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ISOMORPHIC SUBGROUPS OF FINITE p-GROUPS REVISITED

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Several papers of George Glauberman have appeared which analyze the structure of a finite p-group which contains two isomorphic maximal subgroups. The usual setting for an application of these results is a finite group, a p-subgroup, and an isomorphism of this p-group induced by conjugation. In this paper we prove a stronger version of Glauberman's Theorem 8.1 [**1**].

THEOREM A. *Suppose that H is a finite group and P is a weakly closed subgroup of some Sylow p-subgroup of H with respect to H. Assume that for some $h \in H$*

$$\langle P, P^h \rangle = H \quad \text{and} \quad |P : P \cap P^h| = p.$$

Let A be a subgroup of $\mathrm{Aut}(P)$ *that contains the automorphisms induced by conjugation by the elements of $N_H(P)$. Suppose that some element of A does not fix $Q = P \cap P^h$.*

Take $Q^* \subseteq Q$ maximal such that $Q^* \trianglelefteq H$ and such that there exists an element $\alpha \in A$ that fixes Q^* but not Q. Let n be the smallest positive integer such that α^n fixes Q. Let $\bar{P} = P/Q^* = \bar{P}_1$, and $\bar{P}_{i+1} = [\bar{P}_i, \bar{P}]$. Then $\bar{P}_4 = 1$. Furthermore:

(a) *If $\bar{P}_2 = 1$, then $|\bar{P}| \leqq p^n$.*

(b) *If $\bar{P}_2 \neq 1$, $\bar{P}_3 = 1$ and $p = 2$, then P is the direct product of $E(p)$ and an elementary group, and $n = 2$.*

(c) *If $\bar{P}_2 \neq 1$, $\bar{P}_3 = 1$ and $p \neq 2$, then n is a divisor of p, $p - 1$, or $p + 1$, and either $\bar{P} \simeq E(p)$ or $\bar{P} \simeq E(p) \times Z_p$. In the latter case $p = 3$ and there exists $\alpha \in A'$ such that α^p fixes Q but α does not.*

(d) *If $\bar{P}_3 \neq 1$, then $p = 3$, $\bar{P} \simeq E^*(p)$ and $n = 2$.*

Theorem A is exactly Glauberman's Theorem [**1**, 8.1] with the inclusion of "$p = 3$" in conclusion (c). $E(p)$ is the non-Abelian group of order p^3 which is generated by two elements of order p. $E^*(p)$ is a particular p-group of order p^6 defined in [**1**]. We note that Theorem 2 of Glauberman's paper [**1**] is a consequence of his Theorem 8.1. Thus Theorem A will also improve Glauberman's Theorem 2 [**1**].

To illustrate the application of Theorem A we include Theorem B, a corollary of Theorem A. In the statement of Theorem B, property $\mathscr{P}$ is any property of a finite group which is inherited from subgroups. For example $\mathscr{P}$ might be "the group involves $SL(2, p)$".

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THEOREM B. *Suppose G is a finite group, p is an odd prime, and S is a Sylow p-subgroup of G. Assume that some p-local subgroup satisfies $\mathscr{P}$ and choose a p-local subgroup $G^* = N(D)$ satisfying $\mathscr{P}$ with $|G^*|_p$ maximal. Suppose that $S^* = N_S(D)$ is a Sylow p-subgroup of G^*, $S^* \neq S$, $|S^* : D| = p$, and that there exists $h \in G^*$ such that $H = \langle S^*, S^{*h}\rangle$ satisfies $\mathscr{P}$.*

Then S^* is elementary abelian, $E(p)$, $E(3) \times Z_3$ or $E^*(3)$.

Proof. Choose $y \in N_S(S^*) - S^*$ such that $y^p \in S^*$ and let A be the automorphism group induced on S^* by $\langle N_H(S^*), y\rangle$. Pick $D^* \subseteq D$ maximal such that $D^* \trianglelefteq H$ and y fixes D^*. If $D^* \neq 1$ then $N(D^*) \supseteq \langle H, y\rangle$ which contradicts the choice of G^*. Thus $D^* = 1$ and Theorem B follows from Theorem A.

Before proving Theorem A we need one preliminary lemma.

LEMMA 1. *Let $P \simeq E(p) \times Z_p$, suppose $\alpha \in \mathrm{Aut}(P)$ is a p-element and is nontrivial on $P/Z(P)$. Then $\alpha^p = 1$ if $p \geqq 5$.*

Proof. Choose generators of $P/Z(P)$, a and b, such that

$$a^\alpha \equiv ab \quad \text{and} \quad b^\alpha \equiv b \quad (\text{modulo } Z(P)).$$

Let $c = [a, b]$ and choose $d \in Z(P)$ such that $Z(P) = \langle c, d\rangle$. Now $\langle c\rangle \simeq Z_p$ so α centralizes $P' = \langle c\rangle$ as well as $Z(P)/P'$. Thus α may be written as:

$$\alpha: \begin{array}{l} (1,0,0,0) \rightarrow (1, r, s, 1) \\ (0,1,0,0) \rightarrow (0,1,0,0) \\ (0,0,1,0) \rightarrow (0,t,1,0) \\ (0,0,0,1) \rightarrow (0,u,v,1) \end{array}$$

where $(f, g, h, i) = a^f c^g d^h b^i$. An inductive argument shows that for $n \geqq 4$,

$$\alpha^n: \begin{array}{l} (1,0,0,0) \rightarrow \left(1, nr + \binom{n}{2}ts + \binom{n}{3}tv + \binom{n}{2}u, ns + \binom{n}{2}v, n\right) \\ (0,1,0,0) \rightarrow (0,1,0,0) \\ (0,0,1,0) \rightarrow (0,nt,1,0) \\ (0,0,0,1) \rightarrow \left(0, nu + \binom{n}{2}tv, nv, 1\right). \end{array}$$

Therefore if $p \geqq 5$, p divides p, $\binom{p}{2}$, and $\binom{p}{3}$ and so α^p centralizes P. Since $\alpha \in \mathrm{Aut}(P)$, $\alpha^p = 1$.

Proof of Theorem A. We will adopt the notation of Glauberman [**1**]. In particular $\phi(x) = (x^\alpha)^h$ and the conditions of his Section 7 are satisfied by H and P.

By induction on $|Q|$ we may assume that $Q^* = 1$. Applying Glauberman's Theorem 8.1 [**1**] (see comment after Theorem A) we see that $P \simeq E(p) \times Z_p$ and it remains to prove that $p = 3$. To this end suppose that $p \geqq 5$.

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By (c) of Glauberman's 8.1 we may assume that α^p fixes Q but α does not. In particular we may assume that α is a p-element. By Lemma 1, $\alpha^p = 1$.

Now let M be the semidirect product of P by $\langle d, \kappa\rangle$ where κ is the automorphism of P induced by k from Lemma 7.3 [**1**]. Furthermore, in the notation of [**1**], $P = \langle x_1, x_2, x_3, x_4\rangle$, $Z(P) = \langle x_2, x_3\rangle$, $Q = \langle x_2, x_3, x_4\rangle$, $\langle x_3\rangle = Z(H)$, $P' = \langle [x_1, x_4]\rangle$, and $\phi(x) = (x^\alpha)^h$ so $x_{i+1} = (x_i^\alpha)^h$. Finally we let $T = P\langle\alpha\rangle$ so $|T : P| = p$.

By construction $Z(P)$ and P' are normal in M. We claim that $P' = Z(T)$. Since α moves Q, $T/Z(P) \simeq E(p)$ and $Z(T) \subseteq P$. Thus $Z(T) \subseteq Z(P)$. If $Z(T) = Z(P)$, then $\langle x_3\rangle \trianglelefteq H$ and is fixed by α contrary to assumption. Thus $|Z(T)| = p$ and $Z(T) = P'$.

Let $C = C_M(P/Z(P))$ and $C_1 = C_M(Z(P))$. Since $P \subseteq C$,

$$C = P(C \cap \langle\alpha, \kappa\rangle).$$

Now $\langle\alpha, \kappa\rangle/C_1 \cap \langle\alpha, \kappa\rangle$ is isomorphic to a subgroup of $GL(2, p)$ which fixes a one dimensional subspace. Thus it is isomorphic to a group of order $p(p - 1)$. Since neither α nor κ are in C, we conclude that $C \cap \langle\alpha, \kappa\rangle \subseteq C_1$. Since $P \subseteq C_1$ this means that $C \subseteq C_1$.

Now consider M/C. Since M/C is isomorphic to a subgroup of $GL(2, p)$, if $TC/C \neq T^\kappa C/C$ then M/C contains $SL(2, p)$. By construction $M/C_M(P)$ is isomorphic to a subgroup of A. Thus any element of M that moves Q must not fix any subgroup of Q that is normal in H. In particular

$$C \subseteq C_1 \subseteq N_M(\langle x_3\rangle) \subseteq N_M(Q) \subset M.$$

Since $p \geqq 5$ we conclude that $|C_1 : C| = 1$ or 2 by the structure of $SL(2, p)$. But then M/C_1 involves $SL(2, p)$ contrary to the fact that $1 \subset P' \subset Z(P)$ is fixed by M. We conclude that $TC = T^\kappa C = (TC)^\kappa$.

We now study the action of κ. By [**1**, Lemma 7.4] P contains a subgroup S such that $P/Z(P) = S/Z(P) \times Q/Z(P)$ and

$$x^\kappa \equiv x^{i^2} \quad (\text{modulo } Z(P)) \text{ if } x \in S$$
$$x^\kappa \equiv x^{i-1} \quad (\text{modulo } Z(P)) \text{ if } x \in Q,$$

where i is a primitive $(p - 1)$th root of unity in Z_p. Since α fixes S [**1**, 7.6] and $\bar{S} = S/Z(P)$ is of order p, $\bar{S} = Z(\bar{T}) = \bar{T}'$. Pick $x \in Q - Z(P)$; then $[\bar{x}, \bar{\alpha}] \in \bar{S}$. As $\bar{T}$ is not abelian $1 \neq [\bar{x}, \bar{\alpha}]$. Now $\alpha^\kappa = \alpha^m w$ for $w \in C$ and m a positive integer. Then

$$1 \neq [\bar{x}, \bar{\alpha}]^{i^2} = [\bar{x}, \bar{\alpha}]^\kappa = [\bar{x}^\kappa, \bar{\alpha}^\kappa] = [\bar{x}^{i-1}, \bar{\alpha}^m\bar{w}] = [\bar{x}^{i-1}, \bar{\alpha}^m] = [\bar{x}, \bar{\alpha}]^{i^{-1}m}.$$

Thus $\alpha^\kappa = \alpha^{i^3}w$. Now $Z(P) = \langle x_3\rangle P'$ and $P' = Z(T)$ imply that

$$1 \neq [x_3, \alpha] \in P'.$$

Hence

$$[x_3, \alpha]^\kappa = [x_3^\kappa, \alpha^\kappa] = [x_3, \alpha^{i^3}w] = [x_3, \alpha^{i^3}] = [x_3, \alpha]^{i^3}.$$

However by [**1**, Lemma 7.3], $[x_3, \alpha]^\kappa = [x_3, \alpha]^i$. Thus

$$i \equiv i^3 \quad (\text{modulo } p).$$

Since i is a primitive $(p - 1)$th root of unity this implies that $p = 3$ contrary to assumption.

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