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ON SOME GROUPS WITH TRIVIAL MULTIPLICATOR

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For Bernhard Neumann, with respect and affection

Scene:	B.H. Neumann's office, in the University of Manchester; an M.Sc. tutorial.
Date:	late 1954.
Dramatis Personae:	B.H. Neumann and James Wiegold.

B.H.N.:	"So you see, the Schur multiplicator is an important tool for the BFC-problem and others; thou shouldst definitely go forth and multiplicate."
J.W.:	"I Schur should."

I discussed the sort of thing Bernhard had in mind concerning the BFC-problem in [3], an article written for his sixieth birthday. This vigesennial contribution deals with one of the "and others", namely the zero deficiency problem. In his article [1] bearing the same title as this one, Bernhard gave 2-generator 2-relator presentations of some metacyclic groups; if one distils and formalises the process used there, one gets the following result, to be used later in constructing some zero-deficiency finite p-groups.

THEOREM. *Let m and n be positive integers and w a two-variable word. The group $G = \langle a, b \mid a^m = b^n = w(a,b)\rangle$ is finite if and only if G/G' and $H = \langle a, b \mid a^m = b^n = w(a,b) = 1\rangle$ are finite. If G/G' and H are finite π-groups for some set π of primes, so is G.*

PROOF: One way around everything is obvious. Suppose that G/G' and H are finite π-groups. Since a^m is central in G and $G/\langle a^m\rangle \cong H$, it follows that $G' \cap \langle a^m\rangle$ is a finite π-group, being a homomorphic image of the multiplicator of H. But $a^s \in G'$ for some π-number s, and a^{ms} has π-order since $a^{ms} \in G' \cap \langle a^m\rangle$. Thus $\langle a^m\rangle$ is a finite π-group, and therefore so is G. ∎

Of course, the theorem has a version for groups on more than two generators and/or more than one relator. However, the main interest is in zero deficiency groups.

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Robertson [2] has constructed some finite non-metacyclic 2-groups of zero deficiency and high nilpotency class; as far as I know, 2 is the only prime where this has been achieved to date. The theorem allows us to make some suitable 3-groups.

Example 1. *The group* $G = \langle a, b \mid a^{3^n} = b^3 = [a, a^b] \rangle$ *is a finite 3-group of nilpotency class* $2n + 2$.

PROOF: Clearly, G/G' is a finite 3-group. But the corresponding $H = \langle a, b \mid a^{3^n} = b^3 = [a, a^b] = 1 \rangle$ is nothing but $Z_{3^n} \wr Z_3$ thinly disguised; the relation $[a, a^b] = 1$ yields $1 = [a, a^b]^b = [a^b, a^{b^2}]$ and $1 = [a^b, a^{b^2}]^b = [a^{b^2}, a]$, so that $\langle a, a^b, a^{b^2} \rangle$ is a commutative subgroup of H. The class of H is $2n + 1$, and the claim about the class of G follows after a tedious calculation. ∎

Of course, the group obtained by replacing 3 by 2 in Example 1 is a finite 2-group of high class. Here is a slightly more complicated example.

Example 2. *The group* $G = \langle a, b \mid a^{2^n} = b^2 = [a^b, a, a] \rangle$ *is a finite 2-group of class* $n + 2$.

PROOF: Here $H = \langle a, b \mid a^{2^n} = b^2 = [a^b, a, a] = 1 \rangle$ is the second nilpotent wreath product of Z_{2^n} by Z_2; for $1 = [a^b, a, a] \implies 1 = [a^b, a, a]^b = [a, a^b, a^b]$, so that the normal closure $\langle a, a^b \rangle$ of a is nilpotent of class two. I shall once more omit the class-claim. ∎

Like all other methods attempted so far, that presented here appears to be hopeless for constructing zero-deficiency groups of high solubility length. Further, I think there is enough lack of evidence around to make a final point:

CONJECTURE. *For each* $p \geq 5$, *zero-deficiency finite p-groups have p-bounded nilpotency classes.*

I shall try to confirm or deny this in readiness for October 2009.

REFERENCES

[1] B.H. Neumann, 'On some groups with trivial multiplicator', *Publ. Math. Debrecen* **4** (1956), 190–194.

[2] Edmund F. Robertson, 'A comment on finite nilpotent groups of deficiency zero', *Canad. Math. Bull.* **23(3)** (1980), 313–316.

[3] James Wiegold, 'Commutator subgroups of finite p-groups', *J. Austral. Math. Soc.* **1** (1969), 480–484.

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