THE WEAK-STAR CLOSURE OF THE UNIT BALL
IN A HYPERPLANE

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1. Introduction

Let $X$ be a normed linear space. We regard $X$ as a subspace of its bidual $X^{**}$. Polars will always be evaluated in the pair $(X^{**}, X^*)$. We denote the closed unit ball in $X$ by $U$, so that $U^0$, $U^{00}$ are the closed unit balls in $X^*$, $X^{**}$ respectively. The weak topology induced by $X$ on $X^*$ (the "weak-star" topology) will be denoted by $\sigma(X)$, and $\text{cl}(\cdot)$ will denote $\sigma(X)$-closure.

Let $\phi$ be an element of $X^{**}$ that is not in $X$, and let $K$ be the kernel of $\phi$. Then $K$ is, of course, $\sigma(X)$-dense in $X^*$. When $X$ is complete, the Krein-Smul'yan theorem tells us that $K \cap U^0$ is not $\sigma(X)$-closed, but it gives no further information about the set $\text{cl}(K \cap U^0)$. The purpose of this note is to determine $\text{cl}(K \cap U^0)$ as accurately as possible (in doing so, we shall obtain incidentally a very simple proof of the Krein-Smul'yan theorem for hyperplanes). The radius of the largest ball contained in $\text{cl}(K \cap U^0)$ is known to be

$$\inf \{ \| x - \lambda \phi \| : x \in X, \| x \| = 1, \lambda \text{ scalar} \}$$

(1), ch. IV, § 5, ex. 14). This last statement applies, in fact, when $K$ is any $\sigma(X)$-dense linear subspace of $X^*$ (with elements of $K^0$ replacing the multiples $\lambda \phi$), and is generalised to arbitrary linear subspaces in (3). However, $\text{cl}(K \cap U^0)$ is clearly not just a multiple of $U^0$. Denoting by $d(\phi, X)$ the norm-distance from $\phi$ to $X$, we shall prove that

$$A(\phi) \subseteq \text{cl}(K \cap U^0) \subseteq B(\phi),$$

where

$$A(\phi) = \left\{ f \in X^*: \| f \| + \frac{|\langle \phi, f \rangle|}{d(\phi, X)} \leq 1 \right\},$$

$$B(\phi) = \left\{ f \in U^0: |\langle \phi, f \rangle| \leq 2d(\phi, X) \right\}.$$

In the particular case $X = c_0$, we show that $\text{cl}(K \cap U^0)$ is always $A(\phi)$. In general, however, $\text{cl}(K \cap U^0)$ can be either $A(\phi)$, $B(\phi)$ or something between the two.

The appearance of the ratio $|\langle \phi, f \rangle|/d(\phi, X)$ in the description is not as unreasonable as may at first seem, as the following considerations suggest:

(1) One would expect $f$ to have a better chance of being in $\text{cl}(K \cap U^0)$ if $|\langle \phi, f \rangle|$ is small.
(2) $K$ is unchanged if $\phi$ is multiplied by a non-zero scalar. Hence the factor $|\langle \phi, f \rangle|$ will need to be balanced by something else that is multiplied by $|\lambda|$ when $\phi$ is replaced by $\lambda \phi$.

(3) If $d(\phi, X)$ is small, then $\phi$ is not far from being $\sigma(X)$-continuous. Consequently, one might expect $\text{cl}(K \cap U^0)$ to be small.

The author is indebted to the referee for suggesting a better proof of Theorem 1, and for the comment in Note 3 to Theorem 1.

2. The theorems

Theorem 1. If $d(\phi, X)>0$, then $\text{cl}(K \cap U^0)$ contains $A(\phi)$.

Proof. For the moment, let $K$ be any linear subspace of $X^*$. It follows at once from the Hahn-Banach theorem, by extending the restriction to $K$, that $(K \cap U^0)^0 = K^0 + U^{00}$.

Hence $\text{cl}(K \cap U^0)$ is the polar (in $X^*$) of $X \cap (K^0 + U^{00})$. If $K$ is now the kernel of $\phi$, then $K^0$ is the linear span of $\phi$. The result follows if we show that, for any $f \in X^*$,

$$
\sup \{ |\langle x, f \rangle| : x \in X \cap (K^0 + U^{00}) \} \leq \| f \| + \frac{|\langle \phi, f \rangle|}{d(\phi, X)}.
$$

Let $x$ be in $X \cap (K^0 + U^{00})$. Then there exist $\psi$ in $U^{00}$ and a scalar $\lambda$ such that $x = \lambda \phi + \psi$. Then $\| \lambda \phi - x \| \leq 1$, so $|\lambda| \leq 1/d(\phi, X)$, and

$$
|\langle x, f \rangle| = |\langle \lambda \phi + \psi, f \rangle| \leq |\lambda| \cdot |\langle \phi, f \rangle| + \| f \|,
$$

giving the required inequality.

Notes

(1) In particular, if $d(\phi, X)>0$, then $K \cap U^0$ is not $\sigma(X)$-closed. Hence we have proved the Krein-Smul'yan theorem for hyperplanes: if $X$ is complete and $K \cap U^0$ is $\sigma(X)$-closed, then $K$ is $\sigma(X)$-closed.

(2) Similar reasoning can be applied to the common kernel of a finite number of functionals. Let $\phi_1, \ldots, \phi_m$ be elements of $X^{**}$ such that

$$
\inf \left\{ \left\| \sum_{i=1}^m \lambda_i \phi_i - x \right\| : x \in X, \sum_{i=1}^m |\lambda_i| = 1 \right\} = r > 0,
$$

and let $K = \bigcap_{i=1}^m \ker \phi_i$. Then $\text{cl}(K \cap U^0)$ contains

$$
\left\{ f \in X^* : \| f \| + \frac{1}{r} \max |\langle \phi_i, f \rangle| \leq 1 \right\}.
$$

For if $L$ is the linear span of $\phi_1, \ldots, \phi_m$, and $x$ is in $X \cap (L + U^{00})$, then it is easily seen that

$$
|\langle x, f \rangle| \leq \| f \| + \frac{1}{r} \max |\langle \phi_i, f \rangle|.
$$
(3) Let $K$ be a linear subspace of $X^*$. Using the weak compactness of $U^{00}$, it is easy to show that
\[ X \cap (K^0 + U^{00}) = \{ x \in X : d(x, K^0) \leq 1 \}. \]
Hence $\text{cl}(K \cap U^0)$ is precisely the dual unit ball when $X$ is given the seminorm $p$, where $p(x) = d(x, K^0)$. This, of course, is the seminorm induced on $X$ by the quotient norm in $X/K^0$. It is a norm when $K$ is $\sigma(X)$-dense in $X^*$.

(4) The author's original proof of Theorem 1 was similar to the proof of the related result Corollary II, 4, 3 in (2). The proof given above was suggested by the referee.

**Theorem 2.** $\text{cl}(K \cap U^0)$ is contained in $B(\phi)$.

**Proof.** $\text{cl}(K \cap U^0)$ is contained in $U^0$, since $U^0$ is $\sigma(X)$-closed. Write $d(\phi, X) = r$. Suppose that $f \in U^0$ and $|\langle \phi, f \rangle| > 2r$; let $|\langle \phi, f \rangle| = 2r + 3\alpha$. Take $x_0 \in X$ such that $\| \phi - x_0 \| < r + \alpha$. Then $|\langle \phi - x_0, f \rangle| < r + \alpha$, since $\| f \| \leq 1$, so $|\langle x_0, f \rangle| > r + 2\alpha$. For $g \in U^0$, $|\langle \phi - x_0, g \rangle| < r + \alpha$, so if $|\langle x_0, g \rangle| > r + \alpha$, then $g \notin K$. Hence $\{ g \in X^* : |\langle x_0, g - f \rangle| \leq \alpha \}$ is disjoint from $K \cap U^0$.

**Corollary.** If $d(\phi, X) = 0$, then $K \cap U^0$ is $\sigma(X)$-closed.

Hence if $X$ is an incomplete normed space, and $\phi$ is an element of $X^{**} \setminus X$ with $d(\phi, X) = 0$, then $K \cap U^0$ is $\sigma(X)$-closed, though $K$ is $\sigma(X)$-dense in $X^*$ (a fact noted by Kerr (4)).

The set $B(\phi)$ is not necessarily $\sigma(X)$-closed (cf. examples below).

### 3. Two particular cases

(i) $X = c_0$. We show that, in this case, $\text{cl}(K \cap U^0)$ is always equal to $A(\phi)$. Identify $c_0^*$ with $l_1$ and $c_0^{**}$ with $m$. We use the notation $x(n)$ for the $n$th term of a sequence $x$; we continue to use the notation $\langle , \rangle$ for the evaluation of functionals. It is sufficient to consider $\phi \in m$ with $d(\phi, c_0) = 1$. Take an element $f$ of $l_1$ that is not in $A(\phi)$: then
\[ \| f \| + |\langle \phi, f \rangle| = 1 + 3\alpha \]
for some $\alpha > 0$. Since $d(\phi, c_0) = 1$, there exists $N$ such that $|\phi(i)| \leq 1 + \alpha$ for all $i > N$. Choose $N$ so that, also,
\[ \sum_{i=1}^{N} |f(i)| + \left| \sum_{i=1}^{N} \phi(i)f(i) \right| > 1 + 2\alpha. \]
There is a $\sigma(c_0)$-neighbourhood $V$ of $f$ such that, for $g \in V$,
\[ \sum_{i=1}^{N} |g(i)| + \left| \sum_{i=1}^{N} \phi(i)g(i) \right| > 1 + \alpha. \]
If $g \in V$ and $\langle \phi, g \rangle = 0$, then

$$\left| \sum_{N+1}^{\infty} \phi(i)g(i) \right| = \left| \sum_{1}^{N} \phi(i)g(i) \right|.$$  

But

$$\left| \sum_{N+1}^{\infty} \phi(i)g(i) \right| \leq (1 + \alpha) \sum_{N+1}^{\infty} |g(i)|.$$  

By (1) and (2),

$$\|g\| = \sum_{1}^{N} |g(i)| + \sum_{N+1}^{\infty} |g(i)| > \frac{1}{1 + \alpha} = 1.$$

Hence $V$ does not meet $K \cap U^0$.

An example is given in (1) (loc. cit.) of a $\sigma(c_0)$-dense linear subspace $E$ of $l_1$ (necessarily not a hyperplane) such that $\text{cl}(E \cap U^0)$ contains no multiple of $U^0$.

(ii) $X = l_1$. We use the following (more or less standard) notation: $e$ denotes the sequence having 1 in each place, and $e_n$ denotes the sequence having 1 in place $n$ and 0 elsewhere.

Let $L$ be a linear functional on $m$ such that $L(c_0) = \{0\}$, $L(e) = 1$ and $\|L\| = 1$ (i.e. an “extended limit”; the existence of such functionals is guaranteed by the Hahn-Banach theorem). We show that $d(L, l_1) = 1$. Choose $x \in l_1$. For any $\varepsilon > 0$, there exists $N$ such that

$$\sum_{N+1}^{\infty} |x(n)| \leq \varepsilon.$$

Let $f_N = e - (e_1 + \ldots + e_N) \in m$. Then $\|f_N\| = 1$, $L(f_N) = 1$, and $|\langle x, f_N \rangle| \leq \varepsilon$.

Hence $\|L - x\| \geq 1 - \varepsilon$.

First let $\phi = L$. Then $K$ contains $c_0$, from which it follows that

$$\text{cl}(K \cap U^0) = U^0$$

(in general, $U$ is $\sigma(X^*)$-dense in $U^{00}$). In this case, $\text{cl}(K \cap U^0)$ is $B(\phi)$.

Now take $k > 1$, and let $\phi = L + ke_1$. Then $d(\phi, l_1) = 1$. We show that:

(a) There exists $f_0$ in $\text{cl}(K \cap U^0)$ with $\langle \phi, f_0 \rangle = 2$.

(b) Given $\varepsilon > 0$, there exists $f_\varepsilon \notin \text{cl}(K \cap U^0)$ with $|\langle \phi, f_\varepsilon \rangle| \leq \varepsilon$ and $\|f_\varepsilon\| = 1$.

Roughly speaking, this means that $\text{cl}(K \cap U^0)$ goes out as far as $B(\phi)$ in some directions, and only as far as $A(\phi)$ in others.

(a) Let

$$f_0 = e - \left(1 - \frac{1}{k}\right)e_1 = \left(\frac{1}{k}, 1, 1, \ldots\right).$$

Then $\langle \phi, f_0 \rangle = 2$, and $f_0$ is the $\sigma(l_1)$-limit of the sequence $(f_n)$, where

$$f_n = \left(\frac{1}{k}, 1, \ldots, 1, -1, -1, \ldots\right) = -e + \left(1 + \frac{1}{k}\right)e_1 + \sum_{j=2}^{n} 2e_j.$$
Each $f_n$ is in $K \cap U^0$.

(b) We may assume that $\varepsilon \leq k - 1$. Let

$$f_\varepsilon = \left(\frac{1+\varepsilon}{k}, -1, -1, \ldots\right) = \left[1 + \frac{1}{k} (1+\varepsilon)\right]e_1 - e.$$

Then $\langle \phi, f_\varepsilon \rangle = \varepsilon$. If $\|g\| \leq 1$ and $g(1) > \frac{1}{k} \left(1 + \frac{\varepsilon}{2}\right)$, then $\langle ke_1, g \rangle > 1 + \frac{\varepsilon}{2}$, while $|\langle L, g \rangle| \leq 1$, so $\langle \phi, g \rangle \neq 0$. Hence $f_\varepsilon \notin \text{cl}(K \cap U^0)$.

REFERENCES


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