ON EXTREMALITY OF TWO CONNECTED LOCALLY EXTREMAL BELTRAMI COEFFICIENTS

GUOWU YAO

Let $\Omega_1$ and $\Omega_2$ be two domains in the complex plane with a nonempty intersection. Suppose that $\mu_j$ are locally extremal Beltrami coefficients in $\Omega_j$ $(j = 1, 2)$ respectively. In 1980, Sheretov posed the problem: Will the coefficient $\mu$ defined by the condition $\mu(z) = \mu_j(z)$ for $z \in \Omega_j$, $j = 1, 2$, be locally extremal in $\Omega_1 \cup \Omega_2$? We give a counterexample to show that $\mu$ may not be locally extremal and not even be extremal.

1. INTRODUCTION

Let $\mathcal{D}$ be a domain in the complex plane $\mathbb{C}$ with at least two boundary points and let $M(\mathcal{D})$ be the open unit ball of $L^\infty(\mathcal{D})$. Every element $\mu \in M(\mathcal{D})$ can be regarded as an element in $L^\infty(\mathbb{C})$ by putting $\mu$ equal to zero in the outside of $\mathcal{D}$. Every $\mu \in M(\mathcal{D})$ induces a global quasiconformal self-mapping $f$ of the plane which solves the Beltrami equation [1],

$$f_\mu(z) = \mu(z)f(z),$$

and $f$ is defined uniquely up to postcomposition by a complex affine map of the plane. Conversely, any quasiconformal mapping $f$ defined on $\mathcal{D}$ has a Beltrami coefficient $\mu(z) = f_\mu(z)/f_z(z)$ in $M(\mathcal{D})$.

Two Beltrami coefficients $\mu, \nu \in M(\mathcal{D})$ are equivalent if they induce quasiconformal mappings $f$ and $g$ by (1) such that there is a conformal map $c$ from $f(\mathcal{D})$ to $g(\mathcal{D})$ and an isotopy through quasiconformal mappings $h_t$, $0 \leq t \leq 1$, from $\mathcal{D}$ to $\mathcal{D}$ which extend continuously to the boundary of $\mathcal{D}$ such that

1. $h_0(z)$ is identically equal to $z$ on $\mathcal{D},$
2. $h_1$ is identically to $g^{-1} \circ c \circ f,$ and
3. $h_t(p) = g^{-1} \circ c \circ f(p)$ for any $p \in \partial \mathcal{D}.$

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The equivalence relation partitions $M(\mathcal{D})$ into equivalence classes and the space of equivalence classes is by definition the Teichmüller space $T(\mathcal{D})$ of $\mathcal{D}$.

Given $\mu \in M(\mathcal{D})$, we denote by $[\mu]$ the set of all elements $\nu \in M(\mathcal{D})$ equivalent to $\mu$, and set

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$ 

We say that $\mu$ is extremal (in $[\mu]$) if $\|\mu\|_\infty = k_0([\mu])$, $\mu$ is uniquely extremal if $\|\nu\|_\infty > k_0([\mu])$ for any other $\nu \in [\mu]$; the alternative is that $\mu$ is non-uniquely extremal.

We define $A(\mathcal{D})$ as the Banach space of all holomorphic functions $\varphi$ on $\mathcal{D}$ with $L^1$-norm

$$\|\varphi\| = \int_{\mathcal{D}} |\varphi(z)| < \infty.$$ 

As is well known, a necessary and sufficient condition (Hamilton-Krushkal-Reich-Strebel condition) that a Beltrami coefficient $\mu$ is extremal in its class in $T(\mathcal{D})$ is that $[4]$ it has a so-called Hamilton sequence, namely, a sequence $\{\phi_n \in A(\mathcal{D}) : \|\phi_n\| = 1, \ n \in \mathbb{N}\}$, such that

$$\lim_{n \to \infty} \iint_{\mathcal{D}} \mu \phi_n(z) \, dx \, dy = \|\mu\|_\infty. \tag{2}$$

A Beltrami coefficient $\mu$ in $\mathcal{D}$ is called to be locally extremal if for any domain $G \subset \mathcal{D}$ it is extremal in its class in $T(G)$; in other words,

$$\|\mu\|_G := \operatorname{esssup}_{z \in G} |\mu| = \sup \left\{ \frac{\iint_{G} \mu \phi_n(z) \, dx \, dy}{\|\phi\|} : \phi \in A(G) \right\}.$$ 

Obviously, extremality in the whole domain is a prerequisite for a Beltrami coefficient to be locally extremal.

In [6], Sheretov investigated locally extremal Beltrami coefficients and posed the following problem: Let $\Omega_1$ and $\Omega_2$ be two domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that $\mu_j$ are locally extremal Beltrami coefficients in $\Omega_j$ ($j = 1, 2$) respectively. Will the coefficient $\mu$ defined by the condition $\mu(z) = \mu_j(z)$ for $z \in \Omega_j$, $j = 1, 2$, be locally extremal in $\Omega_1 \cup \Omega_2$?

The main purpose of this paper is to give a negative answer to the above problem in a stronger sense. We shall construct certain counterexample in the next section.

2. CONSTRUCTION OF COUNTEREXAMPLE

If $\mu$ in $M(\mathcal{D})$ is uniquely extremal in its class $[\mu]$ in $T(\mathcal{D})$, then it is obviously locally extremal. But the converse is not true for which here we include the example constructed in [2, Theorem 2.2] by Reich.

Reich’s example: We denote the parabolic region $\Omega_0$ by

$$\Omega_0 = \{z = x + iy : x > y^2, \ x > 0\}.$$
In $\Omega_0$, we define $\mu(z) \equiv k$ where $k \in (0,1)$ is a constant. Examining the proof of [2, Theorem 2.2], we find that
\[
\sup \left\{ \frac{\left| \int_G \mu(z) \phi(z) \, dz \, dy \right|}{\int_G |\phi| \, dz \, dy} : \phi(z) \in A(\Omega_0) \right\} = k
\]
for any positive measure subset $G$ of $\Omega_0$. This relation indicates that $\mu$ is locally extremal in $\Omega_0$. But, it is well known that $\mu$ is not uniquely extremal (see [2, 3]).

In our counterexample to Sheretov's problem, $\mu_j$, $j = 1, 2$, are uniquely extremal while $\mu$ may not be locally extremal and not even be extremal in its corresponding class.

EXAMPLE 1. Let $\Delta$ be the unit disk $\{z : |z| < 1\}$. Put
\[
\Omega_1 = \left\{ z \in \Delta : \arg z \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\}, \quad \Omega_2 = \left\{ z \in \Delta : |\arg z| > \frac{\pi}{4} \right\}.
\]
Obviously, $\Omega_1 \cup \Omega_2 = \Delta^* = \Delta - \{0\}$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$. Set $\mu = k|\varphi|/|\varphi|$ on $\Delta$, where $k \in (0,1)$ is a constant and $\varphi(z) = 1/z^2$. Let $\mu_j$ ($j = 1, 2$) be the restrictions of $\mu$ on $\Omega_j$, respectively. We claim that $\mu_j$ are uniquely extremal in their classes in $T(\Omega_j)$, respectively.

Suppose the conformal mapping $z = F(\zeta)$ maps $\Delta_\zeta = \{|\zeta| < 1\}$ onto $\Omega_1$. The question becomes that of determining whether the Beltrami coefficient
\[
\tilde{\mu} = k \frac{|\varphi \circ F'| F'(\zeta)^2}{|\varphi \circ F| F'(\zeta)^2}
\]
is extremal or uniquely extremal in its class in $T(\Delta_\zeta)$. Set
\[
\psi(\zeta) = (\varphi \circ F)F'(\zeta)^2.
\]
Because the conformal mapping $F^{-1}$ transfers the second order pole of $\varphi(z)$ to the second order pole of $\psi(\zeta)$, it is not difficult to see that $\psi(\zeta)$ is holomorphic in $\Delta_\zeta$ and is meromorphic in $\Delta_\zeta$ except that it has a pole of second order at $\zeta = F^{-1}(0)$. Thus, by [5, Theorem 6], $\tilde{\mu}$ is uniquely extremal in its class in $T(\Delta_\zeta)$, and hence $\mu_1$ is uniquely extremal in its class in $T(\Omega_1)$. Similarly, $\mu_2$ is uniquely extremal in its class in $T(\Omega_2)$.

However, $\mu$ is not even extremal in $[\mu]$ in $T(\Delta^*)$. In fact, noting that $\{z^n : n = -1, 0, 1, 2, \ldots\}$ is a base of the Banach space $A(\Delta^*)$ and
\[
\int \int_{\Delta^*} \mu(z) \phi(z) \, dx \, dy = \int \int_{\Delta^*} k \frac{z^2}{|z|^2} z^n \, dx \, dy = 0, \quad n = -1, 0, 1, 2, \ldots,
\]
it follows readily that
\[
\sup \left\{ \left| \int \int_{\Delta^*} \mu(z) \phi(z) \, dx \, dy \right|/\|\phi\| : \phi(z) \in A(\Delta^*) \right\} = 0.
\]
Thus, $\mu$ is not extremal in its class in $T(\Delta^*)$ by the condition of Hamilton sequence. And hence, $\mu$ is not locally extremal in $\Delta^*$. 
Notice that in the above example, \( \Omega_1 \cap \Omega_2 \) contains two connected components. If the condition \( \Omega_1 \cap \Omega_2 \neq \emptyset \) in the original problem replaced by that \( \Omega_1 \cap \Omega_2 \) is connected, what situation should be? Up to the present, we can not find such a counterexample.

**Remark 1.** After the completion of this paper I have become aware of a paper with related result: Zhong Li et al., An extremal problem of quasiconformal maps, to appear in Proc. Amer. Math. Soc.

**References**


