result is a consequence of the Lemma 4.11 with the notation

$$\omega_a = \mathbf{X}(p_A, q_A, p_B, q_B), \quad \omega_b = \mathbf{X}(p_A, q_B, q_A, p_B), \quad \omega_c = \mathbf{X}(p_A, p_B, q_B, q_A),$$
$$z_{\mathtt{m}} = \mathrm{tr}\,\rho(\mathtt{m}), \quad z_{\mathtt{n}} = \mathrm{tr}\,\rho(\mathtt{n}). \qquad \blacksquare$$

4.3 Symmetries of Ideal Tetrahedra

We denote by S_4 the permutation group of a set $\{1, 2, 3, 4\}$.

Definition 4.13 Let p_1, p_2, p_3, and p_4 be four points of $\partial\mathbf{H}^2_{\mathbb{C}}$, and σ an element of S_4. Define Ω^σ to be the vector

$$\Omega^\sigma = [\![p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}, p_{\sigma(4)}]\!] = \begin{bmatrix} \mathbf{X}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}, p_{\sigma(4)}) \\ \mathbf{X}(p_{\sigma(1)}, p_{\sigma(4)}, p_{\sigma(2)}, p_{\sigma(3)}) \\ \mathbf{X}(p_{\sigma(1)}, p_{\sigma(3)}, p_{\sigma(4)}, p_{\sigma(2)}) \end{bmatrix}.$$

Note that
$$\Omega^{\mathrm{id}} = \begin{bmatrix} \omega_a \\ \omega_b \\ \omega_c \end{bmatrix} = [\![p_1, p_2, p_3, p_4]\!].$$

We denote by $(a_1 \cdots a_n)$ the cyclic permutation mapping a_k to a_{k+1} (for $k = 1, \dots, n$). Now let G_1 be the stabilizer of 1 in S_4. The subgroup G_1 is a copy of S_3 and we will identify it with $\mathfrak{S}\{a, b, c\} \sim S_3$ through the bijection $2 \leftrightarrow a$, $3 \leftrightarrow b$, and $4 \leftrightarrow c$. Let V_4 be the normal subgroup of S_4 given by $V_4 = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. The permutation group S_4 is isomorphic to the semi-direct product $G_1 \ltimes V_4$. As a consequence, any element σ of S_4 may be decomposed in a unique way as follows:

$$\sigma = \sigma_1\sigma_2 \text{ with } \sigma_2 \in V_4 \text{ and } \sigma_1 \in \mathfrak{S}\{a, b, c\}.$$

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

We will denote by $Z = \begin{bmatrix} Z_a & Z_b & Z_c \end{bmatrix}^T$ the coordinates on $\mathbb{C}^3$. Define the following applications

$$f_{\mathrm{id}}(Z) = Z,\ f_{(12)(34)}(Z) = \begin{bmatrix} Z_a \\ \overline{Z}_b \\ \overline{Z}_c \end{bmatrix}, \quad f_{(13)(24)}(Z) = \begin{bmatrix} \overline{Z}_a \\ \overline{Z}_b \\ Z_c \end{bmatrix}, \quad f_{(14)(23)}(Z) = \begin{bmatrix} \overline{Z}_a \\ Z_b \\ \overline{Z}_c \end{bmatrix}.$$

It is a direct consequence of the relation (4.1) that for any $\sigma \in V_4$, $\Omega^\sigma = f_\sigma(\Omega)$. The proof of the following proposition is done by repeated use of relation (4.1).

Proposition 4.14 *Let σ be a permutation of S_4. Then*

$$\Omega^\sigma = f_{\sigma_2}\left(\begin{bmatrix} \omega_{\sigma_1(a)}^{\epsilon(\sigma_1)} \\ \omega_{\sigma_1(b)}^{\epsilon(\sigma_1)} \\ \omega_{\sigma_1(c)}^{\epsilon(\sigma_1)} \end{bmatrix} \right)$$

where $\sigma = \sigma_1\sigma_2$, $\sigma_1 \in \mathfrak{S}\{a, b, c\}$, $\sigma_2 \in V_4$, and $\epsilon(\sigma_1)$ is the signature of σ_1.

Definition 4.15 Let $\tau = (p_1, p_2, p_3, p_4)$ be an ideal tetrahedron of $\mathbf{H}^2_{\mathbb{C}}$. A permutation σ of S_4 is said to be a symmetry of τ if there exists an isometry g of $\mathbf{H}^2_{\mathbb{C}}$ preserving τ such that $g(p_i) = p_{\sigma(i)}$ for $i = 1, 2, 3, 4$. We will say that the symmetry σ is *realized* by the isometry g. A symmetry σ of τ is said to be *holomorphic* (resp. *antiholomorphic*) if g is holomorphic (resp. antiholomorphic).

Proposition 4.16 *Let $\tau \subset (\partial\mathbf{H}^2_{\mathbb{C}})^4$ be a non-flat ideal tetrahedron.*

(i) *$\sigma \in S_4$ is a holomorphic symmetry of τ if and only if $\Omega^\sigma = \Omega$.*
(ii) *$\sigma \in S_4$ is an antiholomorphic symmetry of τ if and only if $\Omega^\sigma = \overline{\Omega}$.*

Proof Let us prove (i) If σ is a holomorphic symmetry of τ, then $\Omega = \Omega^\sigma$ since the complex cross-ratio is preserved by holomorphic isometries. Conversely, assume that $\Omega^\sigma = \Omega$. Applying Proposition 4.7, we obtain a holomorphic isometry h such that $h(p_i) = p_{\sigma_i}$ for $i = 1, 2, 3, 4$. The second part is shown in the same way using Corollary 4.8 instead of Proposition 4.7. ■

Proposition 4.17 *Let $\tau = (p_1, p_2, p_3, p_4)$ be a non-flat ideal tetrahedron.*

(i) *The permutation $(12)(34)$ is a antiholomorphic symmetry of τ if and only if $\omega_a = \mathbf{X}(p_1, p_2, p_3, p_4)$ is real and positive. In this case, $(12)(34)$ is realized by an Lagrangian reflection.*
(ii) *The permutation $(12)(34)$ is a holomorphic symmetry of τ if and only if both $\omega_b = \mathbf{X}(p_1, p_4, p_2, p_3)$ and $\omega_c = \mathbf{X}(p_1, p_3, p_4, p_2)$ are real and positive, that is, if and only if $(13)(24)$ and $(14)(23)$ are antiholomorphic symmetries of τ. In this case, $(13)(24)$ and $(14)(23)$ are realized by Lagrangian reflections R_1 and R_2, and $(12)(34)$ is realized by $R_1 \circ R_2$, which is a complex symmetry.*

Proof The first assertion is obtained by applying Propositions 4.14 and 4.16. Now, if there exists an antiholomorphic isometry g swapping x_1 and x_2, and x_3 and x_4, then

g^2 has four fixed points in $\partial\mathbf{H}^2_{\mathbb{C}}$ that are not contained in a totally geodesic subspace. Thus g^2 is the identity, and g is a Lagrangian reflection.

Applying again Propositions 4.14 and 4.16, we obtain that $(12)(34)$ is a holomorphic symmetry if and only if both ω_b and ω_c are real. If one of them were negative, τ would be flat (see Corollary 4.3). Applying Proposition 4.16, we see that $(13)(24)$ and $(14)(23)$ are antiholomorphic symmetries (realized by two Lagrangian reflections R_1 and R_2). Thus $(12)(34)$ is realized by the product $R_1 \circ R_2$, which is holomorphic and has order two. The result follows. ■

Remark 17 Using this method, it is possible to describe all the possible subgroups of S^4 that can occur as symmetry groups of an ideal tetrahedron. This was done in [34].

4.4 Decomposition of Pairs of Loxodromic Isometries

The definition of $\mathbb{R}$- and $\mathbb{C}$-decomposability of a pair of isometries is given in Definition 1.1 in the introduction.

Proposition 4.18 *Let (A, B) be a pair of loxodromic isometries, not stabilizing a common totally geodesic subspace.*

(i) *The pair (A, B) is $\mathbb{R}$-decomposable if and only if the ideal tetrahedron $\tau_{A,B} = (p_A, q_A, p_B, q_B)$ admits the antiholomorphic symmetry $(12)(34)$.*
(ii) *The pair (A, B) is $\mathbb{C}$-decomposable if and only if both A and B admit lifts to $\mathrm{SU}(2,1)$ with real trace greater than 3 and the ideal tetrahedron*

$$\tau_{A,B} = (p_A, q_A, p_B, q_B)$$

admits the holomorphic symmetry $(12)(34)$.

Proof (i) According to Proposition 2.7, a Lagrangian reflection decomposes the pair (A, B) if and only if it swaps simultaneously p_A and q_A, and p_B and q_B. Such a Lagrangian reflection corresponds to a $(12)(34)$ antiholomorphic symmetry of $\tau_{A,B}$.

(ii) The argument is the same as for (i). ■

As a consequence of Propositions 2.7, 4.17, and 4.18, we obtain the following.

Proposition 4.19 *Let $\rho \in \mathcal{M}^{\mathrm{lox}}$ be a representation. Assume that the image of ρ does not stabilize any totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$. Assume that*

$$\Psi([\rho]) = (z_A, z_B, \omega_a, \omega_b, \omega_c),$$

where $[\rho]$ is the class of ρ modulo conjugation in $\mathrm{PU}(2,1)$. Then

(i) *ρ is $\mathbb{R}$-decomposable if and only if ω_a is real and positive;*
(ii) *ρ is $\mathbb{C}$-decomposable if and only if z_A and z_B are both real and greater than 3, and ω_b and ω_c are real and positive.*

5 Relation between Trace Coordinates and KR Coordinates on $\mathcal{M}^{lox}$

We have so far described two systems of coordinates on $\mathcal{M}^{\mathrm{lox}}$. The first one is obtained by restricting to $\mathcal{M}^{\mathrm{lox}}$ the trace coordinates $(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB, \operatorname{tr} A^{-1}B, \operatorname{tr}[A,B])$ on $\mathcal{M}$. The second one is the KR coordinates system $(\operatorname{tr} A, \operatorname{tr} B, \omega_a, \omega_b, \omega_c)$, defined directly on $\mathcal{M}^{\mathrm{lox}}$. The purpose of this section is to pass from one system to the other.

5.1 Normalization of Pairs of Loxodromic Isometries

We first provide a normalization of pairs of loxodromic isometries.

Lemma 5.1 *Any pair of loxodromic isometries is conjugate to a pair* (A, B) *given by*

$$A = \begin{bmatrix} \bar{\mu}^{-1} & \bar{z}_2 g\left(\bar{\mu}^{-1}\right) & z_1 g(\mu) + \bar{z}_1 g(\bar{\mu}^{-1}) \\ 0 & \bar{\mu}\mu^{-1} & z_2 g(\mu) \\ 0 & 0 & \mu \end{bmatrix}, \tag{5.1}$$

$$B = \begin{bmatrix} \nu & 0 & 0 \\ w_2 g(\nu) & \bar{\nu}\nu^{-1} & 0 \\ w_3 g(\nu) + \bar{w}_3 g(\nu^{-1}) & \bar{w}_2 g(\bar{\nu}^{-1}) & \bar{\nu}^{-1} \end{bmatrix},$$

where

- μ *and* ν *are two complex numbers such that* $|\mu| < 1$ *and* $|\nu| < 1$,
- g *is the function defined by* $g(z) = z - \bar{z}z^{-1}$,
- z_1, z_2, w_2, *and* w_3 *satisfy*

$$z_1 + \bar{z}_1 + |z_2|^2 = w_3 + \bar{w}_3 + |w_2|^2 = 0. \tag{5.2}$$

Proof Conjugating if necessary, we may assume that the attractive fixed point p_A of A (resp. p_B of B) lifts to $\mathbf{p}_A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ (resp. $\mathbf{p}_B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$), and that its repulsive fixed point q_A (resp. q_B) lifts to

$$\mathbf{q}_A = \begin{bmatrix} z_1 & z_2 & 1 \end{bmatrix}^T \left(\text{resp. } \mathbf{q}_B = \begin{bmatrix} 1 & w_2 & w_3 \end{bmatrix}\right)^T.$$

Writing that $\langle \mathbf{q}_A, \mathbf{q}_A \rangle = \langle \mathbf{q}_B, \mathbf{q}_B \rangle = 0$ yields the two relations (5.2). It is then a direct computation to check that two isometries A and B fixing these points are as above (they must satisfy relation (2.3). ∎

Remark 18 If A and B are as in the relation (5.1), their inverses are obtained by changing μ into μ^{-1} and ν into $1/\nu^{-1}$. Note, moreover, that if *mu* or ν has modulus 1, then A or B is a complex reflection.

Using the lifts of p_A, q_A, p_B and q_B in the proof of Lemma 5.1, we obtain the following values for the cross-ratios:

$$\omega_a = \mathbf{X}(p_A, q_A, p_B, q_B) = \frac{1 + w_2\bar{z}_2 + w_3\bar{z}_1}{w_3\bar{z}_1}, \tag{5.3}$$

$$\omega_b = \mathbf{X}(p_A, q_B, q_A, p_B) = \frac{1}{1 + z_2\bar{w}_2 + z_1\bar{w}_3}, \tag{5.4}$$

$$\omega_c = \mathbf{X}(p_A, p_B, q_B, q_A) = w_3 z_1. \tag{5.5}$$

5.2 Connection between Traces and Cross-Ratios

The following result is the main technical tool to translate the decomposability criterion (Proposition 4.19) from KR coordinates to trace coordinates.

Proposition 5.2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two loxodromic conjugacy classes in* $\mathrm{SU}(2,1)$. *There exists an (explicit) affine bijection $\Phi\colon \mathbb{R}^4 \to \mathbb{R}^4$ determined by $\mathcal{C}_1$ and $\mathcal{C}_2$ such that for any $(A,B) \in \mathcal{C}_1 \times \mathcal{C}_2$,*

$$(\operatorname{tr} AB, \operatorname{tr} A^{-1}B) = \Phi\Big(\frac{1}{\omega_b}, \omega_c\Big).$$

Proof Recall that g is defined on $\mathbb{C} \setminus \{0\}$ by $g(z) = z - \bar{z}z^{-1}$. Let h be the function defined on $(\mathbb{C} \setminus \{0\})^2$ by $h(x,y) = g(x)g(y)$. Note that h satisfies

$$h(x,y)h\Big(\frac{1}{x}, \frac{1}{y}\Big) - h\Big(\frac{1}{x}, y\Big)\, h\Big(x, \frac{1}{y}\Big) = 0 \ \text{ for all } x, y \in \mathbb{C}. \tag{5.6}$$

Using the special form of an element of $\mathcal{C}_1 \times \mathcal{C}_2$ given by Lemma 5.1 (relation (5.1)), we compute $\operatorname{tr} AB$ and $\operatorname{tr} A^{-1}B$ in terms of μ, ν, z_1, z_2, w_2, and w_3. The three cross-ratios ω_a, ω_b, and ω_c are given by the relations (5.3), (5.4), and (5.5). This yields

$$\operatorname{tr} AB = \omega_c h(\mu,\nu) + \bar{\omega}_c h(\bar{\mu}^{-1}, \bar{\nu}^{-1}) + \frac{1}{\omega_b} h(\mu, \bar{\nu}^{-1}) + \frac{1}{\bar{\omega}_b} h(\bar{\mu}^{-1}, \nu) + \alpha, \tag{5.7}$$

$$\operatorname{tr} A^{-1}B = \omega_c h(\mu^{-1},\nu) + \bar{\omega}_c h(\bar{\mu}, \bar{\nu}^{-1}) + \frac{1}{\omega_b} h(\mu^{-1}, \bar{\nu}^{-1}) + \frac{1}{\bar{\omega}_b} h(\bar{\mu}, \nu) + \beta, \tag{5.8}$$

where α and β are given by

$$\alpha = \frac{\bar{\mu}}{\mu}\left(\nu + \bar{\nu}^{-1}\right) + \frac{\bar{\nu}}{\nu}\left(\mu + \bar{\mu}^{-1}\right) - \frac{\bar{\nu}\bar{\mu}}{\mu\nu},$$
$$\beta = \frac{\mu}{\bar{\mu}}\left(\nu + \bar{\nu}^{-1}\right) + \frac{\bar{\nu}}{\nu}\left(\mu^{-1} + \bar{\mu}\right) - \frac{\bar{\nu}\mu}{\nu\bar{\mu}}.$$

Applying (5.6), we compute now the following two linear combinations:

$$(5.7)\cdot h(\bar{\mu}, \bar{\nu}^{-1}) - (5.8)\cdot h(\bar{\mu}^{-1}, \bar{\nu}^{-1}) \quad \text{and} \quad \overline{(5.7)}\cdot h(\bar{\mu}^{-1}, \bar{\nu}) - \overline{(5.8)}\cdot h(\bar{\mu}, \bar{\nu}),$$

where $\overline{(5.7)}$ and $\overline{(5.8)}$ are the relations obtained as the complex conjugates of (5.7) and (5.8). We obtain this way the two relations:

$$\begin{aligned} &\omega_c\big(h(\mu,\nu)h(\bar{\mu},\bar{\nu}^{-1}) - h(\mu^{-1},\nu)h(\bar{\mu}^{-1},\bar{\nu}^{-1})\big) \\ &\quad + \frac{1}{\omega_b}\big(h(\mu,\bar{\nu}^{-1})h(\bar{\mu},\bar{\nu}^{-1}) - h(\mu^{-1},\bar{\nu}^{-1})h(\bar{\mu}^{-1},\bar{\nu}^{-1})\big) \\ &\qquad = (\operatorname{tr} AB - \alpha)h(\bar{\mu},\bar{\nu}^{-1}) - (\operatorname{tr} A^{-1}B - \beta)h(\bar{\mu}^{-1},\bar{\nu}^{-1}) \end{aligned} \tag{5.9}$$

https://doi.org/10.4153/CJM-2009-067-6 Published online by Cambridge University Press

and

$$\omega_c\big(h(\mu^{-1},\nu^{-1})h(\bar{\mu}^{-1},\bar{\nu}) - h(\mu,\nu^{-1})h(\bar{\mu},\bar{\nu})\big) + \frac{1}{\omega_b}\big(h(\mu^{-1},\bar{\nu})h(\bar{\mu}^{-1},\bar{\nu}) - h(\mu,\bar{\nu})h(\bar{\mu},\bar{\nu})\big) = (\overline{\operatorname{tr} AB} - \bar{\alpha})h(\bar{\mu}^{-1},\bar{\nu}) - (\overline{\operatorname{tr} A^{-1}B} - \bar{\beta})h(\bar{\mu},\bar{\nu}). \tag{5.10}$$

As a consequence, ω_b^{-1} and ω_c are the solutions of the affine system (S) formed by the two equations (5.9) and (5.10). The coefficients of the linear part of (S) only depend on μ and ν, that is, on $\mathcal{C}_1$ and $\mathcal{C}_2$. The determinant of (S) is computed to be

$$\delta = \frac{(|\mu|^2-1)^2|\mu-\bar{\mu}^2|^4(\nu-\bar{\nu}^2)^3(\nu^2-\bar{\nu})(|\nu|^2-1)}{|\mu|^8|\nu|^6\bar{\nu}}.$$

The two factors $(\nu^2-\bar{\nu})$ and $(\mu-\bar{\mu}^2)$ vanish if and only if $\nu^3=1$ or $\mu^3=1$. Thus δ vanishes if and only if μ or ν has unit modulus, that is, if A or B is a complex reflection. Hence, as long as A and B are loxodromic, there exist two complex numbers ω_b and ω_c satisfying the above equations. However, these two numbers may be interpreted as cross-ratios if and only if they satisfy the inequality (4.5). This shows the result. ∎

Note that the two relations (5.7) and (5.8) have appeared already in [22].

Remark 19 Finishing the resolution of the above system, we obtain for ω_b and ω_c the following expressions:

$$\begin{aligned}\frac{1}{\omega_b} = \frac{|\mu|^2|\nu|^2}{D_b}\Big\{&\mu \operatorname{tr}(AB) + \operatorname{tr}(A^{-1}B) + \bar{\nu}\,\overline{\operatorname{tr}(AB)} + \mu\,\bar{\nu}\overline{\operatorname{tr}(A^{-1}B)}\\ &- \mu\nu(\bar{\mu}^2+\mu)(\bar{\nu}^3+1) - \bar{\nu}(\mu^2+\bar{\mu})(|\mu|^2+1)(\bar{\nu}+\nu^2)\Big\},\\ \omega_c = \frac{|\mu|^2|\nu|^2}{D_c}\Big\{&\mu\nu \operatorname{tr}(AB) + \nu \operatorname{tr}(A^{-1}B) + \overline{\operatorname{tr}(AB)} + \mu\,\overline{\operatorname{tr}(A^{-1}B)}\\ &- \mu\bar{\nu}(\bar{\mu}^2+\mu)(\nu^3+1) - \nu(\mu^2+\bar{\mu})(|\mu|^2+1)(\nu+\bar{\nu}^2)\Big\},\end{aligned}$$

where D_b and D_c are given by

$$\begin{aligned}D_b &= \left(|\mu|^2-1\right)\left(|\nu|^2-1\right)\left(\mu^2-\bar{\mu}\right)\left(\bar{\nu}^2-\nu\right),\\ D_c &= \left(|\mu|^2-1\right)\left(|\nu|^2-1\right)\left(\mu^2-\bar{\mu}\right)\left(\nu^2-\bar{\nu}\right).\end{aligned}$$

As we have seen in Section 3, once $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr} AB$, and $\operatorname{tr} A^{-1}B$ are fixed, there are two possible conjugacy classes of groups. Depending on the choice of coordinates (traces or cross-ratios), these two classes are associated with the order 2 indetermination either on ω_a or on $\operatorname{tr}[A,B]$. We establish now the connection between ω_a and $\operatorname{tr}[A,B]$.

The relation (4.4) expresses $\mathrm{Re}\,(\omega_a)$ using ω_b and ω_c, which are obtained in terms of traces as in Remark 19. In order to obtain $\mathrm{Im}\,(\omega_a)$, we compute the trace of the commutator using the normalization given by Lemma 5.1. This yields

(5.11)
$$\begin{aligned}\mathrm{tr}[A,B] = 3 - 2\mathrm{Re}\left(\bar{\omega}_c h(\mu,\nu)h(\mu^{-1},\nu^{-1}) + \frac{1}{\omega_b}h(\bar{\mu},\nu)h(\bar{\mu}^{-1},\nu^{-1})\right)\\ + \left|\left(\omega_c h(\bar{\mu},\bar{\nu}) + \bar{\omega}_c h(\mu^{-1},\nu^{-1}) + \frac{1}{\omega_b}h(\bar{\mu},\nu^{-1}) + \frac{1}{\bar{\omega}_b}h(\bar{\mu}^{-1},\bar{\nu})\right)\right|^2\\ - |\omega_c|^2\left(|h(\mu,\nu)|^2 + |h(\mu^{-1},\nu^{-1})|^2\right)\\ - \frac{1}{|\omega_b|^2}\left(|h(\mu,\nu^{-1})|^2 + |h(\mu^{-1},\nu)|^2\right)\\ + |\omega_c|^2\Big(\omega_a\left(|h(\mu,\nu^{-1})|^2 + |h(\mu^{-1},\nu)|^2\right)\\ + \bar{\omega}_a\left(|h(\mu,\nu)|^2 + |h(\mu^{-1},\nu^{-1}|^2\right)\Big).\end{aligned}$$

Note that only the last two lines of (5.11) involve non-real contributions. Taking the imaginary part yields the following relation between $\mathrm{Im}\,(\mathrm{tr}[A,B])$ and $\mathrm{Im}\,(\omega_a)$:

$$(5.12)\qquad \begin{aligned}\frac{\mathrm{Im}\ \mathrm{tr}[A,B]}{\mathrm{Im}\,(\omega_a)} &= |\omega_c|^2\Big(\left(|h(\mu,\nu^{-1})|^2 + |h(\mu^{-1},\nu)|^2\right)\\ &\qquad - \left(|h(\mu,\nu)^2 + |h(\mu^{-1},\nu^{-1})|^2\right)\Big)\\ &= |\omega_c|^2\left(|g(\mu)|^2 - |g(\mu^{-1})|^2\right)\left(|g(\nu)|^2 - |g(\nu)^{-1}|^2\right).\end{aligned}$$

Now, for any non-zero complex number z, $|g(z)|^2 - |g(z^{-1})|^2 = \left(1 - |z|^2\right)|z^2/\bar{z}+1|^2$. The factor $z^2/\bar{z} + 1$ vanishes if and only if $z^3 = -1$. Since μ and ν have modulus smaller than 1 (see Lemma 5.1), the right-hand side of (5.12) is positive. Hence $\mathrm{Im}\ \mathrm{tr}[A,B]$ and $\mathrm{Im}\,(\omega_a)$ have the same sign.

5.3 Decomposability Results

Now we prove Theorem 1.2, stated in the introduction.

Proof of Theorem 1.2 Proof of (i)

(b) $\Rightarrow$ (a). If the pair (A,B) is $\mathbb{R}$-decomposable, then according to Remarks 3 and 4, we choose three lifts of the I_k's. We obtain in this way three matrices M_1, M_2, and M_3 in $\mathrm{SU}(2,1)$ satisfying $M_k\overline{M}_k = 1$. Following Remark 4, the isometries A and B admit the lifts to $\mathrm{SU}(2,1)$

$$(5.13)\qquad A = M_1\overline{M}_2 \quad\text{and}\quad B = M_3\overline{M}_2.$$

Computing the commutator, we get $[A,B] = M\overline{M}$, where $M = M_1\overline{M}_2M_3$. Hence, the commutator $[A,B]$ has real trace.

(a) $\Rightarrow$ (b). If $\operatorname{tr}[A,B]$ is real, then, as a consequence of relation (5.11), ω_a is real. The result is thus a consequence of Proposition 4.19, and, in fact, ω_a is positive (otherwise $\langle A,B\rangle$ would preserve a complex line).

Proof of (ii)

(b) $\Rightarrow$ (a). If (A,B) is $\mathbb{C}$-decomposable, A, B, AB, and $A^{-1}B$ admit lifts to SU(2, 1) with real trace, since they all are products of two complex symmetries. Next, if the pair (A^2,B^2) is $\mathbb{C}$-decomposable, the proof of Proposition 2.7 shows that both A and B have real eigenvalues. Moreover, since A^2 and B^2 have the same fixed points as A and B, the two cross-ratios ω_b and ω_c are real. Hence, using relation (5.7) and (5.8), we see that $\operatorname{tr} AB$ and $\operatorname{tr} A^{-1}B$ are real.

(a) $\Rightarrow$ (b) If the lifts A and B have real trace, then A and B have real eigenvalues, as seen in the proof of Proposition 2.7. Thus, taking the imaginary part in the two relations (5.7) and (5.8) yields

$$\begin{aligned}(5.14)\qquad 0 &= \operatorname{Im}(\omega_c)\big(h(\mu,\nu)-h(\mu^{-1},\nu^{-1})\big)\\ &\qquad + \operatorname{Im}(\omega_b^{-1})\big(h(\mu,\nu^{-1})-h(\mu^{-1},\nu)\big)\\ (5.15)\qquad 0 &= \operatorname{Im}(\omega_c)\big(h(\mu^{-1},\nu)-h(\mu,\nu^{-1})\big)\\ &\qquad + \operatorname{Im}(\omega_b^{-1})\big(h(\mu^{-1},\nu^{-1})-h(\mu,\nu)\big)\end{aligned}$$

Now, if x and y are real, $h(x,y)=(x-1)(y-1)$. The determinant of the linear system $\{(5.14),(5.15)\}$ is

$$-\frac{(\mu-1)^3(\nu-1)^3(\mu+1)(\nu+1)}{\mu^2\nu^2}.$$

Since $|\mu|\neq 1$ and $|\nu|\neq 1$ because A and B are loxodromic, the above system is non degenerate. As a consequence, ω_b and ω_c are both real (and positive, else G would preserve a complex line). The result is obtained from Propositions 2.7 and 4.19. ■

Remark 20 In the same way, it is a direct computation using the three relations (5.7), (5.8), and (5.11) to check that if the five isometries A, B, AB, $A^{-1}B$, and $[A,B]$ have real trace, ω_a, ω_b, and ω_c are real. Thus, in this case, the fixed points of A and B belong to a totally geodesic subspace, which is preserved by $\langle A,B\rangle$.

We may now characterize the set of classes of $\mathbb{R}$-decomposable representations of F_2 in PU(2, 1) as the fixed point set of an involution on $\mathcal{M}^{\mathrm{lox}}$.

Proposition 5.3 *There exists an involution Σ on $\mathcal{M}$ such that for any Zariski dense representation $\rho\in\mathcal{M}^{\mathrm{lox}}$, $\Sigma([\rho])=[\rho]$ if and only if ρ is $\mathbb{R}$-decomposable.*

Proof Σ corresponds to $(A,B)\mapsto(\overline{A^{-1}},\overline{B^{-1}})$. In trace coordinates, Σ induces the mapping

$$(\operatorname{tr} A,\operatorname{tr} B,\operatorname{tr} AB,\operatorname{tr} A^{-1}B,\operatorname{tr}[A,B])\longrightarrow(\operatorname{tr} A,\operatorname{tr} B,\operatorname{tr} AB,\operatorname{tr} A^{-1}B,\overline{\operatorname{tr}[A,B]}).\qquad ■$$

The same kind of results has been obtained in a different frame by Schaffhauser [26, 27]. If A and B are two loxodromic isometries such that $[A,B]$ is unipotent, lifts of A and B may be chosen such that their commutator has trace 3. Hence, Theorem 1.2 asserts that either the group $G = \langle A, B\rangle$ preserves a complex line, or the pair (A, B) is $\mathbb{R}$-decomposable. We can even be more precise.

Theorem 5.4 *Let A and B be two loxodromic isometries such that $C = [A, B]$ is pure parabolic. Then one of the following two possibilities occurs.*

(i) *The pair (A, B) is $\mathbb{R}$-decomposable, and C is conjugate to a horizontal Heisenberg translation.*

(ii) *The commutator C is conjugate to a vertical translation and G preserves a complex line.*

See Remark 2 about horizontal and vertical translations. The case where the commutator of A and B is parabolic is of special interest for it corresponds to the type-preserving representations of the fundamental group of the 1-punctured torus.

Proof (i) If the pair (A, B) is $\mathbb{R}$-decomposable, then there exist three matrices M_1, M_2, and M_3 in $\mathrm{SU}(2, 1)$ satisfying relation (5.13). The isometry C is the square of the antiholomorphic isometry $\tilde{C}$ given by $Z \to M\overline{Z}$, where $M = M_1\overline{M}_2M_3$. Now C and $\tilde{C}$ have the same fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$, which may be assumed to be ∞, and may thus be represented by the lift $[1 \quad 0 \quad 0]^T$. As a consequence, we may assume that C and M are both upper triangular. It follows then from Lemma 5.5 above that $M\overline{M}$ is either the identity, or conjugate to a non-vertical Heisenberg translation.

(ii) If C is conjugate to a vertical Heisenberg translation, then the pair (A, B) is not $\mathbb{R}$-decomposable (see Lemma 5.5). Thus, G preserves a complex line. ■

Lemma 5.5 *Let M be an element of* $\mathrm{SU}(2, 1)$ *such that $M\overline{M}$ is unipotent. Either $M\overline{M}$ is the identity or $M\overline{M} - \mathrm{Id}$ is nilpotent of order* 3.

Proof Conjugating if necessary, we may assume that M is upper triangular with unit modulus diagonal coefficents. The general form of an such a matrix in U(2,1) is proportional to

$$M = \begin{bmatrix} 1 & -\sqrt{2}\bar{z}e^{i\theta} & -|z|^2 + it \\ 0 & e^{i\theta} & \sqrt{2}z \\ 0 & 0 & 1 \end{bmatrix}.$$

Computing the product $M\overline{M}$ yields the result. ■

Remark 21 Define $\mathcal{P}$ to be the subset of $\mathcal{M}^{\mathrm{lox}}$ containing the classes of those representations of F_2 mapping $[\mathtt{m}, \mathtt{n}]$ to a parabolic isometry, and $\mathcal{P}^{\mathcal{C}}$ to be the subset of $\mathcal{P}$ containing the classes of those representations such that $\rho([\mathtt{m}, \mathtt{n}])$ belongs to a given parabolic conjugacy class $\mathcal{C}$. Then $\mathcal{M}^{\mathrm{lox}}$ is 8-dimensionnal, and $\mathcal{P}$ is 7-dimensionnal. Each of the $\mathcal{P}^{\mathcal{C}}$'s has dimension 6. In [34, 35], we have described a system of coordinates on $\mathcal{P}^{\mathcal{C}_3}$, where $\mathcal{C}_3$ is the class of unipotent parabolics of index 3, *i.e.,* non-vertical Heisenberg translations.

6 Surface Groups with Prescribed Conjugacy Classes

6.1 The Sphere with Three Holes

Let $\mathcal{C}^{\text{lox}}$ be the set of loxodromic conjugacy classes of $\mathrm{PU}(2,1)$. A conjugacy class is fully determined by one complex number λ of modulus greater than 1: its eigenvalue of greater modulus (see Section 2.4 and Remark 8). Therefore $\mathcal{C}^{\text{lox}}$ may be seen as the cyclinder $\{|z|>1\}$. We call the lines of fixed argument $\{\arg\lambda=\theta\}$ in C^{lox} the *vertical lines* of $\mathcal{C}^{\text{lox}}$. Fix $\mathcal{C}_1$ and $\mathcal{C}_2$ two loxodromic conjugacy classes, and define the mapping

$$\begin{aligned}\pi\colon \mathcal{C}_1\times\mathcal{C}_2 &\longrightarrow \mathcal{C}^{\text{lox}}\\ (A,B)&\longmapsto \mathtt{Cl}(AB),\end{aligned}$$

where $\mathtt{Cl}(AB)$ is the conjugacy class of the product AB.

We will show that for any $\mathcal{C}_1$ and $\mathcal{C}_2$, π is onto $\mathcal{C}^{\text{lox}}$, (see also [6]). For this purpose, we begin with the reducible representations, that is, the groups generated by two loxodromic elements preserving a common complex line. In this case, the tetrahedron $\tau_{A,B}$ is flat, contained in the boundary of the stable complex line. We denote by $(\mathcal{C}_1\times\mathcal{C}_2)^{\text{red}}$ the set of reducible pairs of $\mathcal{C}_1\times\mathcal{C}_2$.

Lemma 6.1 *The image of the restriction of π to $(\mathcal{C}_1\times\mathcal{C}_2)^{\text{red}}$ is a vertical line of $\mathcal{C}^{\text{lox}}$.*

Proof Assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ correspond respectively to eigenvalues of modulus greater than 1 having argument α and β. Assume that (A,B) is a reducible pair of loxodromic isometries. Both A and B preserve a complex line C, polar to some positive vector $\mathbf{c}$. This means that the vector $\mathbf{c}$ is an eigenvector of both A and B. In the normalization provided by Lemma 5.1, this condition leads to $z_2=w_2=0$. Therefore, the vector $\mathbf{c}$ is also an eigenvector for AB, and using the normalized form given by Lemma 5.1, we see that the associated eigenvalue is $e^{-2i(\alpha+\beta)}$. The result follows using Proposition 2.5. ■

Theorem 6.2 *Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be three loxodromic conjugacy classes. There exists a representation ρ of F_2 in* PU(2, 1) *such that $\rho(\mathtt{m})\in\mathcal{C}_1$, $\rho(\mathtt{n})\in\mathcal{C}_2$, and $\rho(\mathtt{mn})\in\mathcal{C}_3$.*

Proof The two conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ determine $\operatorname{tr}A$ and $\operatorname{tr}B$. According to Proposition 4.12, $\mathcal{C}_1\times\mathcal{C}_2$ may be seen as the set of isometry classes of ideal tetrahedra, that is,

$$\Big\{(\omega_a,\omega_b,\omega_c)\in\mathbb{C}^3\ \Big|\ |\omega_a\omega_b\omega_c|=1,\\ 2\mathrm{Re}\,(\omega_c)=\frac{1}{|\omega_b|^2}\Big(\Big|1-\frac{1}{\omega_a}\Big|^2-1\Big)+\Big|1-\frac{1}{\omega_b}\Big|^2\Big\}.$$

Let T be the mapping

$$\begin{aligned}\mathcal{C}_1\times\mathcal{C}_2&\longrightarrow\mathbb{C}\\ (A,B)&\longmapsto \operatorname{tr}AB.\end{aligned}$$

The image of T contains all the loxodromic traces. Indeed, according to relation (5.7), this mapping is (real) affine in cross-ratio coordinates. It is thus continuous, open and closed, and according to Lemma 6.1, its image is an open and closed subset of the cylinder $\mathcal{C}^{\text{lox}}$ containing a vertical line. ∎

6.2 The Torus with One Hole

We will show the following.

Theorem 6.3 *Let $\mathcal{C}$ be a non-elliptic conjugacy class. There exists a $\mathbb{C}$-decomposable representation ρ of F_2 in* $\mathrm{PU}(2,1)$ *such that $\rho([\mathtt{m},\mathtt{n}]) \in \mathcal{C}$.*

Proof Let $\mathcal{C}^{\frac{1}{2}}$ be the conjugacy class containing those isometries h such that $h^2 \in \mathcal{C}$. We will show that there exists a symmetric complex triangle group $\langle I_1, I_2, I_3\rangle$ such that $I_1 I_2 I_3 \in \mathcal{C}^{\frac{1}{2}}$. Then if $\rho(\mathtt{m}) = I_1 \circ I_2$ and $\rho(\mathtt{n}) = I_3 \circ I_2$, the commutator is $\rho([\mathtt{m},\mathtt{n}]) = (I_1 \circ I_2 \circ I_3)^2$ and belongs to $\mathcal{C}$. If z is the parameter of a symmetric complex triangle group (see Remark 15), then $\mathrm{tr}\,(I_1 I_2 I_3) = 8z^3 - 12|z|^2 + 3 = \psi(z)$.

Recall that the set of complex numbers z associated with a complex triangle group is $\mathcal{D} = \{x + iy,\ x < -1/2,\ y > 1 + \sqrt{3}x,\ y > 1 - \sqrt{3}x\}$ (see relation (3.5)). Call $\mathcal{H}$ the half-plane $\{\mathrm{Re}\,(z) < -1\}$ of $\mathbb{C}$. We will show that the image of $\psi\colon \mathcal{D} \to \mathbb{C}$ is $\mathcal{H}$. Note that the image of $\partial\mathcal{D}$ under ψ is the line $\mathrm{Re}\,(z) = -1$, and that $\psi(-1) = -17$. Hence, the image of ψ is a connected subset of $\mathcal{H}$. We will show that ψ is open and closed, and thus, that $\psi(\mathcal{D}) = \mathcal{H}$.

Computing the holomorphic and antiholomorphic derivatives of ψ, we see that the only critical point of ψ in $\mathbb{C}$ is $(0,0)$, which does not belong to $\mathcal{D}$. Thus ψ is a local homeomorphism, and is open.

To see that ψ is closed, note that

$$|\psi(z)| = 8|z|^3\Big(1 - \frac{3}{2|z|} - \frac{3}{|z|^3}\Big) \geqslant 4|z|^3, \tag{6.1}$$

as soon as $|z| > R$ for some great enough R. Now if $\psi(p_n)$ converges to some point q (with $p_n \in \mathcal{D}$), $\psi(p_n)$ is bounded and so is p_n by (6.1). The result follows by passing to a converging subsequence of p_n. Now if h is a loxodromic isometry, its trace is a complex number defined up to multiplication by a cube root of 1. At least one of the three possible choices belongs to $\mathcal{D}$ and is reached by ψ. ∎

Acknowledgments I thank Elisha Falbel for his constant encouragement. Gilles Courtois, Pierre-Vincent Koseleff and John Parker gave me much advice., for which I would like to thank them. I warmly thank Patrick Polo for spending time discussing invariants with me. I thank Amadeo Irigoyen for discussing Section 6.2, and Martin Deraux, Masseye Gaye, Florent Schaffhauser, and Julien Paupert for numerous discussions, and Arlo Caine for proofreading part of the paper.

References

[1] A. Beardon, *The Geometry of Discrete Groups.* Graduate Texts in mathematics 91. Springer-Verlag, New York, 1983.

[2] M. Deraux, E. Falbel, and J. Paupert, *New constructions of fundamental polyhedra in complex hyperbolic space.* Acta Math. **194**(2005), no. 2, 155–201.

[3] E. Falbel, *Spherical* CR *structures on the complement of the figure eight knot with discrete holonomy.* J. Differential Geom. **79**(2008), no. 1, 69–110.

[4] E. Falbel and P. V. Koseleff, *Rigidity and flexibility of triangle groups in complex hyperbolic geometry.* Topology **41**(2002), no. 4, 767–786.

[5] E. Falbel and J. Parker, *The moduli space of the modular group in complex hyperbolic geometry.* Inv. Math. **152**(2003), no. 1, 57–88..

[6] E. Falbel and R. Wentworth, *Compacité à la Mumford-Mahler pour les groupes fuchsiens dans un espace symétrique de rang un.* Preprint available at www.insitut.math.jussieu.fr/~falbel.

[7] E. Falbel and V. Zocca, *A Poincaré's polyhedron theorem for complex hyperbolic geometry.* J. Reine Angew. Math. **516**(1999), 133–158.

[8] W. Fenchel, *Elementary Geometry in Hyperbolic Space*. de Gruyter Studies in Mathematics 11. Walter de Gruyter, Berlin, 1989.

[9] N. P. Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*. Lecture Notes in Mathematics 1794. Springer-Verlag, Berlin, 2002.

[10] R. Fricke and F. Klein, *Vorlesungen über die Theorie der Automorphen Funktionen.* Tuebner, Stuttgart, 1965

[11] J. Gilman, *Two-Generator Discrete Subgroups of* $\mathrm{PSL}(2, \mathbb{R})$. Mem. Amer. Math. Soc. **117**(1995), no. 561.

[12] W. Goldman, *An exposition of results of Fricke and Vogt*. www.math.umd.edu/~wmg.

[13] ———, *Topological components of spaces of representations.* Invent. Math. **93**(1988), no. 3, 557–607.

[14] ———, *Complex Hyperbolic Geometry*. Oxford University Press, Oxford, 1999.

[15] W. Goldman and J. Millson, *Local rigidity of discrete groups acting on complex hyperbolic space.* Invent. Math. **88**(1987), no. 3, 495–520.

[16] W. Goldman and J. Parker, *Complex hyperbolic ideal triangle groups.* J. Reine Angew. Math. **425**(1992), 71–86.

[17] N. Gusevskii and J. R. Parker, *Complex hyperbolic quasi-fuchsian groups and Toledo's invariant.* Geom. Dedicata **97**(2003), 151–185.

[18] V. T. Khoi, *On the* SU(2, 1) *representation space of the Brieskorn homology spheres.* J. Math. Sci. Univ. Tokyo **14**(2007), no. 4, 499–510.

[19] A. Koranyi and H. M. Reimann, *The complex cross-ratio on the Heisenberg group.* Enseign. Math. **33**(1987), no. 2-3, 291–300.

[20] S. Lawton, *Generators, relations and symmetries in pairs of* 3x3 *unimodular matrices.* J. Algebra **313**(2007), no. 2, 782–801.

[21] G. D. Mostow, *On a remarkable class of polyhedra in complex hyperbolic space.* Pacific J. Math. **86**(1980), no. 1, 171–276.

[22] J. Parker and I. Platis, *Complex hyperbolic Fenchel-Nielsen coordinates.* Topology **47**(2008), no. 2, 101–135.

[23] A. Pratoussevitch, *Traces in complex hyperbolic triangle groups.* Geom. Dedicata **111**(2005), 159–185.

[24] C. Procesi, *The invariant theory of* $n \times n$ *matrices.* Advances in Math. **19**(1976), no. 3, 306–381.

[25] H. Sandler, *Traces on* SU(2, 1) *and complex hyperbolic ideal triangle groups.* Algebras Groups Geom. **12**(1995), no. 2, 139–156.

[26] F. Schaffhauser, *Decomposable representations and Lagrangian submanifolds of moduli spaces associated to surface groups.* Math. Ann. **342**(2008), no. 2, 405–447.

[27] ———, *Representations of the fundamental group of an l-punctured sphere generated by products of Lagrangian involutions.* Canad. J. Math. **59**(2006), no. 4, 845–879.

[28] R. E. Schwartz, *Degenerating the complex hyperbolic ideal triangle groups.* Acta Math. **186**(2001), no. 1, 105–154.

[29] ———, *Ideal triangle groups, dented tori, and numerical analysis.* Ann. of Math. **153**(2001), no. 3, 533–598.

[30] A. S. Sikora, SL_n*-character varieties as spaces of graphs.* Trans. Amer. Math. Soc. **353**(2001), no. 7, 2773–2804.

[31] D. Toledo, *Representations of surface groups in complex hyperbolic space.* J. Differential Geom. **29**(1989), no. 1, 125–133.

[32] H. Vogt, *Sur les invariants fondamentaux des équations différentielles linéaires du second ordre.* Ann. Sci. Ècole Norm. Sup. **6**(1886), 3–71.

[33] Z. X. Wen, *Relations polynomiales entre les traces de produits de matrices.* C. R. Acad. Sci Paris Sèr I Math. **318**(1994), no. 2, 99–104.

[34] P. Will. *Groupes libres, groupes triangulaires et tore épointé dans* PU(2, 1). Thèse de l'université Paris VI.

[35] ———, *The punctured torus and Lagrangian triangle groups in* PU(2, 1). J. Reine Angew. Math. **602**(2007), 95–121.

Institut Fourier, 100 rue des Maths, 38402 St Martin d'Hères, France
e-mail: pierre.will@ujf-grenoble.fr