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CATEGORY RESULTS FOR TSUJI FUNCTIONS

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1. Introduction and statement of results. Let D be the unit disk, $|z| < 1$, and $H(D)$ the Fréchet space of holomorphic functions on D, provided with the topology of uniform convergence on compact subsets of D. If f is meromorphic in D, we denote by

$$f^*(z) = \frac{|f'(z)|}{1 + |f(z)|^2},$$

the spherical derivative of f. If Γ is a rectifiable curve in D,

$$(1.1) \quad \Lambda(\Gamma) = \Lambda(\Gamma, f) = \int_\Gamma f^*(z)|dz|$$

is the length of the projection of $f(\Gamma)$ on the Riemann sphere. The Tsuji class T_1 consists of the meromorphic functions f on D for which

$$(1.2) \quad \sup_{0<r<1} \Lambda(C_r) < \infty,$$

where C_r is the circle $|z| = r$. The Tsuji class T_2 consists of the meromorphic functions f on D for which there is a sequence $\{J_n\}$ of closed rectifiable curves, depending on f, whose interiors expand to D as n tends to infinity, such that

$$(1.3) \quad \limsup_{n\to\infty} \Lambda(J_n) < \infty.$$

The classes T_1 and T_2 were introduced by Tsuji [**15**] and Hayman [**11**], respectively.

In Section 2 we prove the two principal results of this paper.

THEOREM 1. *The subset of $H(D)$ consisting of functions f for which f^* is integrable on D (with respect to two-dimensional Lebesgue measure) is of first category in $H(D)$. A fortiori, $T_1 \cap H(D)$ is a first category subset of $H(D)$.*

THEOREM 2. *For all f in a residual subset of $H(D)$, there holds*

$$(1.4) \quad \liminf_{n\to\infty} \int_{C(n)} f^*(z)|dz| = 0,$$

where $C(n)$ is the circle $|z| = 1 - 1/n$, $n = 2, 3, \ldots$. A fortiori, $T_2 \cap H(D)$ is a residual subset of $H(D)$.

Theorem 2 settles the question raised in [**10**].

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A holomorphic function f on D is called *strongly annular* if

$$\limsup_{r\to 1-} \min\{|f(z)| : |z| = r\} = \infty.$$

If

$$\limsup_{n\to\infty} \min\{|f(z)| : z \in J_n\} = \infty$$

for a sequence, $\{J_n\}$, of Jordan curves whose interiors expand to D, then f is called *annular* [**3**]. The strongly annular functions are residual in $H(D)$ [**6**]. In Section 3 we indicate how the technique of Section 2 can be used to obtain this and several other known results. In the final section we discuss the question of whether a function can be T_1 and annular.

2. Proofs of Theorems 1 and 2. Some special notation will be helpful. For $n = 2, 3, \ldots$, let $C(n)$ be the circle $|z| = r(n) = 1 - 1/n$. In the proofs of Theorems 1 and 2 we let

$$K(n, f) = \iint_{|z|\leqq r(n)} f^* \, dA; \quad L(n, f) = \Lambda(C(n), f) \tag{2.1}$$

for each $f \in H(D)$, $n = 2, 3, \ldots$.

(In the applications in Section 3, $K(n, \cdot)$ and $L(n, \cdot)$ will denote nonnegative-valued functions defined on $H(D)$ or some of its subspaces.)

It is clear that for each fixed n, $K(n, \cdot)$ and $L(n, \cdot)$ are continuous on $H(D)$. Theorem 1 asserts that only on a set of first category do we have

$$\limsup_{n\to\infty} K(n, f) < \infty, \tag{2.2}$$

while Theorem 2 claims that only on a set of first category do we have

$$\liminf_{n\to\infty} L(n, f) > 0 \tag{2.3}$$

If (2.2) holds for all f in a second category set, there is a nonempty open set $\mathcal{O}$ in $H(D)$ such that $\{K(n, f) : f \in \mathcal{O}, n = 2, 3, \ldots\}$ is bounded ([**4**, p. 19] or [**14**, p. 77].) A parallel argument holds for (2.3). Hence, we can prove Theorems 1 and 2 by establishing appropriate density lemmas.

LEMMA 1. *$\{K(n, \cdot) : n = 2, 3, \ldots\}$ is unbounded on every nonempty open subset of $H(D)$. That is, given $f_0 \in H(D)$ satisfying* (2.2), *a neighborhood $\mathcal{O}$ of f_0, and arbitrary $M > 0$, there exists $f_1 \in \mathcal{O}$ and an integer $q \geqq 2$ such that*

$$K(q, f_1) \geqq M. \tag{2.4}$$

LEMMA 2. *Given $f_0 \in H(D)$, a neighborhood $\mathcal{O}$ of f_0, and arbitrary $l > 0$, there exists $f_1 \in \mathcal{O}$ and an integer $q \geqq 2$ such that*

$$L(q, f_1) < l. \tag{2.5}$$

It is enough to prove the lemmas for a set $\mathcal{O}$ of the type

$$\mathcal{O} = \{f \in H(D) : \max_K |f - f_0| < \epsilon\},$$

where K is an arbitrary fixed compact subset of D, and ϵ is a fixed positive number.

Proof of Lemma 1. Let d be the distance on the Riemann sphere between the parallels of latitude corresponding to $|w| = 1/4$ and $|w| = 3/4$. Choose integers p, q and j so that: $K \subset \{|z| < r(p)\}$; $q > p$; and j is so large that

$$(2.6)\quad 2dj(r(q) - r(p)) > M.$$

Next select j values of θ, $0 < \theta_1 < \theta_3 < \ldots < \theta_{2j-1} < 2\pi$, so that $f_0(z) \neq 0$ for all $z \in S_1 = \{re^{i\theta} : r(p) \leqq r \leqq r(q),\ \theta = \theta_1, \theta_3, \ldots, \theta_{2j-1}\}$. Let $\theta_0 = 0$, $\theta_{2j} = 2\pi$, and $\theta_{2s} = (\theta_{2s-1} + \theta_{2s+1})/2$ for $s = 1, 2, \ldots, j-1$, and $S_2 = \{re^{i\theta} : r(p) \leqq r \leqq r(q), \theta = \theta_0, \theta_2, \ldots, \theta_{2j}\}$. By Mergelyan's theorem there is an entire function g which so closely approximates 1 on K, $1/f_0$ on S_1, and 0 on S_2 that $f_1 = gf_0$ lies in $\mathcal{O}$, $|f_1| > 3/4$ on S_1, and $|f_1| < 1/4$ on S_2. Consequently, for $r(p) < r < r(q)$, $\Lambda(C_r, f_1) > 2dj$, and

$$K(q, f_1) \geqq \int_{r(p)}^{r(q)} \int_0^{2\pi} f^*(re^{i\theta}) r d\theta dr$$
$$> \int_{r(p)}^{r(q)} 2dj dr = 2dj(r(q) - r(p)) > M$$

by (2.6). This establishes (2.4) and proves Theorem 1.

The proof of Lemma 1 can be modified so that $S_1 \cup S_2$ lies in any preassigned sector $A = \{re^{i\theta} : 0 \leqq r < 1,\ |\theta - \phi| < \delta\}$. We say $e^{i\phi}$ is a *Tsuji point* for $f \in H(D)$ if, for some $\delta > 0$,

$$\sup_{0<r<1} \int_{\phi-\delta}^{\phi+\delta} f^*(re^{i\theta}) r d\theta < \infty.$$

If $f \in H(D)$ has a Tsuji point on C, then f^* is integrable over some sector A in D. The *Tsuji set* of a function $f \in H(D)$ is the set of points $\alpha \in D$ for which $f \circ \phi_\alpha \in T_1$, where $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. If $f \in H(D)$ has a nonempty Tsuji set in D, then f^* is integrable over some sector A in D. Thus we can state the following result.

THEOREM 3. *The set of functions f in $H(D)$ for which $\int\int_A f^*(z) r dr d\theta < \infty$, where A is a sector in D, is of first category. In particular, the set of functions in $H(D)$ with no Tsuji points is residual, and the set of functions in $H(D)$ with a nonempty Tsuji set is of first category.*

Proof of Lemma 2. Let p be a positive integer such that $K \subset \{|z| < r(p)\}$ and P be a partial sum of the Maclaurin's series for f_0 such that

$$\max_{|z| \leqq r(p)} |f_0 - P| < \epsilon/2.$$

Now choose an integer $q > p$ so large that both

$$q > 10\pi l \max_{C(p)} \{1 + |P| + |P'|\}$$

holds, and

$$f_1(z) = P(z) + 4\pi q l^{-1}(z/r(p))^q$$

lies in $\mathscr{O}$.

Straightforward calculation shows that, on $C(p)$, $|f_1^*(z)| \leqq l/(2\pi)$, and (2.5) is established.

3. Applications. Many recent category results for $H(D)$ or its subspaces can be viewed as statements that certain sequences of nonnegative functions $\{K(n, \cdot)\}$ (resp. $\{L(n, \cdot)\}$) have lim sup equal to ∞ (resp. lim inf equal to 0) on a residual subset. Such a reformulation emphasizes in each case how the proof rests on the construction of an example for an appropriate density lemma. In this section we list several recent results and indicate the appropriate $K(n, f)$ or $L(n, f)$. We shall not discuss the "density lemma" example in any detail, except in Theorem E where we deduce a stronger conclusion than the original statement.

In each of the subspaces considered, the metric topology used is finer than the relative topology from $H(D)$, and there is no difficulty about the continuity of $K(n, \cdot)$ or $L(n, \cdot)$.

THEOREM A [**6**]. *The strongly annular functions form a residual subset of* $H(D)$.

THEOREM B [**12**]. *In the space* $\{f(z) = \sum_{n=0}^{\infty} \epsilon_n z^n\colon \epsilon_n = \pm 1, z \in D\}$ *with the relative topology from* $H(D)$, *there is a residual subset of strongly annular functions.*

In both Theorems A and B we may take

$$K(n, f) = \min \{|f(z)|\colon z \in C(n)\}.$$

The following three theorems deal with the normed space

$$A = \left\{f \in H(D)\colon f(z) = \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} |a_k| < \infty\right\}$$

with norm $||f|| = \sum_{k=0}^{\infty} |a_k|$.

THEOREM C [**2**]. *The set of functions in* A *which have no Tsuji point is residual.*

If $\{\theta_n\}$ is a dense sequence in $[0, 2\pi)$ and

$$\Gamma(n) = C(n) \cap \{z\colon |\arg z - \theta_n| < 1/\sqrt{n}\},$$

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we let

$$K(n, f) = \Lambda(\Gamma(n)).$$

(Compare Theorem C with Theorem 3 above.)

THEOREM D [**1**, Theorem 1]. *Let Ψ be a real continuous function on* [0, 1) *with $\Psi(r) \uparrow \infty$ as $r \uparrow 1$. There is a residual subset S of A such that, for all f in S,*

$$V(f, \theta, \Psi) = \int_0^1 |f'(re^{i\theta})|\Psi(r)dr = \infty$$

for every θ.

In this case we take

$$K(n, f) = \min_{|\theta| \leqq \pi} \int_0^{r(n)} |f'(re^{i\theta})|\Psi(r)dr.$$

THEOREM E. *Let $\{\phi(n)\}_{n=0}^{\infty}$ be a monotone sequence of positive numbers increasing to infinity arbitrarily slowly. For each $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in A, let $f(z, \phi) = \sum_{n=0}^{\infty} a_n \phi(n) z^n$. If X denotes the set of functions f in A such that $f(z, \phi)$ is strongly annular, then X is residual.*

(Since strongly annular functions do not have bounded characteristic, Theorem E contains [**1**, Theorem 4]).

Proof. For $n = 2, 3, \ldots$, and $f \in A$, we let $K(n,f) = \min \{|f(z, \phi)|: z \in C(n)\}$. It will suffice to verify that the analogue of Lemma 1 holds.

Suppose $f_0 \in A$ and $f_0(z, \phi)$ is not already strongly annular. Let $\epsilon > 0$ and arbitrarily large $M > 0$ be given, and let $\mathcal{O}_A = \{g \in A: ||g - f_0|| < \epsilon\}$.

Let P be a partial sum of the Maclaurin's series of f_0 of degree N so large that $||P - f_0|| < \epsilon/2$. Now choose an integer $p > N$ so that

$$(\epsilon/2)\phi(p) > 2(M + ||P(z, \phi)||),$$

and choose an integer q so that $(r(q))^p > 1/2$. If we set $f_1(z) = P(z) + (\epsilon/2)z^p$, then $||f_1 - f_0|| < \epsilon$, and

$$\begin{aligned} K(q, f_1) &= \min_{C(q)} |P(z, \phi) + (\epsilon/2)\phi(p)z^p| \\ &\geqq (\epsilon/2)\phi(p)(r(q))^p - ||P(z, \phi)|| \\ &> M + ||P(z, \phi)|| - ||P(z, \phi)|| = M. \end{aligned}$$

THEOREM F [**7**]. *If $0 < p < \infty$, in the space H^p, with the usual topology, there is a residual subset of functions f for which every point of $|z| = 1$ is a Picard point for f.*

For each γ, $|\gamma| = 1$, and $f \in H^p$, we let Δ_n be the complement of the set $f(\{z: z \in D, |z - \gamma| < 1/n\})$ and $L(n, f)$ be the diameter of $\Delta_n \cap \{|w| \leqq n\}$. With an analogue of Lemma 2 it is established that H^p contains a residual subset in which each function has γ as a Picard point. Since the set of Picard points is closed, the theorem follows by repeating this argument for a dense sequence on $|z| = 1$.

4. An open question. Theorems 2 and A imply that a residual subset of $H(D)$ consists of functions which are both T_2 and annular. Hayman, [**11**, Theorem 5], constructed a nonconstant function f in $T_2 \cap H(D)$ for which

$$\liminf_{r \to 1} \int_{C_r} f^*(z)|dz| = 0.$$

It is not difficult to see that such a function f must be strongly annular.

However, the following question is not yet resolved.

Question. Does there exist an annular function in the Tsuji class T_1?

Such a function, were one to exist, would have many interesting properties, of which we mention several which follow easily from the known properties of annular functions [**5**], and of the class T_1 [**11**].

1. For each γ, $|\gamma| = 1$, there exists a path $\Gamma(\gamma)$ in D ending at γ along which $f \to \infty$ as $z \to \gamma$. And for almost all θ

$$\lim_{r \to 1} f(re^{i\theta}) = \infty.$$

2. If $\Gamma(0)$ is any path in D ending at 1, for each θ let $\Gamma(\theta)$ be the rotation of $\Gamma(0)$ through angle θ. Then, for each θ in a residual subset of $[0, 2\pi)$, $f(\Gamma(\theta))$ is dense in the complex plane [**9**, p. 76].

3. For any small $\epsilon > 0$, $L(\epsilon) = \{z \in D: |f(z)| < \epsilon\}$ has infinitely many components in every neighborhood of every point of $|z| = 1$.

Let S be the subset of $[0, 2\pi)$ consisting of those θ for which the radius to $e^{i\theta}$ intersects infinitely many components of $L(\epsilon)$. S has measure 0 in $[0, 2\pi)$ but is residual in $[0, 2\pi)$.

Recall that if f is meromorphic in D, a point $e^{i\theta}$ is an *ambiguous* point of f if there exist disjoint paths Γ_1 and Γ_2 in D ending at $e^{i\theta}$ for which $f(\Gamma_1)$ and $f(\Gamma_2)$ have disjoint closures. A point $e^{i\theta}$ is a *normal* point for f if, in some neighborhood N of $e^{i\theta}$ relative to D, $f(N)$ omits three values on the Riemann sphere. Annular functions have neither ambiguous points [**5**, Theorem 4.1, p. 45], nor normal points [**5**, Theorem 4.4, p. 49].

Our last result shows that these properties characterize the annular, T_1 functions, if any exist.

THEOREM 4. *Let f be a T_1 function in $H(D)$. A necessary and sufficient condition that f be annular is that f have no ambiguous points and no normal points.*

Proof. We need show only that if $f \in T_1$ has neither ambiguous points nor normal points, then f is annular.

Since f has no normal points, every point $e^{i\theta}$ is the end of a path in D along which f tends to ∞ [**11**, Thorem 11]. Were $|f|$ bounded along any path in D ending at a point $e^{i\theta}$, f would have $e^{i\theta}$ as an ambiguous point. Consequently, for every $e^{i\theta}$ and any path $\Gamma(\theta)$ in D ending at $e^{i\theta}$, the closure of $f(\Gamma(\theta))$ contains ∞.

A theorem of McMillan, [**13**, Theorem 2], implies that for each $e^{i\theta}$ there exists a sequence, $\{J_n(\theta)\}$, of Jordan arcs in D such that:

(i) J_n has endpoints $e^{i\alpha(n)}$, $e^{i\beta(n)}$ with $\alpha(n) < \theta < \beta(n)$;

(ii) for every $\epsilon > 0$ there exists positive integer $N(\epsilon)$ such that, whenever $n > N(\epsilon)$,

$$J_n \subset \{z \in D: |z - e^{i\theta}| < \epsilon\}$$

and

$$d(f(z), \infty) < \epsilon \quad \text{for all } z \text{ on } J_n$$

where $d(\cdot, \cdot)$ is chordal distance on the Riemann sphere. Consequently, if $\Gamma = \{z(t): 0 \leqq t < 1\}$ with $\lim_{t\to 1} |z(t)| = 1$ is any boundary path in D, the closure of $f(\Gamma)$ contains ∞, and f is annular [**5**, Theorem 4.1, p. 45].

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