WEYL’S THEOREM FOR $p$-HYPONORMAL OR $M$-HYPONORMAL OPERATORS

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Abstract. In 1997, M. Cho, M. Ito and S. Oshiro showed that Weyl’s theorem holds for $p$-hyponormal operators, for any $p > 0$. In this note we give another proof of this result. Also, it is shown that Weyl’s theorem holds for $M$-hyponormal operators. Further, in 1962, Stampfli showed that if $T$ is hyponormal and its Weyl spectrum is $\{0\}$ then $T$ is compact and normal. We show that this result remains true if the hypothesis of hyponormality is replaced by either (a) $p$-hyponormality or (b) $M$-hyponormality.

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1. Introduction. We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. For a $T \in \mathcal{B}(\mathcal{H})$ and for some $p > 0$, if $(T^*T)^p \geq (TT^*)^p$, then $T$ is said to be $p$-hyponormal. $T \in \mathcal{B}(\mathcal{H})$ is called $M$-hyponormal if there exists a positive constant $M$ for each $z \in \mathbb{C}$ such that $(T-zI)(T-zI)^* \leq M^2(T-zI)^*(T-zI)$. If $p = 1$, then $T$ is called simply hyponormal and it is equivalent to the case where $M = 1$.

$T \in \mathcal{B}(\mathcal{H})$ is called a Fredholm operator if $T\mathcal{H}$ is closed and both $\text{Ker}T = \{x \in \mathcal{H} : Tx = 0\}$ and $\text{Ker}T^*$ are finite-dimensional. To any Fredholm operator $T$ there corresponds an integer $i(T) = \dim \text{Ker}T - \dim \text{Ker}T^*$, called the index of $T$. Let $\mathcal{F}_0$ denotes the class of all Fredholm operators in $\mathcal{B}(\mathcal{H})$ of index 0. Then $w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \not\in \mathcal{F}_0\}$ is called the Weyl spectrum of $T$. It is known that, for a $T \in \mathcal{B}(\mathcal{H})$, $w(T)$ is non-empty and $w(T) = \bigcap_{K \in \mathcal{C}(\mathcal{H})} \sigma(T + K)$, where $\sigma(T)$ and $\mathcal{C}(\mathcal{H})$ denote the spectrum of $T$ and the set of all compact operators in $\mathcal{B}(\mathcal{H})$ respectively.

For a $T \in \mathcal{B}(\mathcal{H})$, let $\sigma_p(T)$ and $\pi_0(T)$ denote the point spectrum and the set of all isolated eigenvalues of finite multiplicity of $T$ respectively. According to Coburn [3], we say that Weyl’s theorem holds for $T$ if $\sigma(T) \setminus w(T) = \pi_0(T)$. He showed that Weyl’s theorem holds for hyponormal operators and this result was generalized to $p$-hyponormal operators by Cho-Ito-Oshiro [4]. Also, by Stampfli [8], it is known that if $T$ is hyponormal and if $w(T) = \{0\}$, then $T$ is compact and normal. In this paper, we shall give another proof of the result of Cho-Ito-Oshiro and prove that Weyl’s theorem holds for $M$-hyponormal operators and that Stampfli’s result above is also applicable to $p$-hyponormal or $M$-hyponormal operators.

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2. Preliminaries. According to Berberian [2], we say that $T$ is isoloid if every isolated point of $\sigma(T)$ is in the point spectrum of $T$. Also if every restriction $T|_M$ to its reducing subspace $M$ is isoloid, then we say that $T$ satisfies the condition $(\alpha^{\prime\prime})$.

The following result was given by Berberian [2].

**Theorem A.** If $T \in \mathcal{B}(\mathcal{H})$ satisfies the condition $(\alpha^{\prime\prime})$ and if every finite-dimensional eigenspace of $T$ reduces $T$, then Weyl’s theorem holds for $T$.

**Definition 1.** If $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$, then we say that $T$ is paranormal.

The following results are well known.

**Proposition 1.** If $T$ is paranormal, then $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

**Proposition 2.** (Heinz’s, inequality [6]) If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$, for all $\alpha \in (0, 1]$.

**Proposition 3.** (Hansen’s inequality [5]) If $A \geq 0$ and $\|B\| \leq 1$ then $(B^*AB)^\delta \geq B^* A^\delta B$, for all $\delta \in (0, 1]$.

**Proposition 4.** (Hölder-McCarthy inequality [7]) If $A \geq 0$, then for each $x \in \mathcal{H}$ we have

$$\langle A^r x, x \rangle \begin{cases} \leq \langle Ax, x \rangle^r \|x\|^{2(1-r)} & (0 < r \leq 1), \\ \geq \langle Ax, x \rangle^r \|x\|^{2(1-r)} & (r \geq 1). \end{cases}$$

**Proposition 5.** [12] Let $T$ be $p$-hyponormal with its polar decompositon $T = U|T|$. Then, for any $s$ and $t$ such that $s \geq 0$ and $t \geq 0$,

$$|T|^s U|U|^t \text{ is } \begin{cases} 1\text{-hyponormal} & (\max(s, t) \leq p), \\ \frac{p+\min(s, t)}{s+t}\text{-hyponormal} & (\max(s, t) > p). \end{cases}$$

**Proposition 6.** ([11]) The restriction $T|_M$ of an $M$-hyponormal operator $T$ to its invariant subspace $M$ is also $M$-hyponormal.

**Definition 2.** For a $T \in \mathcal{B}(\mathcal{H})$, we say that $T$ belongs to the class $\mathcal{Y}_\alpha$ for some $\alpha \geq 1$ if there is a positive number $K_\alpha$ such that $|T^*T - TT^*|^\alpha \leq K_\alpha^2(T - zI)^*$ for all $z \in \mathbb{C}$. Also, let $\mathcal{Y} = \bigcup_{\alpha \geq 1} \mathcal{Y}_\alpha$.

The following results are known.

**Proposition 7.** [10] If $T$ is $M$-hyponormal, then $T \in \mathcal{Y}_2 \subseteq \mathcal{Y}$.

**Proposition 8.** [10] If $T \in \mathcal{Y}$, then $Tx = \lambda x$ implies $T^*x = \overline{\lambda} x$.

**Proposition 9.** [10] If $T \in \mathcal{Y}$ and if $\sigma(T) = \{0\}$, then $T = 0$. 

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Corollary 1. If $T$ is $M$-hyponormal and $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.

Proof. Let $T$ be a $M$-hyponormal operator such that $\sigma(T) = \{\lambda\}$, then $T - \lambda I$ is also $M$-hyponormal and $\sigma(T - \lambda I) = \{0\}$. Therefore $T - \lambda I = 0$ by Propositions 7 and 9.

3. Main theorems.

Lemma 1. If $T$ is invertible and $p$-hyponormal, then $T^{-1}$ is also $p$-hyponormal.

Proof. Since $T$ is an invertible $p$-hyponormal operator, we have $|T|^{2p} \geq |T^*|^{2p}$ and $|T|^{-2p} \leq |T^*|^{-2p}$. It is easy to verify the equalities $|T^{-1}|^{2p} = |T|^{-2p}$ and $|T^*|^{-2p} = |T^{-1}|^{-2p}$, and the assertion of Lemma 1 holds.

Remark. It is well known that the inverse operator $T^{-1}$ of an invertible paranormal operator $T$ is also paranormal.

Lemma 2. If $T$ is $p$-hyponormal, then $T$ is paranormal.

Proof. Let $x \in \mathcal{H}$ be an arbitrary non-zero vector. Then we have

\[
\|T^2x\|^2 = \left\langle (T^*T)^{\frac{1}{p}}Tx, Tx \right\rangle \\
\geq \|Tx\|^2(1-\frac{1}{p})\left\langle (T^*T)^{\frac{1}{p}}Tx, Tx \right\rangle^{\frac{1}{p}} \quad \text{(by Proposition 4)} \\
\geq \|Tx\|^2(1-\frac{1}{p})\left\langle T^*TT^*x, Tx \right\rangle^{\frac{1}{p}} \quad \text{(since $T$ is $p$-hyponormal)} \\
= \|Tx\|^2(1-\frac{1}{p})\left\langle T^*T^1+\frac{1}{p}x, x \right\rangle^{\frac{1}{p}} \\
\geq \|Tx\|^2(1-\frac{1}{p})\|x\|^{-2}\|Tx\|^{2(1+\frac{1}{p})} \quad \text{(by Proposition 4)} \\
= \|Tx\|^4\|x\|^{-2}.
\]

Hence, we have $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$, and the proof of Lemma 2 is complete.

Corollary 2. If $T$ is $p$-hyponormal and if $\sigma(T) = \{0\}$, then $T = 0$.

Proof. By Lemma 2 and by Proposition 1, we have the conclusion.

Lemma 3. [9] If $T$ is $p$-hyponormal for a $p$ such that $0 < p \leq 1$, then the restriction $T|_{\mathcal{M}}$ to its invariant subspace $\mathcal{M}$ is also $p$-hyponormal.

Proof. Let $P$ be the orthogonal projection onto $\mathcal{M}$. Then $T|_{\mathcal{M}} = TP$ on $\mathcal{M}$. Thus we obtain

\[
\left\langle (T|_{\mathcal{M}})^\delta(T|_{\mathcal{M}})^\rho \right\rangle = \left\langle (PT^*TP)^\rho \right\rangle \geq P(T^*T)^\rho P \quad \text{(by Proposition 3)},
\]

and
\[\{(T|_M)(T|_M)^\ast\}^p = (TPT^\ast)^p = P(TPT^\ast)^p P \leq P(TT^\ast)^p P \quad \text{(by Proposition 2).}\]

Since \(T\) is \(p\)-hyponormal
\[\{(T|_M)(T|_M)^\ast\}^p \leq P(TT^\ast)^p P \leq P(T^\ast T)^p P \leq \{(T|_M)^\ast(T|_M)\}^p\]
and this inequality shows that \(T|_M\) is \(p\)-hyponormal.

**Theorem 1.** If \(T\) is \(p\)-hyponormal or \(M\)-hyponormal, then \(T\) is isoloid and satisfies the condition \((\alpha^\prime)\) by Lemma 3 or Proposition 6 respectively.

**Proof.** Let \(\lambda\) be an isolated point of \(\sigma(T)\). Then the range of the Riesz projection \(E = \frac{1}{2\pi i} \int_{\partial D} (I - T)^{-1} dz\) is a closed invariant subspace for \(T\) and \(\sigma(T|_EH) = \{\lambda\}\). Here \(D\) is a closed ball with center \(\lambda\) that satisfies \(\sigma(T) \cap D = \{\lambda\}\), and \(\partial D\) is the boundary of \(D\) described once counterclockwise.

First, we prove that every \(M\)-hyponormal operator is isoloid.

If \(T\) is \(M\)-hyponormal, then \(T|_EH\) is also \(M\)-hyponormal, by Proposition 6. Since \(\sigma(T|_EH) = \{\lambda\}\), we have \(T|_EH = \lambda E\), by Corollary 1. Hence, the assertion of Theorem 1 holds for \(M\)-hyponormal operators.

Next, we prove that every \(p\)-hyponormal operator is isoloid.

If \(\lambda = 0\), then \(\sigma(T|_EH) = \{0\}\) and \(T|_EH\) is paranormal, by Lemma 3. \(T|_EH = 0\) by Corollary 2. Therefore 0 is in the point spectrum of \(T\).

If \(\lambda \neq 0\), \(T|_EH\) is an invertible paranormal operator and hence \((T|_EH)^{-1}\) is also paranormal by Lemmas 1 and 2. By Proposition 1, we see that \(\|T|_EH\| = |\lambda|\) and \(\|(T|_EH)^{-1}\| = \frac{1}{|\lambda|}\). Let \(x \in E\mathcal{H}\) be an arbitrary vector. Then
\[\|x\| \leq \|(T|_EH)^{-1}\| \|T|_EHx\| = \frac{1}{|\lambda|} \|T|_EHx\| \leq \frac{1}{|\lambda|} |\lambda| \|x\| = \|x\|.
\]
This implies that \(\frac{1}{\lambda} T|_EH\) is unitary. Therefore \(T|_EH\) is normal and equal to \(\lambda E\). Hence, the assertion of Theorem 1 holds for \(M\)-hyponormal operators. This completes the proof of Theorem 1.

**Lemma 4.** For any \(T \in \mathcal{B}(\mathcal{H})\) with its polar decomposition \(T = U|T|\) and \(\lambda = re^{i\theta} \neq 0\), \((T - \lambda I)x = (T - \lambda I)^*x = 0\) if and only if \((|T| - ri)x = (U - e^{i\theta}I)x = 0\).

**Proof.** If \((T - \lambda I)x = (T - \lambda I)^*x = 0\), then \(|T|^2 x = T^* T = r^2 x\) and \(|T|x = rx\) because \(|T| + ri\) is invertible. By the assumption, \(re^{i\theta}x = Tx = U|T|x = rUx\) and we have \((U - e^{i\theta}I)x = 0\) because \(r \neq 0\).

Conversely, if \((|T| - ri)x = (U - e^{i\theta}I)x = 0\), then \((U - e^{i\theta}I)^*x = 0\) by the general theory and we obtain that \(Tx = U|T|x = Urx = re^{i\theta}x = \lambda x\) and \(T^*x = |T|U^*x = |T|e^{-i\theta}x = re^{-i\theta}x = \lambda x\).

**Lemma 5.** If \(T\) is \(p\)-hyponormal for \(p = \frac{1}{2}\), then \((T - \lambda I)x = 0\) implies that \((T - \lambda I)^*x = 0\).

**Proof.** If \(\lambda = 0\), then the assertion is trivial since \(\text{Ker}T \subseteq \text{Ker}T^*\), for every \(p\)-hyponormal operator \(T\).

For a \(p\)-hyponormal operator \(T\) with \(p = \frac{1}{2}\) and a \(\lambda = re^{-i\theta} \neq 0\), we have
\[(T - \lambda I)^*(T - \lambda I) = (|T| - |\lambda|I)^2 + |\lambda|((U - e^{i\theta}I)|T|(U - e^{i\theta}I)^* + |\lambda|(|T| - |T^*|) \geq (|T| - |\lambda|I)^2.\]

Therefore, if \((T - \lambda I)x = 0\), then \(|T| - |\lambda|I)x = 0\) and hence \((U - e^{i\theta}I)x = 0\) because \(r \neq 0\). We have \((T - \lambda I)^*x = 0\) by Lemma 4.

**Lemma 6.** If \(T\) is \(p\)-hyponormal with the polar decomposition \(T = U|T|\), then every eigenspace of \(U\) is invariant under \(|T|\) and \(U^*\).

**Proof.** If \(\lambda \in \mathbb{C}\) is an eigenvalue of \(U\) and \(x\) is a non-zero eigenvector with respect to \(\lambda\), then \(\lambda = 0\) or \(|\lambda| = 1\), since the range of \(U\) is a subspace of the domain space of \(U\).

If \(\lambda = 0\), then the assertion follows from \(\text{Ker} U = \text{Ker}|T| \subseteq \text{Ker}|T^*| = \text{Ker} U^*\).

If \(\lambda = e^{i\theta}\) and \((U - e^{i\theta}I)x = 0\), then \((U - e^{i\theta}I)^*x = 0\) by the general theory and, since

\[
\left\| (|T|^{2p} - |T^*|^{2p})^\frac{1}{2} x \right\|^2 = \left\langle \left( (|T|^{2p} - |T^*|^{2p})x, x \right) \right\rangle = \left\langle \left( (|T|^{2p} - U|T|^{2p}U^*)x, x \right) \right\rangle = -e^{-i\theta} \left\langle \left( |T|^{2p}x, (U - e^{i\theta}I)^*x \right) \right\rangle = 0,
\]

it follows that \((U - e^{i\theta}I)|T|^{2p}x = -e^{-i\theta}(|T|^{2p} - |T^*|^{2p})x = 0\). Therefore \(\text{Ker}(U - e^{i\theta}I)\) is invariant under \(|T|^{2p}\) and hence invariant under \(|T|\).

**Corollary 3.** If \(T\) is \(p\)-hyponormal, then \((T - \lambda I)x = 0\) implies that \((T - \lambda I)^*x = 0\).

**Proof.** Since \(\text{Ker} T \subseteq \text{Ker} T^*\) for every \(p\)-hyponormal operator \(T\), the assertion holds for \(\lambda = 0\).

By Propositions 5 and 2, if \(T\) is a \(p\)-hyponormal operator and \(\tilde{T} = |T|^{1/2} U |T|^{1/2}\), then \(\tilde{T}\) is always \(p\)-hyponormal with \(p = \frac{1}{2}\) and \(|\tilde{T}^*| \leq |T| \leq |\tilde{T}|\).

Hence if \((T - \lambda I)x = 0\), for some \(\lambda = re^{i\theta} \neq 0\), then \((\tilde{T} - \lambda I)|\tilde{T}|^\frac{1}{2}x = 0\) and \(|\tilde{T}|^\frac{1}{2}x = |\tilde{T}^*||\tilde{T}|^\frac{1}{2}x = r|\tilde{T}|^\frac{1}{2}x\), by Lemmas 5 and 4. From the inequality \(|\tilde{T}^*| \leq |T| \leq |\tilde{T}|\) it is easy to show that \(|T||\tilde{T}|^\frac{1}{2}x = r|\tilde{T}|^\frac{1}{2}x\). Hence we have \((|T| - rI)x \in \text{Ker}|T| = \text{Ker} U\) and \(Ux = \frac{1}{r} U|T|x = \frac{1}{r} T x = \frac{1}{r} \lambda x = e^{i\theta} x\). Also \(x = U^* Ux \in [|T|\mathcal{H}]^{\sim}\).

Since \((|T| - rI)x \in \text{Ker}|T| \cap [|T|\mathcal{H}]^{\sim} = \{0\}\), \(|T|x = rx\) and \(T^*x = |T|U^*x = e^{-i\theta}|T|x = re^{-i\theta}x = \tilde{\lambda}x\).

**Remark.** In the proof of Corollary 3 (the case \(\lambda \neq 0\)), we only used the fact that \(T\) was \(w\)-hyponormal (i.e., \(T\) satisfies the condition \(|\tilde{T}^*| \leq |T| \leq |\tilde{T}|\)). In [1], Aluthge-Wang proved that every \(w\)-hyponormal operator is paranormal and it is easy to show that every \(w\)-hyponormal operator \(T\) with \(\text{Ker} T \subseteq \text{Ker} T^*\) satisfies the condition \((\alpha^2)\). Hence Weyl’s theorem holds also for a \(w\)-hyponormal operator \(T\) which satisfies the property \(\text{Ker} T \subseteq \text{Ker} T^*\), by Berberian’s result (Theorem A).

**Theorem 2.** Weyl’s theorem holds for \(p\)-hyponormal or \(M\)-hyponormal operators.

**Proof.** If \(T\) is \(p\)-hyponormal or \(M\)-hyponormal, then every eigenspace of \(T\) is a reducing subspace of \(T\), by Corollary 3 or Propositions 7 and 8, respectively. Also \(T\)
satisfies the condition $a^\alpha$, by Theorem 1, and therefore Berberian’s result (Theorem A) shows that Weyl’s theorem holds for $T$.

**Theorem 3.** For a w-hyponormal operator $T$, $\sigma(T) \backslash w(T) \subseteq \pi_{00}(T)$. Moreover, Weyl’s theorem holds for $T$ if $\text{Ker} T_{|[TH]} = \{0\}$.

**Proof.** Firstly, we shall show that $\sigma(T) \backslash w(T) \subseteq \pi_{00}(T)$, for every w-hyponormal operator $T$. It follows from the Remark after Corollary 3 that we have $\sigma(T) \backslash \{w(T) \cup \{0\}\} = \pi_{00}(T) \cup \{0\}$. Also it suffices to show that if $0 \in \sigma(T) \backslash \{w(T) \cup \{0\}\}$, then $0 \in \pi_{00}(T)$. Assume that $0 \notin \pi_{00}(T)$. Since $0 < \text{dim Ker} T < \infty$ and $TH$ is closed because $0 \in \sigma(T) \backslash w(T)$, our assumption implies that $0$ is a cluster point of $\sigma(T)$. Since $T \in \mathcal{F}_0$, there is $s > 0$ with $\{z \in \mathbb{C} : 0 < |z| < s\} \cap \sigma(T) \subseteq \sigma(T) \backslash \{w(T) \cup \{0\}\} \subseteq \pi_{00}(T)$ and therefore $\{z \in \mathbb{C} : 0 < |z| < s\} \cap \sigma(T)$ is a countable infinite set whose only cluster point is $0$. Put $\{\lambda_n : n \in \mathbb{N}\} = \{z \in \mathbb{C} : 0 < |z| < s\} \cap \sigma(T)$. Then each $\lambda_n$ is an eigenvalue of $T$, satisfying $(T - \lambda_n I)x = 0$ implies $(T - \lambda I)^*x = 0$, with finite multiplicity and $\lambda_n \to 0$ as $n \to \infty$. Let $M = \oplus_n \text{Ker}(T - \lambda_n I)$ and let $E_n$ be the orthogonal projection onto $\text{Ker}(T - \lambda_n I)$. Then $M$ is an infinite dimensional subspace of $T$ and the restriction $T|_M = \oplus_n \lambda_n E_n$ is a compact normal operator with $\text{Ker} T|_M = \{0\}$. Hence $TM$ is not closed and this contradicts the fact that $TH$ is closed. Thus we have $0 \in \pi_{00}(T)$ and this completes the proof of the first part of this theorem.

Next, we shall show that Weyl’s theorem holds for w-hyponormal operators which satisfy the condition $\text{Ker} T_{|[TH]} = \{0\}$.

Since $\sigma(T) \backslash \{w(T) \cup \{0\}\} \subseteq \{\sigma(T) \backslash \{w(T) \cup \{0\}\}\} \cup \{0\}$, it suffices to show that if $0 \in \pi_{00}(T)$, then $0 \in \sigma(T) \backslash \{w(T) \cup \{0\}\}$.

Let

$$
T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = [TH]^{\perp} \oplus \text{Ker} T^*,$$

be w-hyponormal with $\text{Ker} A = \{0\}$. If $0 \in \pi_{00}(T)$, then $0 \notin \sigma(A)$ or $0$ is an isolated point of $\sigma(A)$ because $\sigma(A) \subseteq \sigma(T) \subseteq \sigma(A) \cup \{0\}$. We see that $A$ is isolated because $A$ is paranormal. Hence if $0$ is an isolated point of $\sigma(A)$, then $0 \in \sigma_p(A)$ and this contradicts $\text{Ker} A = \{0\}$. Also we have $0 \notin \sigma(A)$. It is easy to see that $\text{Ker} T = \{ -A^{-1} Su \oplus u : 0 \oplus u \in \text{Ker} T^* \}$ and this implies $\text{dim Ker} T^* = \text{dim Ker} T < \infty$. Since the closedness of $TH$ follows from the invertibility of $A$, we have $0 \in \sigma(T) \backslash \{w(T) \cup \{0\}\}$, and this completes the proof of the second part of this theorem.

**Theorem 4.** If $T$ is p-hyponormal or M-hyponormal and if $w(T) = \{0\}$, then $T$ is compact and normal.

**Proof.** Since Weyl’s theorem holds for $T$, by Theorem 2, and $w(T) = \{0\}$, by the assumption, every non-zero point of $\sigma(T)$ is an isolated point of $\sigma(T)$ with finite dimensional eigenspace which reduces $T$, by Corollary 3 or Propositions 7 and 8, respectively. Hence $\sigma(T) \backslash \{w(T) \cup \{0\}\}$ is a finite set or a countably infinite set whose only cluster point is $0$. Let $\sigma(T) \backslash \{w(T) \cup \{0\}\}$ with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots > 0$ and let $E_n$ be the orthogonal projection onto $\text{Ker}(T - \lambda_n I)$. Then $TE_n = E_nT = \lambda_n E_n$ and $E_nE_m = 0$ if $n \neq m$. Put $E = \oplus_n E_n$. Then $T = \oplus_n \lambda_n E_n \oplus T_{|[I-E]\mathcal{H}}$ with the property $\sigma(T)_{|[I-E]\mathcal{H}} = \{0\}$. Since $T_{|[I-E]\mathcal{H}}$ is also p-hyponormal or M-hyponormal by Lemma 3 or by
Proposition 6, respectively, $T|_{(d-E)H} = 0$, by Corollary 2 or by Corollary 1 respectively. Hence $T = \bigoplus_0 \lambda_n E_n$ is normal. The compactness of $T$ follows from the finiteness or the countability of $\{\lambda_n\}_n$ satisfying $|\lambda_n| \downarrow 0$. Also each $E_n$ is a finite rank projection.

**Corollary 4.** If $T$ is $w$-hyponormal and if $w(T) = \{0\}$, then $T$ is compact and normal.

*Proof.* For every w-hyponormal operator $T$, $\sigma(T)|_{w(T)} \subseteq \pi_{00}(T)$, by Theorem 3. The proof is similar to that of Theorem 4.

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