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ON EXTREMALITY OF TWO CONNECTED LOCALLY EXTREMAL BELTRAMI COEFFICIENTS

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Let Ω_1 and Ω_2 be two domains in the complex plane with a nonempty intersection. Suppose that μ_j are locally extremal Beltrami coefficients in Ω_j ($j = 1, 2$) respectively. In 1980, Sheretov posed the problem: Will the coefficient μ defined by the condition $\mu(z) = \mu_j(z)$ for $z \in \Omega_j$, $j = 1, 2$, be locally extremal in $\Omega_1 \cup \Omega_2$? We give a counterexample to show that μ may not be locally extremal and not even be extremal.

1. INTRODUCTION

Let $\mathfrak{D}$ be a domain in the complex plane $\mathbb{C}$ with at least two boundary points and Let $M(\mathfrak{D})$ be the open unit ball of $L^\infty(\mathfrak{D})$. Every element $\mu \in M(\mathfrak{D})$ can be regarded as an element in $L^\infty(\mathbb{C})$ by putting μ equal to zero in the outside of $\mathfrak{D}$. Every $\mu \in M(\mathfrak{D})$ induces a global quasiconformal self-mapping f of the plane which solves the Beltrami equation [**1**],

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{1}$$

and f is defined uniquely up to postcomposition by a complex affine map of the plane. Conversely, any quasiconformal mapping f defined on $\mathfrak{D}$ has a Beltrami coefficient $\mu(z) = f_{\bar{z}}(z)/f_z(z)$ in $M(\mathfrak{D})$.

Two Beltrami coefficients μ, $\nu \in M(\mathfrak{D})$ are equivalent if they induce quasiconformal mappings f and g by (1) such that there is a conformal map c from $f(\mathfrak{D})$ to $g(\mathfrak{D})$ and an isotopy through quasiconformal mappings h_t, $0 \leqslant t \leqslant 1$, from $\mathfrak{D}$ to $\mathfrak{D}$ which extend continuously to the boundary of $\mathfrak{D}$ such that

1. $h_0(z)$ is identically equal to z on $\mathfrak{D}$,
2. h_1 is identically to $g^{-1} \circ c \circ f$, and
3. $h_t(p) = g^{-1} \circ c \circ f(p)$ for any $p \in \partial\mathfrak{D}$.

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The equivalence relation partitions $M(\mathfrak{D})$ into equivalence classes and the space of equivalence classes is by definition the Teichmüller space $T(\mathfrak{D})$ of $\mathfrak{D}$.

Given $\mu \in M(\mathfrak{D})$, we denote by $[\mu]$ the set of all elements $\nu \in M(\mathfrak{D})$ equivalent to μ, and set

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

We say that μ is extremal (in $[\mu]$) if $\|\mu\|_\infty = k_0([\mu])$, μ is uniquely extremal if $\|\nu\|_\infty > k_0([\mu])$ for any other $\nu \in [\mu]$; the alternative is that μ is non-uniquely extremal.

We define $A(\mathfrak{D})$ as the Banach space of all holomorphic functions φ on $\mathfrak{D}$ with L^1-norm

$$\|\varphi\| = \iint_{\mathfrak{D}} |\varphi(z)| < \infty.$$

As is well known, a necessary and sufficient condition (Hamilton-Krushkal-Reich-Strebel condition) that a Beltrami coefficient μ is extremal in its class in $T(\mathfrak{D})$ is that [4] it has a so-called Hamilton sequence, namely, a sequence $\{\phi_n \in A(\mathfrak{D}) : \|\phi_n\| = 1,\ n \in \mathbb{N}\}$, such that

$$\lim_{n\to\infty} \iint_{\mathfrak{D}} \mu\phi_n(z)\, dx\, dy = \|\mu\|_\infty. \tag{2}$$

A Beltraim coefficient μ in $\mathfrak{D}$ is called to be locally extremal if for any domain $G \subset \mathfrak{D}$ it is extremal in its class in $T(G)$; in other words,

$$\|\mu\|_G := \operatorname*{essup}_{z\in G} |\mu| = \sup\left\{\frac{|\iint_G \mu\phi_n(z)\, dx\, dy|}{\|\phi\|} : \phi \in A(G)\right\}.$$

Obviously, extremality in the whole domain is a prerequisite for a Beltrami coefficient to be locally extremal.

In [**6**], Sheretov investigated locally extremal Beltrami coefficients and posed the following problem: Let Ω_1 and Ω_2 be two domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that μ_j are locally extremal Beltrami coefficients in Ω_j $(j = 1, 2)$ respectively. Will the coefficient μ defined by the condition $\mu(z) = \mu_j(z)$ for $z \in \Omega_j$, $j = 1, 2$, be locally extremal in $\Omega_1 \cup \Omega_2$?

The main purpose of this paper is to give a negative answer to the above problem in a stronger sense. We shall construct certain counterexample in the next section.

2. Construction of counterexample

If μ in $M(\mathfrak{D})$ is uniquely extremal in its class $[\mu]$ in $T(\mathfrak{D})$, then it is obviously locally extremal. But the converse is not true for which here we include the example constructed in [**2**, Theorem 2.2] by Reich.

Reich's example: We denote the parabolic region Ω_0 by

$$\Omega_0 = \{z = x + iy : x > y^2,\ x > 0\}.$$

In Ω_0, we define $\mu(z) \equiv k$ where $k \in (0,1)$ is a constant. Examining the proof of [**2**, Theorem 2.2], we find that

$$\sup\left\{\frac{|\iint_G \mu(z)\phi(z)\,dx\,dy|}{\iint_G |\phi|\,dx\,dy} : \phi(z) \in A(\Omega_0)\right\} = k$$

for any positive measure subset G of Ω_0. This relation indicates that μ is locally extremal in Ω_0. But, it is well known that μ is not uniquely extremal (see [**2, 3**]).

In our counterexample to Sheretov's problem, μ_j, $j = 1,2$, are uniquely extremal while μ may not be locally extremal and not even be extremal in its corresponding class.

EXAMPLE 1. Let Δ be the unit disk $\{z : |z| < 1\}$. Put

$$\Omega_1 = \left\{z \in \Delta : \arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}, \ \Omega_2 = \left\{z \in \Delta : |\arg z| > \frac{\pi}{4}\right\}.$$

Obviously, $\Omega_1 \cup \Omega_2 = \Delta^* = \Delta - \{0\}$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$. Set $\mu = k\overline{\varphi}/|\varphi$ on Δ, where $k \in (0,1)$ is a constant and $\varphi(z) = 1/z^2$. Let μ_j $(j = 1,2)$ be the restrictions of μ on Ω_j, respectively. We claim that μ_j are uniquely extremal in their classes in $T(\Omega_j)$, respectively.

Suppose the conformal mapping $z = F(\zeta)$ maps $\Delta_\zeta = \{|\zeta| < 1\}$ onto Ω_1. The question becomes that of determining whether the Beltrami coefficient

$$\widetilde{\mu} = k\frac{\overline{\varphi \circ F}}{|\varphi \circ F|}\frac{\overline{F'(\zeta)^2}}{|F'(\zeta)|^2}$$

is extremal or uniquely extremal in its class in $T(\Delta_\zeta)$. Set

$$\psi(\zeta) = (\varphi \circ F)F'(\zeta)^2.$$

Because the conformal mapping F^{-1} transfers the second order pole of $\varphi(z)$ to the second order pole of $\psi(\zeta)$, it is not difficult to see that $\psi(\zeta)$ is holomorphic in Δ_ζ and is meromorphic in $\overline{\Delta_\zeta}$ except that it has a pole of second order at $\zeta = F^{-1}(0)$. Thus, by [**5**, Theorem 6], $\widetilde{\mu}$ is uniquely extremal in its class in $T(\Delta_\zeta)$, and hence μ_1 is uniquely extremal in its class in $T(\Omega_1)$. Similarly, μ_2 is uniquely extremal in its class in $T(\Omega_2)$.

However, μ is not even extremal in $[\mu]$ in $T(\Delta^*)$. In fact, noting that $\{z^n : n = -1,0,1,2,\ldots\}$ is a base of the Banach space $A(\Delta^*)$ and

$$\iint_{\Delta^*} \mu(z)\phi(z)\,dx\,dy = \iint_{\Delta^*} k\frac{z^2}{|z|^2}z^n\,dx\,dy = 0, \ n = -1,0,1,2,\ldots,$$

it follows readily that

$$\sup\left\{\left|\iint_{\Delta^*} \mu(z)\phi(z)\,dx\,dy\right|/\|\phi\| : \phi(z) \in A(\Delta^*)\right\} = 0.$$

Thus, μ is not extremal in its class in $T(\Delta^*)$ by the condition of Hamilton sequence. And hence, μ is not locally extremal in Δ^*.

Notice that in the above example, $\Omega_1 \cap \Omega_2$ contains two connected components. If the condition $\Omega_1 \cap \Omega_2 \neq \emptyset$ in the original problem replaced by that $\Omega_1 \cap \Omega_2$ is connected, what situation should be? Up to the present, we can not find such a counterexample.

REMARK 1. After the completion of this paper I have become aware of a paper with related result: Zhong Li et al., An extremal problem of quasiconformal maps, to appear in Proc. Amer. Math. Soc.

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