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Brownian Motion and Harmonic Analysis on Sierpinski Carpets

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Abstract. We consider a class of fractal subsets of $\mathbb{R}^d$ formed in a manner analogous to the construction of the Sierpinski carpet. We prove a uniform Harnack inequality for positive harmonic functions; study the heat equation, and obtain upper and lower bounds on the heat kernel which are, up to constants, the best possible; construct a locally isotropic diffusion X and determine its basic properties; and extend some classical Sobolev and Poincaré inequalities to this setting.

1 Introduction

We begin by considering a class of fractal subsets of $\mathbb{R}^d$ formed by the following generalization of the construction of the Cantor ternary set. Let $d \geq 2$ and let $F_0 = [0,1]^d$. Let $l_F \geq 3$ be an integer and divide F_0 into $(l_F)^d$ equal subcubes. Next remove a symmetric pattern of subcubes from F_0 and call what remains F_1. Now repeat the procedure: divide each subcube that is contained in F_1 into l_F^d equal parts, remove the same symmetric pattern from each as was done to obtain F_1 from F_0, and call what remains F_2. Continuing in this way we obtain a decreasing sequence of (closed) subsets of $[0,1]^d$. Let $F = \bigcap_{n=0}^{\infty} F_n$; we call F a *generalized Sierpinski carpet* (GSC) or simply, a *carpet*. The standard SC (see [Sie]) is the GSC for which $d = 2$, $l_F = 3$, and F_1 consists of F_0 minus the central square. Let m_F be the number of subcubes remaining in F_1, and let $d_f = \log m_F / \log l_F$. Then the Hausdorff dimension of F is d_f. For an example of a GSC in $\mathbb{R}^3$, see the picture of the Menger sponge in [Man, p. 145].

We will also be interested in two other related sets, which have a large-scale structure similar to the small-scale structure of F. The first, which following [O1] we call the *pre-carpet*, is the set $\tilde{F}_0 = \bigcup_{n=0}^{\infty} l_F^n F_n$. (Here and throughout this paper we write $\lambda G = \{\lambda x : x \in G\}$). Note that $\tilde{F}_0 \subset \mathbb{R}_+^d$, and that $\tilde{F}_0 \cap [0, l_F^n]^d$ consists of $[0, l_F^n]^d$ with a number of (possibly adjacent) cubical holes removed, of sides varying from 1 to l_F^{n-1}. Write $\Gamma = \mathrm{int}(\tilde{F}_0)$: then Γ is a (non-empty) domain in $\mathbb{R}^d$ with a piecewise linear boundary—see Figure 2. It is easy to check that Γ satisfies the volume doubling condition. The second related set is the unbounded GSC $\tilde{F} = \bigcup_{n=0}^{\infty} l_F^n F$.

We may regard these sets as idealized models of a region with obstacles of many different sizes. Our purpose in this paper is to study the Laplace and heat equations on the spaces F, $\tilde{F}_0$, and $\tilde{F}$. In particular we:

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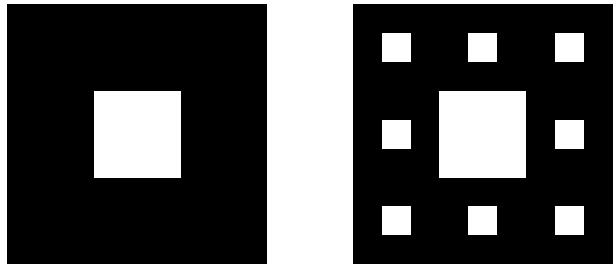

Figure 1: The first two stages of the construction of the standard Sierpinski carpet in two dimensions.

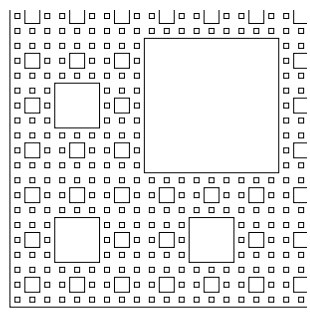

Figure 2: (Part of) the pre-carpet. The small squares have side 1.

(1) prove a uniform Harnack inequality for positive harmonic functions on $\tilde{F}_0$;
(2) study the heat equation on $\tilde{F}$ and F, and obtain upper and lower bounds on the heat kernel which are, up to constants, the best possible;
(3) construct a locally isotropic diffusion X on F and determine its basic properties;
(4) extend some classical Sobolev and Poincaré inequalities to this setting.

Just as the Euclidean dimension enters the standard heat kernel estimates and Sobolev inequalities, the Hausdorff dimension d_f of F plays a role in the analogues for F. What makes the fractal case intriguing, however, is that there is another parameter, called the spectral dimension d_s, which is much more significant. For example, the Sobolev inequality in $\mathbb{R}^d$ for $d > 2$ states that if $p = 2d/(d-2)$ and f and its gradient are in $L^2(\mathbb{R}^d)$, then

$$\|f\|_p \le c_1 \|\nabla f\|_2. \tag{1.1}$$

The corresponding inequality for a GSC (see Theorem 1.5) has the L^p norm of f on the left hand side, but now with $p = 2d_s/(d_s - 2)$; the dimension d_f does not enter into the inequality.

The Hausdorff dimension of F, d_f can be calculated easily from l_F and the Lebesgue measure of F_1. On the other hand the spectral dimension d_s appears to be a "physical" or "analytic" constant rather than a geometric one; we know of no simple formula for d_s in terms of the geometry of F, and believe that none exists. Rather, d_s is defined via the properties of harmonic functions on the sets F_n. While the exact determination of d_s seems

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to be a hard problem, it is quite easy to obtain certain bounds, and we have in particular that $1 < d_s \leq d_f < d$.

Any argument based purely on the geometry of F will inevitably lead to results involving geometric constants. Since all the key inequalities relating to F involve d_s, we cannot expect to be able to derive them by, say, starting with an isoperimetric inequality, as is possible in $\mathbb{R}^d$. Other methods suited to $\mathbb{R}^d$ also fail, and in fact, even very basic tools (like the cut-off functions used in Moser [M]) do not work.

We therefore had to develop some new techniques. Our basic approach is probabilistic: we construct a diffusion X on $\tilde{F}$, which, because it is locally isotropic, we call a "Brownian motion" on $\tilde{F}$. We can then use properties of X to derive bounds on its transition density (which solves the heat equation on F). Given these bounds, we can then derive Sobolev and Poincaré inequalities on $\tilde{F}$ and $\tilde{F}_0$.

The starting point of our analysis, and the hardest result in this paper, is a uniform (elliptic) Harnack inequality. Let B be an open set in $\mathbb{R}^d$. We say that h is *harmonic on* $B \cap F_N$ if $\Delta h(x) = 0$ for $x \in B \cap \mathrm{int}(F_N)$, and the normal derivative of h is 0 on $B \cap \partial F_N$ almost everywhere with respect to surface measure on ∂F_N. Write $B(x, r)$ for the usual open ball in $\mathbb{R}^d$ with centre x and radius r.

Theorem 1.1 *There exists c_1 not depending on N, such that if $x \in F_N$, $r > 0$, and h is positive and harmonic on $B(x, 2r) \cap F_N$, then writing $A = B(x, r) \cap F_N$,*

$$\sup_A h(x) \leq c_1 \inf_A h(y). \tag{1.2}$$

A similar result holds for the pre-carpet $\tilde{F}_0$.

Of course, since F_N is a Lipschitz domain, for each N the standard Harnack inequality guarantees there exists $c_1(N)$ such that (1.2) holds. The point of this theorem is that c_1 can be taken to be independent of N.

In an earlier paper [BB1] we proved a uniform Harnack inequality in the case $d = 2$. The proof used a "path-crossing" argument which cannot be generalized to the case $d \geq 3$. Kusuoka and Zhou [KZ] extended this result to fractals satisfying $d_s < 2$, but their method is also tied to the low-dimensional case. Standard approaches to Harnack inequalities in higher dimensions, such as Moser's iterative technique [M] or the Nash-Fabes-Stroock method [FS], do not appear to work for GSCs, and we were therefore forced to use a different approach. Our proof of (1.2) uses the probabilistic technique of coupling. (See [Lv] for a general introduction to this method). More precisely, given distinct points $x, y \in F_N$, we construct two reflecting Brownian motions on F_N, starting at x, y, such that the two processes meet with a positive probability p which is independent of N. Using this, we then establish a Harnack inequality with constants independent of N. We believe that our use of coupling to prove Harnack inequalities is new; it may also be applicable to a variety of other situations.

Given the Harnack inequality, we modify methods developed by us in earlier work on two-dimensional Sierpinski carpets (see [BB1]–[BB4]). We can construct the process X as the limit of (suitably accelerated) reflecting Brownian motions on the sets F_N. Let μ be (a multiple of) the Hausdorff x^{d_f}-measure on $\tilde{F}$.

Theorem 1.2 *There exists a nondegenerate continuous strong Markov process X_t whose state space is $\tilde{F}$. X_t has transition densities which have the strong Feller property and which are*

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μ-symmetric. The law of the process $(X_t, t \geq 0)$ is locally invariant under local isometries of $\tilde{F}$.

Write P_t for the semigroup associated with X, and let $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ be the infinitesimal generator of P_t; we call $\mathcal{L}$ the Laplacian on $\tilde{F}$. The heat equation on $\tilde{F}$ then becomes

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{L}u(x,t), \qquad x \in \tilde{F}, \quad t > 0. \tag{1.3}$$

The fundamental solutions to the heat equation are given by the transition densities $p(t,x,y)$ for the process X_t on $\tilde{F}$. The spectral dimension d_s is defined from the sequence R_n of electrical resistances across the sets $l_F^n F_n$. Let $d_w = 2d_f/d_s$; note that as $d_s \leq d_f$ we have $d_w \geq 2$.

Theorem 1.3 *$p(t,x,y)$ is symmetric and jointly continuous on $(0,\infty) \times \tilde{F} \times \tilde{F}$, and for each x, y the function $p(t,x,y)$ is C^∞ in t. There exist c_1, c_2, c_3, c_4 such that for all $x, y \in \tilde{F}$ and $t > 0$,*

$$\begin{aligned} &c_1 t^{-d_s/2} \exp\left(-c_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ &\qquad \leq p(t,x,y) \leq c_3 t^{-d_s/2} \exp\left(-c_4 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right). \end{aligned} \tag{1.4}$$

Let W_t be Brownian motion on the pre-carpet $\tilde{F}_0$, with normal reflection on $\partial\tilde{F}_0$, and let $q(t,x,y)$ be its transition density with respect to Lebesgue measure on $\tilde{F}_0$. These transition densities are the fundamental solutions to the heat equation $\partial u/\partial t = \frac{1}{2}\Delta u$ on $\tilde{F}_0$ with Neumann boundary conditions.

Since $\tilde{F}_0$ is locally similar to $\mathbb{R}^d$, but has a "fractal" global structure, we would expect $q(t,x,y)$ to have different behavior for small and large t. We would also expect, in view of standard large-deviation theory for Brownian motion, that, if $|x-y|$ is large in comparison with t then $q(t,x,y)$ will exhibit Gaussian behavior. (Very roughly, if $|x-y|$ is much larger than t, then for the process W to move from x to y in time t, it will with high probability stay close to the shortest path connecting x and y, and it will have no time to feel the fractal structure of $\tilde{F}_0$).

Theorem 1.4 *There exist $c_1, \ldots, c_8 \in (0,\infty)$ such that if $x, y \in \tilde{F}_0$ and*

(a) $t \geq \max(1, |x-y|)$, then

$$\begin{aligned} &c_1 t^{-d_s/2} \exp\left(-c_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ &\qquad \leq q(t,x,y) \leq c_3 t^{-d_s/2} \exp\left(-c_4 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right); \end{aligned} \tag{1.5}$$

(b) if $t \leq 1$, then

$$c_5 t^{-d/2} \exp(-c_6|x-y|^2/t) \leq q(t,x,y) \leq c_7 t^{-d/2} \exp(-c_8|x-y|^2/t). \tag{1.6}$$

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(c) if $t \geq 1$, $|x-y| > t$, then

$$c_5 t^{-d_s/2} \exp(-c_6|x-y|^2/t) \leq q(t,x,y) \leq c_7 t^{-d_s/2} \exp(-c_8|x-y|^2/t). \tag{1.7}$$

As remarked above, the set $\tilde{F}_0$ satisfies the usual volume doubling condition and an elliptic Harnack inequality. However, since $q(t, x_0, .)$ is parabolic on the whole space $(0,\infty)\times\tilde{F}_0$ it follows easily from the bounds in Theorem 1.4 that the (usual) parabolic Harnack inequality on $\tilde{F}_0$ fails for any GSC for which $d_w > 2$. (See Proposition 7.11 for details, and Remark 5.4 for examples of GSCs for which it is known that $d_w > 2$). This answers a question raised in Grigor'yan [Gr]. Essentially the point is that whereas an elliptic Harnack inequality such as Theorem 1.1 contains no information on the space-time scaling of the process W, this scaling information appears explicitly in the parabolic Harnack inequality.

To obtain the parabolic estimates above from Theorem 1.1 some additional information on the process W is needed. This is provided by exploiting the close connection between resistances and crossing times. For graphs this was proved in [CRRST]—see also [Tet], and for Sierpinski carpets see [BB2], [BB4]. Using this, the resistance bounds in [BB4] and [McG], and the Harnack inequality, we can obtain good bounds on the time taken by W to escape from a region of the form $B(x,r)\cap\tilde{F}_0$. These bounds then enable us to derive estimates for expressions like $\int_0^\infty e^{-\lambda t} t^p q(t,x,y)\,dt$ for suitable p, and using these we prove the upper bounds in Theorem 1.4. This method is also new and could easily be modified to give new proofs of the upper bounds of Aronson [A] for the heat kernels of uniformly elliptic operators in divergence form on $\mathbb{R}^d$.

For the key estimate for the lower bound for $q(t,x,y)$ we again use coupling; this may also have applications outside the fractal context. The bounds in Theorem 1.3 follow easily from those in Theorem 1.4

The relationship between bounds on the behavior of the norm of P_t and Sobolev and other analytic inequalities has been explored very extensively in recent years, following the basic papers [V1] and [CKS]. Write $\|f\|_p^p = \int_{\tilde{F}} |f|^p\,d\mu$. As X is μ-symmetric, we can associate a Dirichlet form $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ with P_t (see [FOT]). Since Theorem 1.3 implies easily that

$$\|P_t f\|_\infty \leq c_1 t^{-d_s/2}\|f\|_1,$$

we can apply the theory mentioned above to immediately deduce a Sobolev inequality on $\tilde{F}$.

Theorem 1.5 *Suppose $d_s > 2$. There exists c_1 such that for all $f \in \mathcal{D}(\mathcal{E})$*

$$\|f\|_p \leq c_1 \mathcal{E}(f,f)^{\frac{1}{2}}, \quad p = 2d_s/(d_s-2).$$

We also obtain a Sobolev inequality on $\tilde{F}_0$, and Poincaré and mass-capacity inequalities for both $\tilde{F}$ and $\tilde{F}_0$; see Section 7.

Let G be the graph whose vertices are the centers of those unit squares that lie in $\tilde{F}_0$. Two vertices x and y will be connected by an edge if $|x-y| = 1$. G is called the graphical SC. In [BB6] the results of this paper are used to obtain Poincaré, Sobolev, and mass-capacity inequalities for G and to obtain transition probability estimates for the symmetric random walk on G.

Initial interest in the study of random walks or diffusions on fractals came from mathematical physicists working in the theory of disordered media—see [RT], [AO], and for a

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survey from a physical viewpoint [HBA]. The initial mathematical work was on the simplest non-trivial regular connected fractal, the Sierpinski gasket, in [Kus1], [Go], [BP]. In particular, [BP] obtained bounds similar to those in Theorem 1.3 for the transition densities of the Brownian motion on the Sierpinski gasket. The Sierpinski gasket G is finitely ramified, that is, it can be disconnected by removing a finite number of points. This means that many subsets of G have a finite boundary, and so a Harnack inequality can be proved in an elementary way. Subsequently many other finitely ramified fractals have been treated in a similar fashion—see for example [L], [Kig1], [Kig2], [Kum1], and [Fuk]. Note that while some of these papers use probability theory, others employ a purely analytic approach, and construct the Dirichlet form directly. However, no purely analytic derivation of the bounds on the heat kernel is known in the fractal context. While it possible for a finitely ramified fractal to have $d_s > 2$ (see [Kum2] or [Ny]) these spaces are in some sense quite inhomogeneous. Indeed, Proposition 4.42 of [Bar1] implies that if bounds of the form (1.4) hold on a finitely ramified fractal then $d_s < 2$. So if one wishes to study regular higher dimensional fractal spaces which are not simple products one is led to consider infinitely ramified fractals.

Generalized Sierpinski carpets provide a reasonably simple but general family of infinitely ramified fractals. See [GAM], [BAH] for early work by mathematical physicists, and [HHW1], [HHW2] for an approximate approach to the calculation of the spectral dimension. GSCs in two dimensions have been studied in [BB1], [BB2], [BBS], [BB3], [BB4], but as $d_s \le d_f < d$ these sets also have $d_s < 2$. There are only a few previous papers on GSCs with $d \ge 3$. The fractals studied by Kusuoka and Zhou in [KZ] include GSCs, and that paper contains a Harnack inequality similar to Theorem 1.1 above for GSCs with $d_s < 2$. (They also have some results for more general GSCs). (See Section 9 of this paper for an example of a GSC with $d = 3$ but $d_s < 2$.) In addition, Osada [O1] has proved an isoperimetric inequality, and used it to deduce that

$$q(t,x,y) \le c_1 t^{-d_i/2}, \quad x, y \in \tilde{F}_0, \quad t \ge 1, \tag{1.8}$$

where d_i is the "isoperimetric dimension" of $\tilde{F}_0$. Comparing (1.8) with (1.5) we see that $d_s \ge d_i$, and we expect that in general the inequality is strict.

For surveys of work on finitely ramified fractals see [Kus2] or [Bar1], and for Sierpinski carpets see [Bas3].

The layout of this paper is as follows. Section 2 introduces the notation we will use together with a few basic facts. Section 3 contains the coupling argument: as this is quite long and hard we give here a very brief summary of the essential ideas. See also [BB5], where these results were announced.

Let $N \ge 0$. Let $\mathcal{S}_n^*$ be the set of cubes of side length $2l_F^{-n}$ with vertices in $2l_F^{-n}\mathbb{Z}^d$. Let us say $x \overset{m}{\sim} y$ if $x \in S_1 \in \mathcal{S}_m^*$, $y \in S_2 \in \mathcal{S}_m^*$, and there is an isometry from S_1 to S_2 that takes x to y. Given $x \overset{m}{\sim} y$, we construct two reflecting Brownian motions W_t^x and W_t^y on F_N starting from x and y, respectively, such that $W_t^x \overset{m}{\sim} W_t^y$ for all t, and such that with some positive probability $c_1 > 0$ we have $W_{\sigma_1}^x \overset{m-1}{\sim} W_{\sigma_1}^y$, where σ_1 is the first time either W^x or W^y moves more than a few cubes in $\mathcal{S}_m$ away from their starting points. This construction uses the symmetry of F_1 very heavily. If (n_m) is a sequence of integers and σ_n is the n-th time W^x or W^y has moved more than a few cubes in $\mathcal{S}_m$, then a renewal-type argument tells us that

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$W^x(\sigma_{n_m}) \overset{m-1}{\sim} W^y(\sigma_{n_m})$ with probability at least $1-(1-c_1)^{n_m}$. Repeating this argument for $m-1, m-2, \ldots$, we see that there is positive probability that $W^x \overset{0}{\sim} W^y$ at some time before either W^x or W^y hits ∂F_0.

In Section 4 we derive the uniform Harnack inequality from the coupling result. Section 5 contains the construction of the Brownian motion. In Section 6 we establish the bounds on the heat kernel for both $\tilde{F}$ and $\tilde{F}_0$, and in Section 7 we consider Sobolev and Poincaré inequalities. In Section 8 we establish a number of basic properties of the process X_t. These include transience and recurrence, moduli of continuity, the Hausdorff dimension of the range, the existence of local times, self-intersections, rates of escape, and a zero-one law. The paper is concluded in Section 9 by some examples and open problems.

2 Notation and Preliminaries

We begin by setting up our notation. We use the letter c with subscripts to denote constants which depend only on the dimension d and the carpet F. We renumber the constants for each lemma, proposition, theorem, and corollary.

Let $d \geq 2$, $F_0 = [0,1]^d$, and let $l_F \in \mathbb{N}$, $l_F \geq 3$ be fixed. For $n \in \mathbb{Z}$ let $\mathcal{S}_n$ be the collection of closed cubes of side l_F^{-n} with vertices in $l_F^{-n}\mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set

$$\mathcal{S}_n(A) = \{S : S \subset A, S \in \mathcal{S}_n\}.$$

For $S \in \mathcal{S}_n$, let Ψ_S be the orientation preserving affine map which maps F_0 onto S.

We now define a decreasing sequence (F_n) of closed subsets of F_0. Let $1 \leq m_F < l_F^d$ be an integer, and let F_1 be the union of m_F distinct elements of $\mathcal{S}_1(F_0)$. We impose the following conditions on F_1:

Hypotheses 2.1

(H1) (Symmetry) F_1 is preserved by all the isometries of the unit cube F_0.

(H2) (Connectedness) $\mathrm{Int}(F_1)$ is connected, and contains a path connecting the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$.

(H3) (Non-diagonality) Let B be a cube in F_0 which is the union of 2^d distinct elements of $\mathcal{S}_1$. (So B has side length $2l_F^{-1}$). Then if $\mathrm{int}(F_1 \cap B)$ is non-empty, it is connected.

(H4) (Borders included) F_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_d = 0\}$.

We may think of F_1 as being derived from F_0 by removing the interiors of $l_F^d - m_F$ squares in $\mathcal{S}_1(F_0)$. Given F_1, F_2 is obtained by removing the same pattern from each of the squares in $\mathcal{S}_1(F_1)$. Iterating, we obtain a sequence (F_n), where F_n is the union of m_F^n squares in $\mathcal{S}_n(F_0)$. Formally, we define

$$F_{n+1} = \bigcup_{S \in \mathcal{S}_n(F_n)} \Psi_S(F_1) = \bigcup_{S \in \mathcal{S}_1(F_1)} \Psi_S(F_n), \quad n \geq 1.$$

We call the set $F = \bigcap_{n=0}^{\infty} F_n$ a *generalized Sierpinski carpet (GSC)*. Let $\dim(\cdot)$ denote Hausdorff dimension; by [Hu] $\dim(F) = \log m_F / \log l_F$.

Remark 2.2 These conditions are natural higher-dimensional analogues of the ones given in [BB4, (2.1)]. Since we are interested in constructing continuous processes on F, the hypothesis (H2) is essential. It would be interesting to be able to consider GSCs for which

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the symmetry condition (H1) was either weakened or dispensed with entirely. However, (H1) plays a vital role in this paper, namely, in the key coupling argument in Section 3. Indeed, we do not expect the Harnack inequality Theorem 4.3 to remain true without strong symmetry assumptions on F_1.

The other two hypotheses, (H3) and (H4), are not so essential. We expect that results similar to those in this paper still hold for GSCs which do not satisfy (H3). However, the natural state space of the limiting process X may no longer be F, and the added generality would significantly increase the complexity of the arguments. We include (H4) for simplicity—it ensures that the shortest path metric and the Euclidean metric on F are comparable. See [BB4, Sect. 8] for some remarks on GSCs which do not satisfy (H4), and [FHK], [Kum1] for constructions of such a shortest path metric in the case of nested fractals.

We will be interested in unbounded analogues of F. Set $F_k = F_0$ for $k < 0$ and for $n \in \mathbb{Z}$ let

$$\tilde{F}_n = \bigcup_{r=0}^{\infty} l_F^r F_{n+r}, \tag{2.1}$$

and $\tilde{F} = \bigcap_{n=0}^{\infty} \tilde{F}_n$. In particular we call $\tilde{F}_0$ the *pre-carpet* (see [O1]). Let

$$\mu_n(dx) = m_F^n 1_{\tilde{F}_n}(x)dx,$$

and let μ be the weak limit of the μ_n: μ is a constant multiple of the Hausdorff $x^{\log m_F/\log l_F}$-measure on $\tilde{F}$.

We need notation for a block of cubes that have a point x near the center. For $x = (x_1, \dots, x_d)$, let $\phi(x_i)$ be the integer j such that $l_F^{-r}\big(j - (\frac{1}{2})\big) \le x_i < l_F^{-r}\big(j + (\frac{1}{2})\big)$ and let

$$D_r(x) = \left[\frac{\phi(x_1) - 1}{l_F^r}, \frac{\phi(x_1) + 1}{l_F^r}\right] \times \cdots \times \left[\frac{\phi(x_d) - 1}{l_F^r}, \frac{\phi(x_d) + 1}{l_F^r}\right]. \tag{2.2}$$

Observe that $D_r(x)$ decreases as r increases. Note also that $D_r(x)$ is a cube of side length $2l_F^{-r}$.

For distance on the sets $\tilde{F}_n$ we will frequently find it convenient to use the $\|.\|_\infty$ norm, since in this norm the unit ball is a cube with sides parallel to the axes. We denote by $B_\infty(y, \varepsilon)$ the set $\{x \in \mathbb{R}^d : \|x - y\|_\infty < \varepsilon\}$, and use $B(x, \varepsilon)$ to denote the usual open balls in $\mathbb{R}^d$. Note the following:

Lemma 2.3

(a) If $y \in D_n(x)$ then $\|x - y\|_\infty \le (3/2)l_F^{-n}$.
(b) If $\|x - y\|_\infty \le (1/2)l_F^{-n}$ then $y \in D_n(x)$.
(c) If $x \in \tilde{F}$ and $n \in \mathbb{Z}$ then $m_F^{-n} \le \mu\big(D_n(x)\big) \le 2^d m_F^{-n}$.
(d) If $x \in \tilde{F}_0$ and $n \le 0$ then $m_F^{-n} \le \mu_0\big(D_n(x)\big) \le 2^d m_F^{-n}$.
(e) There exist constants c_1, c_2 such that for $x \in \tilde{F}_0$,

$$c_1 r^d \le \mu_0\big(B(x, r)\big) \le c_2 r^d, \quad 0 < r < 1,$$
$$c_1 r^{d_f} \le \mu_0\big(B(x, r)\big) \le c_2 r^{d_f}, \quad r > 1.$$

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(f) There exist constants c_1, c_2 such that for $x \in \tilde{F}$,

$$c_1 r^{d_f} \le \mu\big(B(x,r)\big) \le c_2 r^{d_f}, \quad r > 1.$$

In particular it follows immediately from (e) that $(\tilde{F}_0, |\cdot - \cdot|, \mu_0)$ satisfies the volume doubling condition (see [Gr]):

$$\mu_0\big(B(x,r)\big) \le c\mu_0\big(B(x,2r)\big), \quad \textit{for } x \in \tilde{F}_0, r > 0.$$

A similar volume doubling condition also holds for $\tilde{F}$.

We write

$$H_i(t) = \{x \in \mathbb{R}^d : x_i = t\}, \quad t \in \mathbb{R}. \tag{2.3}$$

We write $\mathcal{B}(G)$ for the Borel subsets of the set G. For a Borel set A and process X on $\mathbb{R}^d$ we write

$$\mathcal{T}_A = \mathcal{T}(A) = \mathcal{T}_A(X) = \mathcal{T}_A^X = \inf\{t > 0 : X_t \in A\} \tag{2.4}$$

for the hitting time of A and

$$\tau_A = \tau(A) = \tau_A(X) = \tau_A^X = \mathcal{T}(A^c) \tag{2.5}$$

for the exit time of A. We also let

$$\sigma_r(x) = \sigma_r^X(x) = \sigma_r(x, X) = \inf\{t > 0 : X_t \notin D_r(x)\}. \tag{2.6}$$

We define the coupling time for two processes:

Definition 2.4 Let X_t^k, $k = 1, 2$, be processes on $\mathbb{R}^d$. The *coupling time* of X^1, X^2 is defined by

$$T_C = T_C(X^1, X^2) = \inf\{t \ge 0 : X_t^1 = X_t^2\}. \tag{2.7}$$

We say the X^k are *coupled* if $X_t^1 = X_t^2$ for $t \ge T_C$.

Let D be an (open) Lipschitz domain in $\mathbb{R}^d$. We call a process X_t a *reflecting Brownian motion on D*, or RBM(D) for short, if X is a $\overline{D}$-valued diffusion which is locally a Brownian motion on D, with normal reflection on ∂D. If D is a closed set in $\mathbb{R}^d$, $D = \overline{\mathrm{int}(D)}$, and $\mathrm{int}(D)$ satisfies the conditions above, then we say X is a RBM(D) if X is a RBM$\big(\mathrm{int}(D)\big)$. The existence (and uniqueness in law) of such processes is proved in [BH, Sect. 4], and [Bas1].

In the remainder of this section we will give some basic properties of RBM(D), where D is a Lipschitz domain. We write W for RBM($\tilde{F}_0$).

In Section 3 we will make frequent use of the following fact.

Lemma 2.5 *Let D be a Lipschitz domain in $\mathbb{R}^d$, and let X be a* RBM(D). *If $A \subset \mathbb{R}^d$ and A is polar for standard d-dimensional Brownian motion, then $A \cap D$ is polar for X.*

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Note in particular that subspaces of $\mathbb{R}^d$ of codimension greater than or equal to 2 are polar for X.

Proof Suppose $d \geq 3$ and A is polar for d-dimensional Brownian motion. If $x \in \overline{D}$, then there exists $r_x > 0$ such that $D \cap B(x, r_x) = D' \cap B(x, r_x)$, where D' is the region above the graph of a Lipschitz function in some coordinate system. Since $\overline{D}$ can be covered by a countable union of balls of this kind and a countable union of polar sets is polar, it is sufficient to consider the case when D is the region above the graph of a Lipschitz function and A is bounded.

Let v be the Green function for X on D, and w be the Green function for Brownian motion on $\mathbb{R}^d$. By [BH], Corollaries 3.3 and 3.5, there exist c_1 and c_2 such that

$$c_1|y-z|^{2-d} \leq v(y,z) \leq c_2|y-z|^{2-d}, \qquad y,z \in D. \tag{2.8}$$

Suppose ν is a non-zero measure supported on A with $\int v(y,z)\,\nu(dz) \leq 1$ for all y. Then $\int w(y,z)\,\nu(dz)$ is bounded, which implies that A is not polar for d-dimensional Brownian motion. So no such measure ν exists and A is polar for X.

If $d = 2$, we consider $\tilde{X}_t = (X_t, Z_t)$ in $D \times \mathbb{R}$, where Z_t is an independent one-dimensional Brownian motion and use the above argument to show that $A \times \mathbb{R}$ is polar for $\tilde{X}$ since it is polar for 3-dimensional Brownian motion. ∎

We need to extend some results that were proved in [BH] for RBM in regions above the graph of a Lipschitz function to RBM in $\tilde{F}_0$. We begin with a support theorem for RBM($\tilde{F}_0$).

Proposition 2.6 *Let $\varepsilon > 0$ and let $\psi\colon [0,1] \to \tilde{F}_0$ be a differentiable curve. There exists $c_1 > 0$ depending only on ε and $\|\psi'\|_\infty$ such that*

$$\mathbb{P}^{\psi(0)}\big(\sup_{s\leq 1}|W_s - \psi(s)| < \varepsilon\big) \geq c_1.$$

Proof Let $x = \psi(0)$. By Proposition 3.6 of [BH] and scaling, there exist δ_1 and δ_2 depending on ε and $\|\psi'\|_\infty$ such that with probability at least c_2 we have $\sup_{s\leq\delta_1}|W_s - \psi(s)| < \varepsilon/3$ and $\operatorname{dist}(W_{\delta_1}, \partial D) > \delta_2$. By the Markov property, it thus suffices to show

$$\mathbb{P}^y\big(\sup_{\delta_1\leq s\leq 1}|W_s - \psi(s)| < \varepsilon\big) \geq c_3 \tag{2.9}$$

when $|y - \psi(\delta_1)| < \varepsilon/3$ and $\operatorname{dist}(y, \partial D) > \delta_2/2$. However (2.9) follows by applying the support theorem for standard d-dimensional Brownian motion [Bas2, p. 59] with ε replaced by $\varepsilon/3$ and ψ replaced by a curve $\overline{\psi}$ starting at y that is always within $\varepsilon/3$ of ψ but such that $\overline{\psi}$ never gets closer than a distance $(\varepsilon \wedge \delta_2)/2$ to ∂D. ∎

Fix $x_0 \in \tilde{F}_0$. Let $m \in \mathbb{Z}$. As $\tilde{F}_0 \cap D_m(x_0)$ is a bounded Lipschitz domain, by Lemma 4.3 of [BH] and its proof, a RBM$\big(\tilde{F}_0 \cap D_m(x_0)\big)$ W has a continuous transition density $\overline{q}_{D_m(x_0)}(t,x,y) = \overline{q}(t,x,y)$ with respect to Lebesgue measure on $\tilde{F}_0 \cap D_m(x_0)$. By Theorem 2.3 and Remark 3.11 of [BH],

$$\overline{q}(t,x,y) \leq c_1(m)t^{-d/2}\exp\big(-c_2(x-y)^2/t\big), \quad x,y \in \tilde{F}_0 \cap D_m(x_0), \quad t \leq 1. \tag{2.10}$$

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Since $\overline{q}$ is the transition density of a symmetric process, $\overline{q}(t,x,y) = \overline{q}(t,y,x)$. By [Bas2, Sect. 4], there is an eigenvalue expansion for $\overline{q}$:

$$\overline{q}(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\overline{\varphi}_{i,m}(x)\overline{\varphi}_{i,m}(y), \tag{2.11}$$

where the convergence is absolute and uniform. This and (2.10) imply

$$\overline{q}(t,x,y) \le c_3 t^{-d/2}\exp(-c_4 t), \quad x,y \in \tilde{F}_0 \cap D_m(x_0), \quad t > 0, \tag{2.12}$$

where c_3 and c_4 depend on m. Cauchy-Schwarz and (2.11) imply

$$\overline{q}(t,x,y) \le \overline{q}(t,x,x)^{\frac{1}{2}}\overline{q}(t,y,y)^{\frac{1}{2}} \tag{2.13}$$

and also that $\overline{q}(t,x,x)$ is nonincreasing in t for each $x \in \tilde{F}_0$.

3 Coupling of Brownian Motions

The coupling argument given in this section is the foundation of the results in this paper. The argument is quite long, and requires several steps. A great deal of notation, especially for various special subsets of $\mathbb{R}^d$, will be required; however, none of the notation introduced in this section will be used elsewhere. The reader may find it helpful to focus on the case $d = 3$.

Lemma 3.1 **(A Reflection Principle)** *Let D^0 be a Lipschitz domain in $\mathbb{R}^d$, let $D = \overline{D^0}$ and let W be a* RBM(D). *Let H be a hyperplane, $g\colon \mathbb{R}^d \to \mathbb{R}^d$ be reflection across H, and J_1, J_2 be the two half spaces determined by H. Let $A \subset \partial D$, and B_1, B_2 be subsets of A.*

Suppose that

$$\begin{aligned} &\text{(i)} && g\colon D \to D \\ &\text{(ii)} && g(B_1) = B_2 \\ &\text{(iii)} && B_i \subseteq \overline{J_i}, \quad i = 1,2 \\ &\text{(iv)} && g(A \cap J_1) \subset A. \end{aligned} \tag{3.1}$$

Then

$$\mathbb{P}^x(W_{\mathcal{T}_A} \in B_1) \ge \mathbb{P}^x(W_{\mathcal{T}_A} \in B_2) \quad \textit{for } x \in J_1 \cap D. \tag{3.2}$$

Proof Let $C_2 = A \cap J_2 - B_2$, and $C_1 = g(C_2)$. Note that C_1 is not necessarily contained in A. Write $u_i(x) = \mathbb{P}^x(W_{\mathcal{T}_A} \in B_i)$, and $v = u_2 - u_1$. Since W_t killed on hitting A has continuous paths, the maximum principle holds for v. Let $\beta = \sup_{x \in J_1 \cap D} v(x)$. Suppose $\beta > 0$. As $v(x) \le 0$ for $x \in A \cap J_1$, by the maximum principle we have

$$\sup_{x \in J_1 \cap D} v(x) = \sup_{y \in H \cap D} v(y).$$

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Set $S = \mathcal{T}_{B_1} \wedge \mathcal{T}_{B_2} \wedge \mathcal{T}_{C_1} \wedge \mathcal{T}_{C_2}$. Then for $y \in H \cap D$,

$$u_i(y) = \mathbb{P}^y(W_S \in B_i) + \mathbb{E}^y 1_{(W_S \in C_1)} u_i(W_S), \quad i = 1, 2.$$

Since by symmetry $\mathbb{P}^y(W_S \in B_1) = \mathbb{P}^y(W_S \in B_2)$, it follows that

$$v(y) = \mathbb{E}^y 1_{(W_S \in C_1)} v(W_S).$$

However, again by symmetry, $\mathbb{P}^y(W_S \in C_1) = \mathbb{P}^y(W_S \in C_2)$, and so $\mathbb{P}^y(W_S \in C_1) \leq 1/2$. Hence for $y \in H \cap D$, $v(y) \leq \mathbb{P}^y(W_S \in C_1)\beta \leq \beta/2$. Thus $\beta \leq \beta/2$, a contradiction to our assumption that $\beta > 0$. ■

The following definition of a sequence of stopping times will be used several times.

Definition 3.2 Let $\mathcal{H} = \{H_\alpha, \alpha \in I\}$ be a family of closed non-empty subsets of $\mathbb{R}^d$ with the property that $\{\alpha : H_\alpha \cap B(x, 1) \neq \varnothing\}$ is finite for all x. (Note this implies that I is countable). Write $H = \bigcup_{\alpha \in I} H_\alpha$. Let $(X_t, t \geq 0)$ be a continuous process on $\mathbb{R}^d$ such that $H_\alpha \cap H_\beta$ is polar for X, for all $\alpha, \beta \in I$. Then *successive disjoint hits by X on $\mathcal{H}$* are the sequence of stopping times $(T_n, n \geq 0)$ defined by:

$$\begin{aligned} T_0 &= \inf\{t \geq 0 : X_t \in H\}, \\ \Gamma_n &= \{\alpha \in I : X_{T_n} \in H_\alpha\}, \\ T_{n+1} &= \inf\Big\{t \geq T_n : X_t \in H - \bigcup_{\alpha \in \Gamma_n} H_\alpha\Big\}. \end{aligned} \tag{3.3}$$

Lemma 3.3 *Let $\mathcal{H}$, X, (T_n) be as above. Then $\lim_{n\to\infty} T_n = +\infty$, a.s.*

Proof Suppose $T_n < \infty$. Since $B\big(X_{T_n}(\omega), 1\big)$ intersects at most finitely many H_α, Γ_n is a finite set. (In fact, if $T_n > 0$ then as $H_\alpha \cap H_\beta$ is polar, Γ_n contains only one element). Therefore $X_{T_n}(\omega)$ is a positive distance from $H - \bigcup_{\alpha \in \Gamma_n} H_\alpha$. Hence $T_{n+1} > T_n$. Now suppose $\lim_n T_n(\omega) = S(\omega) < \infty$. Then as $\{\alpha : H_\alpha \cap B\big(X_S(\omega), 1\big) \neq \varnothing\}$ is finite, there must exist $\alpha, \beta \in I$ such that $X_{T_n}(\omega) \in H_\gamma$ for infinitely many n, for $\gamma = \alpha, \beta$. So $X_S(\omega) \in H_\alpha \cap H_\beta$, and hence, by the polarity assumption, we deduce $\mathbb{P}(S < \infty) = 0$. ■

Definition 3.4 Let $D = F_{n_0}$, for some $n_0 \geq 1$, and let W be RBM(D). We define

$$H_{ij} = \{x : x_i + x_j = 1\}$$

$$L_i = H_i(0) \cap [0, 1/2]^d,$$

$$M_{ij} = \{x \in [0, 1]^d : x_i = 0, 1/2 \leq x_j \leq 1, 0 \leq x_k \leq 1/2, k \neq j\}, \quad \text{for } i \neq j,$$

$$\tau = \inf\Big\{t \geq 0 : W_t \in \bigcup_{i=1}^d H_i(1)\Big\}.$$

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The hyperplane $H_i(t)$ was defined in (2.3). For any hyperplane $H \subset \mathbb{R}^d$, let $g_H \colon \mathbb{R}^d \to \mathbb{R}^d$ be reflection in H. If $x = (x_1, \dots, x_d)$ note that

$$g_{H_i(\frac{1}{2})}(x) = (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_d),$$
$$g_{H_{12}}(x) = (1 - x_2, 1 - x_1, x_3, \dots, x_d).$$

We now use the symmetry of D, and the invariance of W under certain isometries of D, to deduce lower bounds for certain hitting probabilities of W. (These are higher-dimensional analogues of the "corner" and "knight's" moves in [BB1]). In what follows we use q_i to denote strictly positive reals which depend only on the dimension d.

Proposition 3.5 *Let $i, j \in \{1, \dots, d\}$. Then*

$$\mathbb{P}^x(\mathcal{T}_{L_j} \le \tau) \ge q_1 > 0, \quad \textit{for } x \in L_i. \tag{3.4}$$

Proof Set $T = \mathcal{T}\big(H_j(0) \cup \bigcup_{k=1}^d H_k(1)\big)$. We will actually prove that $\mathbb{P}^x(W_T \in L_j) \ge q_1$ for $x \in L_i$. By the symmetry of D we can take $j = 1$, $i = 2$. (If $i = j$ the result is trivial). Fix $x \in L_2$.

Now apply Lemma 3.1 with $H = H_1(1/2)$, $A = \big(H_1(0) \cup \bigcup_{k=1}^d H_k(1)\big) \cap D$, $B_1 = H_1(0) \cap D$, $B_2 = H_1(1) \cap D$, $J_1 = \{x : x_1 < 1/2\}$, $J_2 = \{x : x_1 > 1/2\}$. Hypotheses (3.1)(i)–(iv) are easily verified, $L_2 \subset J_1 \cap D$, and so we deduce

$$\mathbb{P}^x\big(W_T \in H_1(0)\big) \ge \mathbb{P}^x\big(W_T \in H_1(1)\big). \tag{3.5}$$

We now use Lemma 3.1 again, with $H = H_{1k}$ $(k \ne 1)$, A as before, $B_1 = H_1(0) \cap D$, $B_2 = H_k(1) \cap D$, $J_1 = \{x : x_1 + x_k < 1\}$, $J_2 = \mathbb{R}^d - (H \cup J_1)$. Once again (3.1)(i)–(iv) are easily verified, and so

$$\mathbb{P}^x\big(W_T \in H_1(0)\big) \ge \mathbb{P}^x\big(W_T \in H_k(1)\big), \quad k \ne 1. \tag{3.6}$$

Combining (3.5) and (3.6), and using that fact that

$$\mathbb{P}^x\big(W_T \in H_1(0)\big) + \sum_{k=1}^d \mathbb{P}^x\big(W_T \in H_k(1)\big) \ge 1,$$

we obtain

$$\mathbb{P}^x\big(W_T \in H_1(0)\big) \ge (1 + d)^{-1}. \tag{3.7}$$

Now set $G_n = \big(H_1(0) \cap D\big) \cap \bigcap_{k=2}^n \{x_k \le 1/2\}$, for $2 \le n \le d$. Write $G_1 = H_1(0)$. Let $1 \le n \le d - 1$, set $G_n' = \overline{G_{n-1} - G_n}$, and apply Proposition 3.1 with $H = H_n(1/2)$, $B_1 = G_n$, $B_2 = G_n'$, $J_1 = \{x : x_n < 1/2\}$, $J_2 = \{x : x_n > 1/2\}$, and A as before. We deduce that

$$\mathbb{P}^x(W_T \in G_n) \ge \mathbb{P}^x(W_T \in G_n').$$

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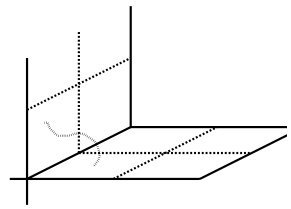

Figure 3: A corner move.

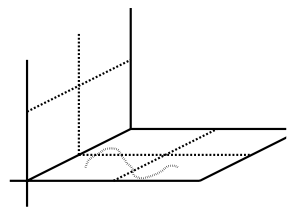

Figure 4: A slide.

Hence

$$2\mathbb{P}^x(W_T \in G_n) \geq \mathbb{P}^x(W_T \in G_n) + \mathbb{P}^x(W_T \in G_n') \geq \mathbb{P}^x(W_T \in G_{n-1}).$$

Since $G_d = L_1$,

$$\mathbb{P}^x(W_T \in L_1) \geq 2^{-(d-1)}\mathbb{P}^x\big(W_T \in H_1(0)\big),$$

and this proves the proposition with $q_1^{-1} = (1+d)2^{d-1}$. ■

Remark 3.6 We call a piece of the path of W in which it moves from L_i to L_j a *corner move*. The other kind of move we will need is from L_i to M_{ij}, which we will call a *slide*. The next few results lead up to the proof of the following.

Proposition 3.7 *Let $1 \leq i_0, j_0 \leq d$, with $i_0 \neq j_0$. Then*

$$\mathbb{P}^x(\mathcal{T}_{M_{i_0 j_0}} \leq \tau) \geq q_2 > 0, \quad \textit{for } x \in L_{i_0}.$$

By symmetry it is sufficient to prove this in the case $i_0 = 1$, $j_0 = 2$. Write $\Lambda_i^0 = \{0 \leq x_i \leq 1/2\}$, $\Lambda_i^1 = \{1/2 \leq x_i \leq 1\}$, and let

$$\begin{aligned} K_1^l &= D \cap H_1(1) \cap \Lambda_2^l, \quad l = 0, 1, \\ K_j^l &= D \cap H_j(1) \cap \Lambda_1^l, \quad l = 0, 1, \quad 2 \leq j \leq d, \\ \Gamma &= \bigcup_{i=1}^{d} H_i(1), \\ K &= H_1(0) \cap \Lambda_2^1 \cap D. \end{aligned} \tag{3.8}$$

Lemma 3.8 $\mathbb{P}^x(\mathcal{T}_{M_{12}} \leq \tau) \geq 2^{2-d}\mathbb{P}^x(\mathcal{T}_K \leq \tau)$, $x \in L_1$.

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Proof Set $G_{d+1} = K$, and let

$$G_n = K \cap \bigcap_{i=n}^{d} \Lambda_i^0, \quad 1 \le n \le d.$$

So $G_3 = D \cap H_1(0) \cap \Lambda_2^1 \cap \bigcap_3^d \Lambda_i^0 = M_{12}$. Now let $3 \le n \le d$, and apply Proposition 3.1 with $H = H_n(1/2)$, $B_1 = G_n$, $B_2 = g_H(G_n)$, $J_1 = \{x_n \le 1/2\}$, $J_2 = \{x_n > 1/2\}$, and $A = \Gamma \cup B_1 \cup B_2 = \Gamma \cup G_{n+1}$.

We verify (3.1)(i)–(iv). (i) and (ii) are obvious, while (iii) is immediate from the observation $G_n \subset \Lambda_n^0 \subset \overline{J_1}$. To prove (3.1)(iv) let $x \in A \cap J_1$. If $x \in B_1$, $g_H(x) \in B_2$, so suppose $x \in \Gamma$. Then $x \in \bigcup_{i \ne n} H_i(1)$, and so $g_H(x) \in \bigcup_{i \ne n} H_i(1)$. Then $g_H(A \cap J_1) \subset A$, as required. Finally, note that $L_1 \subset J_1$. So by (3.2),

$$\mathbb{P}^x(W_{\mathcal{T}_A} \in G_n) \ge \mathbb{P}^x\big(W_{\mathcal{T}_A} \in g_H(G_n)\big), \quad x \in L_1.$$

Hence, since $G_{n+1} = G_n \cup g_H(G_n)$, we deduce

$$\mathbb{P}^x(W_{\mathcal{T}(\Gamma \cup G_{n+1})} \in G_n) \ge \frac{1}{2}\mathbb{P}^x(W_{\mathcal{T}(\Gamma \cup G_{n+1})} \in G_{n+1}) = \frac{1}{2}\mathbb{P}^x(\mathcal{T}_{G_{n+1}} < \mathcal{T}_\Gamma).$$

Since $\mathbb{P}^x(\mathcal{T}_{G_n} < \mathcal{T}_\Gamma) \ge \mathbb{P}^x(W_{\mathcal{T}_{\Gamma \cup G_{n+1}}} \in G_n)$, it follows that $\mathbb{P}^x(\mathcal{T}_{G_n} < \mathcal{T}_\Gamma) \ge (1/2)\mathbb{P}^x(\mathcal{T}_{G_{n+1}} < \mathcal{T}_\Gamma)$ for $3 \le n \le d$, and the result follows immediately. ■

Lemma 3.9 *Set $A = K \cup \bigcup_{i=1}^d H_i(1)$. For $x \in L_1$,*

(a) $\mathbb{P}^x(W_{\mathcal{T}_A} \in K) \ge \mathbb{P}^x(W_{\mathcal{T}_A} \in K_2^0)$,
(b) $\mathbb{P}^x(W_{\mathcal{T}_A} \in K_2^0) \ge \mathbb{P}^x(W_{\mathcal{T}_A} \in K_2^1)$.

Proof These follow from Proposition 3.1 with $H = H_{12}$, for (a), and with $H = H_1(1/2)$, for (b).

Lemma 3.10 $\mathbb{P}^x\big(W_\tau \in H_2(1)\big) \ge q_3 > 0$ *for $x \in L_1$.*

Proof For $1 \le i \le d$ set

$$F_i = \{W \text{ hits } L_2, L_3, \dots, L_d \text{ in order before } \tau, \text{ and } W_\tau \in H_i(1)\}.$$

Let $F = \bigcup_{1 \le i \le d} F_i$. By Proposition 3.5 $\mathbb{P}^x(F) \ge q_1^{d-1}$. Set $q_4 = q_1^{d-1}$. Now let N be a random variable independent of $(W_t, t \ge 0)$ with $\mathbb{P}(N = i) = (d-1)^{-1}$ for $i \in \{1, 3, \dots, d\}$.

Let $\Lambda_i = \{x : x_2 = x_i\}$ for $i \ne 2$, let $g_i = g_{\Lambda_i}$, and define

$$W_t^* = \begin{cases} W_t & 0 \le t \le \mathcal{T}_{\Lambda_N}, \\ g_N(W_t) & \mathcal{T}_{\Lambda_N} \le t. \end{cases}$$

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Then W^* is also a RBM(D). Note that $\mathcal{T}_{\Lambda_N} \le \max_{i\neq 2} \mathcal{T}_{\Lambda_i} < \tau$ on F, and that on $F_k \cap \{N = k\}$, $W^*_\tau = g_N(W_\tau) = g_k(W_\tau)$, so that as $g_k\colon H_k(1) \to H_2(1)$, we have $W^*_\tau \in H_2(1)$. Therefore

$$\begin{aligned}\mathbb{P}^x\big(W_\tau \in H_2(1)\big) &= \mathbb{P}^x\big(W^*_\tau \in H_2(1)\big)\\ &\ge \mathbb{P}^x\Big(\bigcup_{k\neq 2} F_k \cap \{N = k\}\Big)\\ &= \sum_{k\neq 2} \mathbb{P}^x(F_k)\cdot \mathbb{P}(N = k)\\ &= (d-1)^{-1}\mathbb{P}^x(F) \ge (d-1)^{-1} q_4.\end{aligned}$$

■

Proof of Proposition 3.7 From Lemma 3.10 we have, writing $A = K \cup \bigcup_{i=1}^d H_i(1)$,

$$\begin{aligned}q_3 &\le \mathbb{P}^x\big(W_\tau \in H_2(1)\big)\\ &= \mathbb{P}^x\big(W_\tau \in H_2(1), \mathcal{T}_K < \tau\big) + \mathbb{P}^x\big(W_\tau \in H_2(1), \mathcal{T}_K = \tau\big)\\ &= \mathbb{P}^x(W_{\mathcal{T}_A} \in K) + \mathbb{P}^x\big(W_{\mathcal{T}_A} \in H_2(1)\big).\end{aligned}$$

From Lemma 3.9,

$$\mathbb{P}^x\big(W_{\mathcal{T}_A} \in H_2(1)\big) \le \mathbb{P}^x(W_{\mathcal{T}_A} \in K_2^0) + \mathbb{P}^x(W_{\mathcal{T}_A} \in K_2^1) \le 2\mathbb{P}^x(W_{\mathcal{T}_A} \in K).$$

So Proposition 3.7 follows, with $q_2 = q_3/3$. ■

Definition 3.11 A set $A \subset \mathbb{R}^d$ is a *half-face* if there exists $i \in \{1, \dots, d\}$, $a = (a_1, \dots, a_d) \in \frac{1}{2}\mathbb{Z}^d$ with $a_i \in \mathbb{Z}$ such that

$$A = \{x : x_i = a_i,\ a_j \le x_j \le a_j + 1/2 \text{ for } j \neq i\}.$$

For A as above set $\iota(A) = i$. Let $\mathcal{A}$ be the collection of half-faces, and set

$$A^* = \bigcup\{A : A \in \mathcal{A}\}, \quad A_0^* = \bigcup\{A \cap B; A, B \in \mathcal{A}, A \neq B\}. \tag{3.9}$$

Note that $\dim(A_0^*) = d-2$, so that A_0^* is polar for RBM(D) by Lemma 2.5 for any Lipschitz domain $D \subset \mathbb{R}^d$. Recall the definition of $\tilde{F}_0$. Set

$$\mathcal{A}_F = \{A \in \mathcal{A} : A \subset \tilde{F}_0\}.$$

We define a graph structure on $\mathcal{A}$ by taking $\{A, B\}$ to be an edge if

$$\dim(A \cap B) = d-2, \quad \text{and} \quad A \cup B \subset C \quad \text{for some } C \in \mathcal{S}_0. \tag{3.10}$$

Let $\mathcal{E}(\mathcal{A})$ be the set of edges, and let

$$\mathcal{E}(\mathcal{A}_F) = \{\{A, B\} \in \mathcal{E}(\mathcal{A}) : A, B \in \mathcal{A}_F\} \tag{3.11}$$

be the edges in the subgraph $\big(\mathcal{A}_F, \mathcal{E}(\mathcal{A}_F)\big)$. Write d_F for the natural graph distance on $\mathcal{A}_F$. Since $\tilde{F}_0$ is connected we deduce immediately

Lemma 3.12 *The graph $\big(\mathcal{A}_F, \mathcal{E}(\mathcal{A}_F)\big)$ is connected.*

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We will need terminology for the various types of edges in $\mathcal{E}(\mathcal{A})$. We call an edge $\{A, B\}$ an $i - j$ *corner* if $\iota(A) = i$, $\iota(B) = j$, and $i \neq j$ and call $\{A, B\}$ an $i - j$ *slide* if $\iota(A) = \iota(B) = i$, and the line joining the centers of A and B is parallel to the x_j axis. We say two edges are of the same *type* if they are both $i - j$ corners, or both $i - j$ slides, for some pair (i, j). Note that the move (L_i, L_j) is an $i - j$ corner, and (L_i, M_{ij}) is an $i - j$ slide.

Now set

$$\mathcal{A}_F^{(2)} = \{(A, B) \in \mathcal{A}_F : \iota(A) = \iota(B)\},$$

$$\mathcal{E}(\mathcal{A}_F^{(2)}) = \{\{(A, A'), (B, B')\} : \{A, B\}, \{A', B'\} \text{ are edges of the same type in } \mathcal{E}(\mathcal{A}_F)\}. \tag{3.12}$$

The graph $\big(\mathcal{A}_F^{(2)}, \mathcal{E}(\mathcal{A}_F^{(2)})\big)$ is not connected, and this will cause us some additional trouble. Write $d_F^{(2)}$ for the graph distance (with values in $\mathbb{Z}_+ \cup \{+\infty\}$) on $\mathcal{A}_F^{(2)}$.

Let $g_i^R \colon \mathbb{R}^d \to \mathbb{R}^d$ be reflection in the hyperplane $H_i(0)$, and let $\mathcal{G}_R$ be the group generated by the g_i^R; thus $\mathcal{G}_R$ is the set of transformations that can be obtained by a sequence of reflections parallel to the axes. Note that $\mathcal{G}_R$ is commutative. For $n \in \mathbb{Z}$ let $\mathcal{S}_n^*$ be the collection of cubes of side $2l_F^{-n}$ with vertices in $2l_F^{-n}\mathbb{Z}^d$. For $C \in \mathcal{S}_n^*$ let ψ_C be the translation which maps $\mathcal{C}$ onto $[-l_F^{-n}, l_F^{-n}]^d$, and let

$$\mathcal{G}(C, D) = \{\psi_D^{-1} \circ g \circ \psi_C : g \in \mathcal{G}\}.$$

For $x, y \in \mathbb{R}^d$ write $x \overset{n}{\sim} y$ if there exist $C, D \in \mathcal{S}_n^*$, and $g \in \mathcal{G}(C, D)$ such that $x \in C$, $y \in D$, and $g(x) = y$. Similarly, for $A, B \in \mathcal{A}$ write $A \overset{n}{\sim} B$ if there exist $C, D \in \mathcal{S}_n^*$, and $g \in \mathcal{G}(C, D)$, such that $g(A) = B$. We write $\sim$ for $\overset{0}{\sim}$.

Proposition 3.13 *Let $A, B \in \mathcal{A}_F$, with $A \sim B$. Let $C_0 \in \mathcal{S}(\tilde{F}_0)$ with $A \subset C_0$. Suppose $B = B_0, B_1, \ldots, B_n = A$ is a chain in $\mathcal{A}_F$ with $\{B_{i-1}, B_i\} \in \mathcal{E}(\mathcal{A}_F)$ for $1 \leq i \leq n$. Then there exist $A_0, A_1, \ldots, A_n$ in $\mathcal{A}_F$ such that*

(a) $\{(A_{i-1}, B_{i-1}), (A_i, B_i)\} \in \mathcal{E}(\mathcal{A}_F^{(2)})$ *for* $1 \leq i \leq n$.
(b) $A_i \subset C_0$ *for* $0 \leq i \leq n$.
(c) $A_i \sim B_i$ *for* $0 \leq i \leq n$.
(d) $(A_0, B_0) = (A, B)$, $(A_n, B_n) = (A, A)$.

In particular, $d_F^{(2)}\big((A, B), (A, A)\big) = d_F(A, B)$.

Proof Let C_1 be a cube in $\mathcal{S}_0(\tilde{F}_0)$ such that $B_0 \cup B_1 \subset C_1$. For $i = 0, 1$, let D_i be the unique cube in $\mathcal{S}_0^*$ containing C_i. Let $A_0 = A$. As $A_0 \sim B_0$, there exists $g \in \mathcal{G}(D_0, D_1)$ with $g(A_0) = B_0$. If $g(C_0) \neq C_1$, then since $B_0 \subset C_1 \cap g(C_0)$ there exists an isometry $h \in \mathcal{G}(D_1, D_1)$ mapping $g(C_0)$ to C_1 and preserving B_0. If $C_1 = g(C_0)$ take h to be the identity. Let $g' = h \circ g$: then $g'(A_0) = B_0$, and $g'(C_0) = C_1$. Set $A_1 = (g')^{-1}(B_1)$. Since $A_1 \subset C_0 \subset \tilde{F}_0$, $A_1 \in \mathcal{A}_F$. It is clear that the edges (A_0, A_1), (B_0, B_1) in $\mathcal{A}_F$ are of the same type; therefore $\{(A_0, A_1), (B_0, B_1)\} \in \mathcal{E}(\mathcal{A}_F^{(2)})$. Also, since $g'(A_1) = B_1$, we have $A_1 \sim B_1$.

Continuing in this way we can construct a sequence A_i, $0 \leq i \leq n$, in $\mathcal{A}_F$ satisfying conditions (a), (b), (c) above. To prove (d), note that since $A_n \sim B_n = A$, we have $A_n \sim A$. However, since A and A_n are both contained in C_0, this implies that $A_n = A$.

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This argument also proves that $d_F^{(2)}\big((A,B),(A,A)\big) \le d_F(A,B)$; the reverse inequality is evident. ■

Let

$$\widehat{F}_N = \bigcup_{x\in\mathbb{Z}^d} (x + F_N). \tag{3.13}$$

Let $G \subset \mathbb{R}^d$ be a union (finite or infinite) of cubes in $\mathcal{S}_0$: we will assume G is connected. We now construct a reflecting Brownian motion on $G \cap \widehat{F}_N$ from a driving process ξ on $\widehat{F}_N$. We begin with a deterministic construction.

Let $\xi(t)$, $t \ge 0$ be a continuous path on $\widehat{F}_N$ with $\xi(0) = z$ which satisfies the conditions $\xi(t) \notin A_0^*$ for any $t \ge 0$. Let $x \in G \cap \widehat{F}_N$ with $x \sim z$, and let $\eta = (\eta_0, \eta_1, \dots) \in \{0,1\}^{\mathbb{Z}_+}$. We construct from ξ and η a continuous path $w(t)$ on $\tilde{F}_N$, with $w(0) = x$.

Let T_n, $n \ge 0$ be successive disjoint hits by $\xi(\cdot)$ on $\mathcal{A}$. (We can of course take the process X in Definition 3.2 to be deterministic.) If $z \notin A^*$ then $T_0 > 0$. Then each of z, x lies in exactly one cube in $\mathcal{S}_0^*$: call these cubes D^x, D^z, and let $\psi \in \mathcal{G}(D^z, D^x)$ be such that $\psi(z) = x$. Define

$$w(t) = \psi\big(\xi(t)\big), \quad 0 \le t \le T_0.$$

Let $z' = \xi(T_0)$, $x' = w(T_0)$; we have $x', z' \in A^* - A_0^*$. Thus each of x', z' lies in exactly two cubes in $\mathcal{S}_0$: call these cubes S_0^x, S_1^x, S_0^z, S_1^z. Using lexicographic ordering of the cubes in $\mathcal{S}_0$, we can ensure these labels are uniquely specified. As $x' \in G$, there are two possibilities:

(3.14)
(i) Exactly one of S_0^x, S_1^x is contained in G,
(ii) S_0^x, S_1^x are both contained in G.

For $y = x, z$, $i = 0, 1$ let $D_i^y \in \mathcal{S}_0^*$ satisfy $S_i^y \subset D_i^y$. For i, $j = 0, 1$ there exists a unique map $\psi_{ij} \in \mathcal{G}(D_i^z, D_j^x)$ such that $\psi_{ij}(z') = x'$, and $\psi_{ij}(S_i^z) = S_j^x$. The uniqueness is evident; the existence is proved as in Proposition 3.13. Note that $\psi_{0j} = \psi_{1j}$ on $S_0^z \cap S_1^z$.

(i) Let $S_i^x \subset G$, so $(\operatorname{int} S_{1-i}^x) \cap G = \varnothing$. Now define

$$\varphi\colon S_0^z \cup S_1^z \to S_i^x \quad \text{by } \varphi = \psi_{ij}|_{S_j^z} \quad \text{for } j = 0, 1,$$

and let $w(t) = \varphi\big(\xi(t)\big)$, $T_0 \le t \le T_1$.

(ii) For $k = 0, 1$ define $\varphi_k\colon S_0^z \cup S_1^z \to S_0^x \cup S_1^x$ by

$$\varphi_k = \psi_{ij}|_{S_j^z}, \quad i = 0, 1;$$

here $j = j(i,k) = i + k \pmod 2$. Set $w(t) = \varphi_{\eta_0}\big(\xi(t)\big)$, $T_0 \le t \le T_1$.

Note that in either case, for each $t \in [T_0, T_1]$, there exists i, j such that $w(t) = \psi_{ij}\big(\xi(t)\big)$, so that $w(t) \sim \xi(t)$ for $T_0 \le t \le T_1$. (We also have $w(t) \sim \xi(t)$ on the initial segment $[0, T_0]$.)

The same construction can now be repeated on each of the time intervals $[T_{i-1}, T_i]$, using, as above, the index η_{i-1} to make a choice of maps each time case (ii) arises. The path $w \in C(\mathbb{R}_+, G \cap \widehat{F}_N)$ is a function of G, ξ, η and x only; we write

$$w = \Gamma_0(x, G, \xi, \eta). \tag{3.15}$$

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We can now define a pair of Brownian motions on $\tilde{F}$. Recall the definition of A_0^* from (3.9). The following theorem follows in a straightforward fashion from the properties of Γ_0.

Theorem 3.14 *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a probability space carrying a reflecting Brownian motion ξ_t on $\widehat{F}_N$, and independent sequences $(\eta_i^1, i \geq 0)$, $(\eta_i^0, i \geq 0)$ of i.i.d. Bernoulli random variables. Let each of G_k, $k = 1, 2$, be a union of cubes in $\mathcal{S}_0$. Suppose $\xi_0 = z \notin A_0^*$, and let $x_k \in G_k \cap \widehat{F}_N$, satisfy $x_1 \sim x_2 \sim z$. Let T_i, $i \geq 0$ be successive disjoint hits by ξ on $\mathcal{A}$, and suppose that $\eta_i^k \in \mathcal{F}_{T_i}$, $k = 1, 2$, $i \geq 0$. Set*

$$X_t^k = \Gamma_0(x_k, G_k, \xi, \eta^k), \quad k = 1, 2. \tag{3.16}$$

Then

(a) X^k is a RBM$(G_k \cap \hat{F}_N)$, *with $X_0^k = x_k$.*
(b) $X_t^k \sim \xi_t$ for $t \geq 0$.
(c) X^1 and X^2 are conditionally independent given ξ.
(d) If $T_i(X^k)$, $i \geq 0$ denote successive disjoint hits by X^k on $\mathcal{A}$, then $T_i(X^k) = T_i$ for $i \geq 0$.

Proof Note that A_0^* is polar for ξ_t, so that (T_i) and X^k are well-defined. (b), (c), (d) are all evident from the definition of Γ_0.

For (a), let $S_j^{x_i}$, S_j^z denote the cubes in $\mathcal{S}_0$ given in the construction of Γ_0. Fix k. We have

$$X_t^k = \varphi(\xi_t), \quad 0 \leq t \leq T_1,$$

where $\varphi\colon S_0^z \cup S_1^z \to \tilde{F}_N$. In case (3.14)(ii) φ is an isometry between $(S_0^z \cup S_1^z) \cap \hat{F}_N$ and $(S_0^{x_k} \cup S_1^{x_k}) \cap \widehat{F}_N$, so that X_t^k is a RBM$(\widehat{F}_N)$ on the time interval $[0, T_1]$. In case (3.14)(i), if $S_k^{x_1} \subset \tilde{F}_0$, then φ can be written in the form $\varphi = \varphi_0 \circ \varphi_1$, where φ_1 is reflection in the hyperplane containing $S_0^{x_k} \cap S_1^{x_k}$, and φ_0 is an isometry of $\mathbb{R}^d$. Again, it is clear that X_t is a RBM$(\widehat{F}_N)$ on $[0, T_1]$. ■

Remark We will call a pair of processes defined in this way *linked* RBMs.

The next sequence of results will extend the lower bounds on the probabilities of certain moves, given in Propositions 3.3 and 3.7, to joint moves by a pair of RBM X^k, defined by (3.16). We begin by introducing some further notation.

Definition 3.15 Let $J = \{-1, 1\}^d$, and let $\mathcal{E}(J) = \{\{x, y\} \in J : |x - y| = 2\}$. Then $\big(J, \mathcal{E}(J)\big)$ is the natural graph of vertices of the hypercube. For $a = (a_1, \ldots, a_d) \in J$, let

$$C_a = \{x \in [-1, 1]^d : 0 \leq x_i a_i \leq 1\}$$

be the portion of $[-1, 1]^d$ that is in the orthant determined by a. For example, if $d = 3$ and $a = (1, -1, 1)$ then

$$C_a = [0, 1] \times [-1, 0] \times [0, 1].$$

Let J_1, J_2 be non-empty, connected subsets of J, and let

$$D_i = \bigcup_{a \in J_i} (C_a \cap \hat{F}_N), \quad i = 1, 2.$$

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For $e = \{a, b\} \in \mathcal{E}(J)$, let

$$G_e = C_a \cap C_b, \quad G = \bigcup_{e \in \mathcal{E}(J)} G_e, \quad L_e = G_e \cap [-\frac{1}{2}, \frac{1}{2}]^d.$$

Thus $L_e \in \mathcal{A}$; for $j \neq \iota(L_e)$ let M_{ej} be the unique half face contained in G_e which is obtained by translating L_e a distance $1/2$ parallel to the j-axis.

We remark that

$$\begin{aligned} L_e \sim L_f \text{ if and only if } \iota(L_e) = \iota(L_f), \\ M_{ei} \sim M_{fj} \text{ if and only if } i = j \text{ and } \iota(L_e) = \iota(L_f). \end{aligned} \tag{3.17}$$

Now let $x_k \in D_k$, $k = 1, 2$, satisfy $x_1 \sim x_2$, with $x_k \notin A_0^*$, let η^k, ξ, $(\mathcal{F}_t)$ be as in Theorem 3.14, and let

$$X^k = \Gamma_0(x_k, D_k, \xi, \eta^k), \quad k = 1, 2, \tag{3.18}$$

be a pair of linked RBMs on D_1, D_2, respectively. Let

$$\tau = \inf\{t \geq 0 : \xi_t \in \partial[-1, 1]^d\},$$

and let $(T_n, n \geq 0)$ be successive disjoint hits by ξ on $\mathcal{H} = \{G_e, e \in \mathcal{E}(J)\} \cup \{\partial[-1, 1]^d\}$. Note that $\tau = T_M$ for some (random) $M \geq 0$. Let $\pi\colon [-1, 1]^d \to [0, 1]^d$ be defined by $\pi(x_1, \ldots, x_d) = (|x_1|, \ldots, |x_d|)$. Since

$$\pi(X_t^1) = \pi(X_t^2) = \pi(\xi_t), \quad 0 \leq t \leq \tau,$$

we see that if $(T_n^k, n \geq 0, k = 1, 2)$ are successive disjoint hits by X^k on $\mathcal{H}$, then $T_n^k = T_n$ for $0 \leq n \leq M$.

We now define a number of processes associated with X^k and ξ. For $n < M$, let $I_n \in \{1, \ldots, d\}$ be such that $\xi_{T_n} \in H_{I_n}(0)$, and for $a \in J$ let $A_n(a)$ be the unique element of J such that $C_a \cap C_{A_n(a)} \subset H_{I_n}(0)$. Set $\mathcal{G}_t = \sigma(\xi_s, s \leq t)$; note that I_n and $A_n(a)$ are $\mathcal{G}_{T_n}$ measurable on $\{n < M\}$. For each $0 \leq n < M$, $X_{T_n}^k$ lies in exactly two of the cubes C_a, while $X_{T_M}^k$ lies in exactly one cube, a.s. For $1 \leq n < M$ let Z_n^k be the unique element of J_k such that $C_{Z_n^k} \supset \{X_{T_{n-1}}^k, X_{T_n}^k\}$. Choose a_k such that $x_k \in C_{a_k}$ and let $Z_0^k = a_k$.

Set, for $n \geq 0$,

$$\begin{aligned} p_n(a, b) &= \mathbb{P}^x(Z_n^1 = a, Z_n^2 = b \mid \mathcal{G}_{T_n}) 1_{(n<M)}, \\ p_n^k(a) &= \mathbb{P}^x(Z_n^k = a \mid \mathcal{G}_{T_n}) 1_{(n<M)}. \end{aligned} \tag{3.19}$$

Lemma 3.16 $p_n(a, b) = p_n^1(a) p_n^2(b)$.

Proof Write $\mathcal{F}_t^{(k)}$ for the natural filtration of X^k. Then $Z_n^k \in \mathcal{F}_{T_n}^{(k)}$, while $\{M < n\} \in \mathcal{G}_{T_n}$. By (3.18) $\mathcal{F}_t^{(1)}$ and $\mathcal{F}_t^{(2)}$ are conditionally independent given $\mathcal{G}_{T_n}$; the result follows immediately. ∎

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Now write

$$V_n^k(a) = 1_{J_k}\big(A_n^k(a)\big), \quad a \in J. \tag{3.20}$$

Lemma 3.17 *For $a \in J_k$,*

$$p_{n+1}^k(a) = 1_{(n+1<M)}\left(p_n^k(a)\big(1 - V_n^k(a)\big) + \frac{1}{2}\Big(p_n^k(a) + p_n^k\big(A_n(a)\big)\Big)V_n^k(a)\right), \quad a.s. \tag{3.21}$$

To simplify notation, in the next two proofs we will omit the superscript k from X, p_n, *etc.*

Proof Note first that since $X_{T_n} \in C_{Z_n} \cap C_{A_n(Z_n)}$, we have $Z_{n+1} \in \{Z_n, A_n(Z_n)\}$. So also $Z_n \in \{Z_{n+1}, A_n(Z_{n+1})\}$, and

$$\{Z_n, A_n(Z_n)\} = \{Z_{n+1}, A_n(Z_{n+1})\}, \quad \text{on } \{n+1 < M\}. \tag{3.22}$$

Suppose first that $A_n(a) \notin J_k$. Then (3.22) implies that $Z_{n+1} = a$ and $n+1 < M$ if and only if $Z_n = a$, and $n+1 < M$. So

$$1_{(n+1<M)}\big(1 - V_n(a)\big)p_{n+1}(a) = 1_{(n+1<M)}\big(1 - V_n(a)\big), \quad \text{a.s.} \tag{3.23}$$

Now suppose that $A_n(a) \in J_k$. Then to have $Z_{n+1} = a$ it is necessary that $X_{T_n} \in C_a$, while if $X_{T_n} \in C_a$ then $Z_{n+1} = a$ if and only if $n+1 < M$ and $X_{T_{n+1}} \in C_a$. Since the set $C_a \cup C_{A_n(a)}$ is symmetric about the hyperplane $H_{I_n}(0)$, it follows that

$$1_{(n+1<M)}V_n(a)p_{n+1}(a) = 1_{(n+1<M)}V_n(a)\left(\frac{1}{2}p_n(a) + \frac{1}{2}p_n\big(A_n(a)\big)\right). \tag{3.24}$$

Combining (3.23) and (3.24) gives (3.21). ■

Now let $U_n^k = \{a : p_n^k(a) > 0\}$, $u_n^k = \#(U_n^k)$, and

$$q_n^k = 1_{(n<M)} \min\{p_n(a) : p_n(a) > 0\}. \tag{3.25}$$

Lemma 3.18 *For $n \geq 0$, $q_n^k 1_{(n<M)} \geq 2^{-2^d} 1_{(n<M)}$.*

Proof Note first that as $p_0(a_k) = 1$, $q_0 = 1$. From (3.21) we have that on $\{n+1 < M\}$ either $p_{n+1}(a) = p_n(a)$, or $p_{n+1}(a) = \frac{1}{2}\Big(p_n(a) + p_n\big(A_n(a)\big)\Big)$.

Suppose $n+1 < M$, and $U_n = U_{n+1}$. Choose $a \in U_{n+1}$ such that $q_{n+1} = p_{n+1}(a)$. If $A_n(a) \notin J_k$, then $p_{n+1}(a) = p_n(a) \geq q_n$. If $A_n(a) \in J_k$ then $p_n\big(A_n(a)\big) > 0$, since otherwise $p_{n+1}\big(A_n(a)\big) = p_n(a)/2 > 0$, so that $A_n(a) \in U_{n+1} - U_n$. Therefore

$$p_{n+1}(a) \geq \frac{1}{2}\Big(p_n(a) + p_n\big(A_n(a)\big)\Big) \geq q_n.$$

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So if $U_n = U_{n+1}$ we have $q_{n+1} \geq q_n$, on $\{n+1 < M\}$.

If $n+1 < M$, and $U_n \neq U_{n+1}$, we have $u_{n+1} - u_n \geq 1$. Again choose a such that $q_{n+1} = p_{n+1}(a)$; since at least one of a or $A_n(a)$ must be in U_n we deduce $q_{n+1} \geq q_n/2$.

So, in all cases we have

$$q_{n+1} \geq 2^{-(u_{n+1}-u_n)} q_n \quad \text{on } \{n+1 < M\},$$

and since $1 \leq u_1 \leq u_n \leq 2^d$, for all n, the result follows. ■

For the RBM$(\cdot)$ X^k defined above, and $A_1, A_2 \in \mathcal{A}$, set

$$T(A_1, A_2) = \inf\{t \geq 0 : (X_t^1, X_t^2) \in (A_1, A_2)\}. \tag{3.26}$$

Given two linked processes X_t^k, $k = 1, 2$, we let $\mathbb{P}^{(x_1,x_2)}$ denote the joint law of the pair (X_t^1, X_t^2) with X_t^k started at x_k, $k = 1, 2$. When the starting points are clear, we just write $\mathbb{P}$.

We can now give a lower bound on the probability of certain joint moves.

Theorem 3.19 *There exists a constant $p_1 = p_1(d) > 0$, with the following properties. Suppose for $k = 1, 2$, $e_k, f_k \in \mathcal{E}(J)$, $x_k \in L_{e_k}$, $x_1 \sim x_2$, $x_k \notin A_0^*$, with $\iota(L_{f_1}) = \iota(L_{f_2}) = i$, and $G_{f_k} \subset D_k$. Then*

(a) $\mathbb{P}\big(T(L_{f_1}, L_{f_2}) < \tau\big) \geq p_1$.

(b) If $j \neq i$, then

$$\mathbb{P}\big(T(M_{f_1 j}, M_{f_2 j}) < \tau\big) \geq p_1.$$

Proof Let $\overline{\xi}_t = \pi(\xi_t)$; note that T_i, $0 \leq i \leq M$, are also successive disjoint hits by $\overline{\xi}$ on $\mathcal{H}$. For $r \geq 1$ define sets B_r by $B_r = H_r(0) \cap [0, 1/2]^d$, $1 \leq r \leq d$, and $B_{kd+r} = B_r$, for $k \geq 1, 1 \leq r \leq d$. Let

$$S_0 = 0, \quad S_{r+1} = \inf\{t \geq S_r : \overline{\xi}_t \in B_{r+1}\}, \quad r \geq 1.$$

Thus the stopping times $(S_0, S_1, \dots)$ form a subsequence of $(T_0, T_1, \dots)$. For $r \geq 0$, let N_r be such that $T_{N_r} = S_r$.

Suppose that $S_r < \tau$, and that $U_{N_r}^k \neq J_k$. Then (since J_k is connected) there exists $a \in U_{N_r}^k$, $b \in J_k - U_{N_r}^k$ such that $\{a, b\} \in \mathcal{E}(J)$. Let $j = \iota(C_a \cap C_b)$. If $S_{r+d} < \tau$, then for some $i \in \{0, \dots, d-1\}$, $\overline{\xi}_{S_{r+1}}$ lies in the hyperplane $H_j(0)$, so that $A_{N_{r+1}}^k(a) = b$. Hence, by (3.21), $p_{1+N_{r+1}}^k(b) > 0$, so that $b \notin U_{N_{r+d}}^k$. So, if $S_{d2^d} < \tau$ we must have $p_{N_{d2^d}}^k(a) > 0$ for each $a \in J_k$. Therefore, by Lemmas 3.16 and 3.18,

$$p_{N_m}(a, b) \geq 4^{-2^d} \quad \text{on } \{N_m < M\}, \tag{3.27}$$

for each $a \in J_1$, $b \in J_2$, and any $m \geq d2^d$.

Since each move from B_r to B_{r+1} is a corner move of the type considered in Proposition 3.3, we also have from (3.4) that

$$\mathbb{P}^{x_k}(T_{N_m} < \tau) \geq q_1^m. \tag{3.28}$$

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Let $d2^d \le m \le (d+1)2^d$ be such that $\iota(B_m) = \iota(L_{f_1})$. Choose $b_k \in J_k$ such that $L_{f_k} \subset C_{b_k}$. Then note that $X^k_{T_{N_m}} \in L_{f_k}$ on the event $\{N_m < M, Z^k_{N_m} = b_k\}$, so that, using (3.27) and (3.28),

$$\begin{aligned}\mathbb{P}\big(T(L_{f_1}, L_{f_2}) < \tau\big) &\ge \mathbb{P}(X^k_{T_{N_m}} \in L_{f_k}, T_{N_m} < \tau)\\ &\ge \mathbb{E}\big(p_{N_m}(b_1, b_2)1_{(N_m<M)}\big)\\ &\ge 4^{-2^d} q_1^{(d+1)2^d},\end{aligned}$$

which proves (a).

To prove (b), let $S' = \inf\{t \ge T_m : \overline{\xi}_t \in M_{ij}\}$. Then by Proposition 3.7

$$\mathbb{P}(S' < \tau \mid T_m < \tau) \ge q_2.$$

We have $S' = T_{N'}$ for some $N' \ge m$, on $\{S' < \tau\}$. Then $X^k_{T_{N'}} \in M_{f_k j}$ on the event $\{N' < M, Z^k_{N'} = b_k\}$, and so

$$\begin{aligned}\mathbb{P}\big(T(M_{f_1 j}, M_{f_2 j}) < \tau\big) &\ge \mathbb{P}(Z^k_{N'} = b_k, k = 1, 2, N' < M)\\ &\ge 4^{-2^d}\mathbb{P}(N' < M)\\ &= 4^{-2^d}\mathbb{P}(N' < M \mid T_M < \tau)\mathbb{P}(T_m < \tau)\\ &\ge 4^{-2^d} q_2 q_1^m \ge 4^{-2^d} q_2 q_1^{(d+1)2^d},\end{aligned}$$

proving (b). ∎

For the next result, we recall from Section 2 the definition of the cubes $D_n(x)$ and the stopping times $\sigma_n(x)$.

Corollary 3.20 *Let ξ, η^1, η^2, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be as above, and let $x_1 \sim x_2$, with $x_k \notin A_0^*$. Set $X^k = \Gamma_0(x_k, \tilde{F}_N, \xi, \eta^k)$. Let $x_k \in A_k \in \mathcal{A}$, and let $\{(A_1, A_2), (B_1, B_2)\} \in \mathcal{E}(\mathcal{A}_F^{(2)})$. Then*

$$\mathbb{P}\big(T(B_1, B_2) < \sigma_0(x_1, X^1) \wedge \sigma_0(x_2, X^2)\big) \ge p_1.$$

Proof This is immediate from (3.17) and Theorem 3.19, by mapping the cubes $D_0(x_i)$ to $[-1, 1]^d$. ∎

Given processes X^1_t, X^2_t on $\tilde{F}_N$, set for $m \in \mathbb{Z}$,

$$\kappa_m(X^1, X^2) = \inf\{t \ge 0 : X^1_t \overset{m}{\sim} X^2_t\}. \tag{3.29}$$

Theorem 3.21 *Let $N \ge 0$. Let $x_1 \overset{n}{\sim} x_2$, with $x_k \notin l_F^{-n} A_0^*$, $x_k \in \tilde{F}_N$. There exist a pair of process (W^1_t, W^2_t) on $\tilde{F}_N$ with $W^k_0 = x_k$ such that*

(i) *W^k are* RBM$(\tilde{F}_N)$, $k = 1, 2$.
(ii) *$W^1_t \overset{n}{\sim} W^2_t$ for $t \ge 0$.*

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(iii) Writing $\kappa_m = \kappa_m(W^1, W^2)$ *then*

$$\mathbb{P}\Big(\sup_{0\le t\le \kappa_{n-1}} \max_{k=1,2} \|W_t^k - x_k\|_\infty \le (1+l_F)l_F^{-n}, \kappa_{n-1} < \infty\Big) \ge p_2 > 0 \tag{3.30}$$

for some $p_2 = p_2(d, l_F)$.

Proof By scaling, it is enough to prove the result for $n = 0$. Let $A_k \in \mathcal{A}_F$, $C_k \in \mathcal{S}_1(\tilde{F}_{-1})$, $D_k \in \mathcal{S}^*_{-1}$ be such that $x_k \in A_k \subset C_k \subset D_k$. Then there exists $g \in \mathcal{G}(D_1, D_2)$ such that $g(C_1) = C_2$. Let $x_1' = g(x_1)$; as $x_1' \overset{-1}{\sim} x_1 \sim x_2$, we have $x_1' \sim x_2$. Set $A_1' = g(A_1)$; we also have $A_1' \sim A_2$.

The restriction of the graph $\big(\mathcal{A}_F, \mathcal{E}(\mathcal{A}_F)\big)$ to C_2 is connected, so A_1' and A_2 are connected by a chain $A_1' = \alpha_0', \alpha_1', \dots, \alpha_m' = A_2$ where $\alpha_i' \subset C_2$, $\alpha_i' \in \mathcal{A}_F$, and where $m \le c_1(d, l_F)$. Let $C' \in \mathcal{S}_0(\tilde{F}_0)$ be such that $A_2 \subset C'$. Then, as in Proposition 3.13, there exists B_i, $0 \le i \le m$, such that $B_i \subset C'$, $B_i \in \mathcal{A}_F$, $B_0 = B_m = A_2$, and $\{(\alpha_i', B_i), (\alpha_{i+1}, B_{i+1})\} \in \mathcal{E}(\mathcal{A}_F^{(2)})$ for $0 \le i \le m-1$. Set $\alpha_i = g^{-1}(\alpha_i')$. Then $\{(\alpha_i, B_i), (\alpha_{i+1}, B_{i+1})\} \in \mathcal{E}(\mathcal{A}_F^{(2)})$ also, and $(\alpha_0, B_0) = (A_1, A_2)$, $(\alpha_m, B_m) = \big(g^{-1}(A_2), A_2\big)$, which implies that $\alpha_m \overset{-1}{\sim} B_m$.

Let ξ, η^k, $(\mathcal{F}_t)$ be as in Theorem 3.14, and let $W^k = \Gamma_0(x_k, \tilde{F}_N, \xi, \eta^k)$. Set for $i \ge 1$

$$S_0 = 0,$$
$$S_i = \inf\{t \ge S_{i-1} : W_t^1 \in \alpha_i \text{ and } W_t^2 \in B_i\},$$
$$R_i = \inf\{t \ge S_{i-1} : W_t^1 \notin D_0(W^1_{S_{i-1}}) \text{ or } W_t^2 \notin D_0(W^2_{S_{i-1}})\}.$$

By Corollary 3.20 $\mathbb{P}(S_i < R_i \mid \mathcal{F}_{S_{i-1}}) \ge p_1$, and therefore if $G = \bigcap_{i=1}^m \{S_i < R_i\}$ we have $\mathbb{P}(G) \ge p_1^m$. Note that on the event G, $W^k_{S_i} \in C_k$, so that $\|W_t^k - x_k\|_\infty \le l_F + 1$ for $0 \le t \le S_m$. Write $Y_k = W^k_{S_m}$; on G we have $Y_2 \in A_2$, $Y_1 \in \alpha_m = g^{-1}(A_2)$, so that $g(Y_1)$ and Y_2 both lie in A_2. However $Y_1 \sim Y_2$, and $Y_1 \overset{-1}{\sim} g(Y_1)$, so that $g(Y_1) \sim Y_2$. Hence (since $Y_k \notin A_0^*$) we have $g(Y_1) = Y_2$, so that $Y_1 \overset{-1}{\sim} Y_2$. Thus $\kappa_{-1} \le S_m$, and taking $p_2 = p_1^{c_1(d,l_F)}$ this proves the theorem. ∎

The following result is used to start off the final coupling given in Theorem 3.25.

Lemma 3.22 *Let* $n \ge 0$, *and let* $x_k \in \tilde{F}_n$, $k = 1, 2$. *There exists a constant* $p_3 = p_3(d) > 0$ *and processes* W_t^k *on* $\tilde{F}_n$ *with* $W_0^k = x_k$ *such that writing* $\kappa_n = \kappa_n(W^1, W^2)$,

(i) W_t^k *are* $\mathrm{RBM}(\tilde{F}_n)$, $k = 1, 2$,
(ii) $\mathbb{P}\big(\sup_{0\le t\le\kappa_n} \max_k \|W_t^k - x_k\|_\infty \le 2l_F^{-n}, \kappa_n < \infty\big) \ge p_3$.

Recalling the definition of κ_n from (3.29), the lemma says that $W_t^1 \overset{n}{\sim} W_t^2$ at time $t = \kappa_n$ and neither W_t^1 nor W_t^2 has moved too far from its starting point.

Proof By scaling it is enough to consider the case $n = 0$. First we note the following property of a Brownian motion on $\mathbb{R}^d$, which is connected with "reflection coupling". Let $B = B(0, 1/4) \subset \mathbb{R}^d$ and let y_1, $y_2 \in B(0, 1/8)$ with $y_1 = -y_2$. Let H be the hyperplane through 0 perpendicular to the line connecting y_1, y_2, and let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ be reflection in

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H. Then if V_t^1, $t \geq 0$, is a Brownian motion on $\mathbb{R}^d$ with $V_0^1 = y_1$, $\mathbb{P}\big(\mathcal{T}_H(V^1) < \mathcal{T}_{B^c}(V^1)\big) \geq c_1(d) > 0$. (Here c_1 depends only on the dimension d.) So if $V^2 = \psi(V^1)$, we deduce

$$\mathbb{P}\big(T_C(V^1, V^2) < \mathcal{T}_{B^c}(V^1) \wedge \mathcal{T}_{B^c}(V^2)\big) \geq c_1 > 0. \tag{3.31}$$

Now fix $x_1, x_2 \in \tilde{F}_0$, and let $S_k^* \in \mathcal{S}_0^*$, $S_k \in \mathcal{S}_0(\tilde{F}_0)$ satisfy $x_k \in S_k \subset S_k^*$. Let $g \in \mathcal{G}(S_2^*, S_1^*)$ be such that $g\colon S_2 \to S_1$. Let z_k be the center of S_k, and write $B_k = B(z_k, 1/8)$. For $k = 1, 2$ let U_t^k, $t \geq 0$ be independent RBM($\tilde{F}_0$) with $U_0^k = x_k$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let θ_t be the standard shift operators on Ω: $U_s^k(\theta_t\omega) = U_{s+t}^k(\omega)$. Let

$$A_k = \{U_1^k \in B_k,\ \sigma(x_k, U^k) > 1\}.$$

Note that on A_k we have $\sup_{0\leq s\leq 1} \|U_s^k - x_k\|_\infty \leq 3/2$. By Proposition 2.6 there exists $c_2 = c_2(d) > 0$, (not depending on x_k), such that

$$\mathbb{P}^{x_k}(A_k) \geq c_2.$$

Let $Y_1 = U_1^1$, $Y_2 = g(U_1^2)$. Set $Z = (Y_1 + Y_2)/2$, and let $B' = B(Z, 1/4)$. On $A_1 \cap A_2$ we have $|Y_k - z_1| \leq 1/8$, so that $|Z - z_1| \leq 1/8$, and $|Z - Y_k| \leq 1/8$. In particular $B' \subset S_1$. Let H be the hyperplane containing Z and perpendicular to the direction $Y_2 - Y_1$, and let $h\colon \mathbb{R}^d \to \mathbb{R}^d$ be reflection in H. Set $W^1 = U^1$, and let $\tilde{U}_t^2 = h(U_t^1)$, $t \geq 0$. Set

$$W_t^2 = U_t^2 1_{(A_1\cap A_2)^c} + 1_{A_1\cap A_2}\big[1_{[0,1]}(t)U_t^2 + 1_{(1,\infty)}(t)g^{-1}(\tilde{U}_t^2)\big].$$

If $S = \inf\{t \geq 1 : (W_t^1, W_t^2) \notin S_1 \times S_2\}$ then W^2 is a RBM($\tilde{F}_0$) started at x_2 for $0 \leq t \leq S$. It is then straightforward to extend W^2 to a RBM($\tilde{F}_0$) on $[0, \infty)$.

Write $T_C' = T_C(U^1, \tilde{U}^2) \circ \theta_1$, $\tau' = \big(\tau_{B'}(U^1) \wedge \tau_{B'}(U^2)\big) \circ \theta_1$, and let $A_3 = \{T_C' < \tau'\}$. By (3.31) we have

$$\mathbb{P}(T_C' < \tau' \mid A_1 \cap A_2) \geq c_1,$$

so that, writing $A = A_1 \cap A_2 \cap A_3$, $\mathbb{P}(A) \geq c_1c_2^2$. On A we have, by the definition of W^2, that $W_{T_C'}^1 \sim W_{T_C'}^2$, so that $\kappa_0 \leq T_C'$. Also, on A, $\max_k \sup_{1\leq t\leq T_C'} \|W_t^k - W_1^k\|_\infty \leq 1/4$, so that $\max_k \sup_{0\leq t\leq T_C'} \|W_t^k - x_k\|_\infty \leq 3/2 + 1/4 < 2$. ∎

Recall from (2.6) the definition of the exit times $\sigma_r(x)$.

Proposition 3.23 *Let N, m, $n \in \mathbb{Z}$, $m \geq n+3$, $x \in \tilde{F}_N$, and z be the center of $D_n(x)$. Suppose $x_1 \overset{m}{\sim} x_2$, $x_k \notin A_0^*$, and $\|x_k - z\|_\infty \leq l_F^{-n}(1 - l_F^{-1})$, $k = 1, 2$. Then there exist* RBM($\tilde{F}_N$) W_t^k, *with $W_0^k = x_k$, such that if $n_0 = d2^d(2l_F)^{(m-n)d}$, then*

$$\mathbb{P}\big(T_C(W^1, W^2) < \sigma_n(z, W^1) \wedge \sigma_n(z, W^2)\big) \geq p_1^{n_0}.$$

Proof Suppose first that $m = 0$. (So $|n| = -n \geq 3$.) Let $x_k \in A_k \in \mathcal{A}_F$, and let $C_k \in \mathcal{S}_0(\tilde{F}_0)$ be such that $A_k \subset C_k$. Note that in the $\|\cdot\|_\infty$ norm C_k is at least a distance $l_F^{|n|-1} - 1 \geq l_F^2 - 1 \geq l_F + 1$, from $D_n(x)^c$. Hypotheses 2.1 (H1)–(H4) imply we can find a chain of cubes in $\mathcal{S}_0(\tilde{F}_0)$ of length less than or equal to $n_1 = (2l_F)^{|n|d}$ connecting C_1 and C_2, and contained in the cube center z and side $2(l_F^{|n|} - 2)$. If $A, B \subset C$, where $C \in \mathcal{S}_0(\tilde{F}_0)$, then

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certainly $d_F(A,B) \le d2^d$, the total number of half-faces contained in C. So we can find a chain $A_1 = \alpha_0, \alpha_1, \dots, \alpha_r = A_2$ in $\mathcal{A}_F$, where $r \le n_0 = d2^d(2l_F)^{-nd}$, and where each α_i is a distance at least 2 from $D_n(x)^c$.

We have $A_1 \sim A_2$; we can therefore as in Proposition 3.13 find a chain (α_i, B_i) in $\mathcal{A}_F^{(2)}$ such that $(\alpha_0, B_0) = (A_1, A_2)$, $(\alpha_r, B_r) = (A_2, A_2)$, and each $B_i \subset C_2$. Using Corollary 3.20, we therefore have that if $W^k = \Gamma_0(x_k, \tilde{F}_N, \xi, \eta^k)$, with ξ, η^k as above, then the process (W^1, W^2) moves along the chain (α_i, B_i) with probability at least $p_1^r \ge p_1^{n_0}$. As the half-faces in the chain are all a distance at least 2 from $D_n(x)^c$, and since $C_2 \subset D_0(x_2) \subset D_{n+3}(x_2)$, it follows that

$$\mathbb{P}\big(T_{D_{n+3}(x_2)}(W^1) < \sigma_n(z)(W^1), T_C(W^1, W^2) < \sigma_0(z, W^1) \wedge \sigma_0(z, W^2)\big) \ge p_1^{n_0}. \tag{3.32}$$

This implies the result when $m = 0$; if $m \ne 0$, we can scale by l_F^m, and replace (N, m, n) by $(N - m, 0, n - m)$. ∎

The following lower bound on the probability of hitting small cubes is an essential ingredient in the Harnack inequality Theorem 4.3.

Corollary 3.24 *Let $N, n \in \mathbb{Z}$, $x \in \tilde{F}_N$, and z be the center of $D_n(x)$. There exists $\delta = \delta(l_F, d)$ such that if $y_1, y_2 \in B(z, \frac{1}{2}l_F^{-n})$, and W is a* RBM$(\tilde{F}_N)$, *then*

$$\mathbb{P}^{y_1}\big(\sigma_{n+j+2}(y_2, W) < \sigma_n(z)\big) \ge \delta^j, \quad j \ge 1.$$

Proof We begin with the case $j = 1$. Take $m = n + 3$; as in the previous proof we can suppose $m = 0$. Let $x_1 = y_1$, and choose $x_2 \in D_0(y_2)$ such that $x_1 \sim x_2$, and C_2 so that $C_2 \subset D_0(y_2)$. Then (3.32) implies that, writing $n_1 = d2^d(2l_F)^{3d}$, $\delta = p_1^{n_1}$,

$$\mathbb{P}\big(\sigma_0(y_2) < \sigma_n(z)\big) \ge \delta.$$

We now iterate. Write $S_k = \sigma_{n+k+2}(y_2)$, and let z_k be the center of $D_k(y_2)$. Since W_{S_1}, $y_2 \in B_\infty(z_{n+1}, l_F^{-n-1})$, and $\sigma_{n+1}(z_{n+1}) \le \sigma_n(z_n)$, we have

$$\mathbb{P}^{y_1}\big(S_2 < \sigma_n(z)\big) \ge \mathbb{E}^{y_1}\Big(1_{(S_1 < \sigma_n(z))}\mathbb{P}^{W_{S_1}}\big(S_2 < \sigma_{n+1}(z_{n+1})\big)\Big) \ge \delta^2.$$

This gives the case $j = 2$, and by induction we obtain the general case. ∎

Theorem 3.25 *Let $N, n \in \mathbb{Z}$, $x_0, x, y \in \tilde{F}_N$, $\varepsilon > 0$. There exist constants $p_4 = p_4(d, l_F) > 0$, $\delta = \delta(\varepsilon, d, l_F) > 0$, and coupled* RBM$(\tilde{F}_N)$ *W_t^x, W_t^y, with $W_0^x = x$, $W_0^y = y$, with the following properties.*

(a) If $x, y \in D_{n+2}(x_0)$, then

$$\mathbb{P}\big(T_C(W^x, W^y) < \sigma_n(x, W^x) \wedge \sigma_n(x, W^y)\big) > p_4. \tag{3.33}$$

(b) If $\|x - y\|_\infty \le l_F^{-n}\delta$, then

$$\mathbb{P}\big(T_C(W^x, W^y) < \sigma_n(x, W^x) \wedge \sigma_n(x, W^y)\big) > 1 - \varepsilon.$$

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Proof By scaling it is sufficient to prove this for $n = 0$.

Set $A^*_{\mathbb{Z}} = \bigcup_{n\in\mathbb{Z}} l_F^{-n} A^*_0$, and note that $A^*_{\mathbb{Z}}$ is polar for any RBM($\tilde{F}_N$); this ensures that the various sequences of successive disjoint hits below are well-defined. Write $x_1 = x$, $x_2 = y$. Let $m_0 \geq 0$ (to be chosen below), and let $M \geq \max(N, m_0)$. We construct processes W^k_t successively on the intervals $[0, \kappa_M]$, $[\kappa_M, \kappa_{M-1}], \ldots$, where the κ are given by (3.29), as follows.

Suppose first that W^k_t, $0 \leq t \leq \kappa_m = \kappa_m(W^1, W^2)$, $k = 1, 2$, have been defined. Use Theorem 3.21 applied conditionally on $\sigma(W^k_s, s \leq \kappa_m, k = 1, 2)$ to define RBM($\tilde{F}_N$) $X^{k,m,1}_t$, $t \geq 0$, such that

$$\begin{aligned} X^{k,m,1}_0 &= W^k_{\kappa_m}, \quad k = 1, 2, \\ X^{1,m,1}_t &\overset{m}{\sim} X^{2,m,1}_t, \quad t \geq 0, \end{aligned} \tag{3.34}$$

and the estimate (3.30) holds. Set

$$S_{m,1} = \inf\{t \geq 0 : \max_k \|X^{k,m,1}_t - W^k_{\kappa_m}\|_\infty \geq (1 + l_F) l_F^{-m}\}.$$

If $S_{m,1} > \kappa_{m-1}(X^{1,m,1}, X^{2,m,1})$ we say that "trial $(m, 1)$ succeeds", and otherwise we say "trial $(m, 1)$ fails"; (3.30) implies the probability of success is at least p_2. We now define W^k on an additional interval in $[0, \infty)$ by

$$W^k_{\kappa_m + t} = X^{k,m,1}_t, \quad 0 \leq t \leq S_{m,1} \wedge \kappa_{m-1}(X^{1,m,1}, X^{2,m,1}).$$

If trial $(m, 1)$ succeeds we have defined W^k on $[0, \kappa_{m-1}]$, and are ready to repeat the same construction at level $m - 1$. If trial $(m, 1)$ fails, we repeat the attempt, using Theorem 3.21 to define RBM($\tilde{F}_N$), $X^{k,m,2}_t$, $t \geq 0$, satisfying (3.34), (3.30), and with initial conditions

$$X^{k,m,2}_0 = X^{k,m,1}_{S_{m,1}}, \quad k = 1, 2.$$

Continuing in this way, we make trials (m, i), $i = 1, 2, \ldots$ at achieving coupling at level $m - 1$, until one succeeds. As the probability of success (conditional on the past) is at least p_2 for each trial, we have $\kappa_{m-1} < \infty$ a.s.

We use a similar argument to define W^k on the initial segment $[0, \kappa_M]$, except that here we use Lemma 3.22 instead of Theorem 3.21. Finally, we note that $T_C(W^1, W^2) = \kappa_n$ for all sufficiently small n, and define W^k on $\big[T_C(W^1, W^2), \infty\big)$ by taking W^1_t to be a RBM($\tilde{F}_n$) for $t \geq T_C(W^1, W^2)$, and setting $W^2_t = W^1_t$.

Set $p = \min(p_2, p_3)$, and let $a_0, a_1, \ldots, a_M, \ldots$ be positive integers, to be chosen below. Let Y_i, $0 \leq i \leq M + 1$, be the number of the trial at level i that first succeeds. From (3.30) we have

$$\mathbb{P}(Y_i > r \mid Y_{i+1}, \ldots, Y_{M+1}) < (1 - p)^r.$$

Let $m_0 \geq 0$, and let

$$G = \{Y_i \leq a_i, m_0 \leq i \leq M + 1\}.$$

Therefore

$$\mathbb{P}(G^c) \leq \sum_{i=m_0}^{M+1} (1 - p)^{a_i},$$

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while on G we have

$$\|V_t^k - x_k\|_\infty \le \sum_{i=m_0}^{M+1} a_i(1+l_F)l_F^{-i}, \quad \text{for } 0 \le t \le \kappa_{m_0}.$$

Now take $a_i = (2l_F/3)^i$; then as $l_F \ge 3$ we have $a_i \ge i$, so that

$$\mathbb{P}(G^c) \le p^{-1}(1-p)^{m_0},$$

while

$$\sum_{i=m_0}^{\infty} a_i(1+l_F)l_F^{-i} \le 3(1+l_F)(2/3)^{m_0}.$$

Now choose m_0 to be the smallest integer such that $m_0 \ge 3$, $3(1+l_F)l_F^{-m_0} \le (2l_F)^{-1}$, and $p^{-1}(1-p)^{m_0} \le 1/2$. Note that m_0 depends only on d and l_F. Let z be the center of the cube $D_0(x)$. On G we have, for $k = 1, 2$,

$$\|x_k - z\|_\infty \le \|x_0 - z\|_\infty + \|x_k - x_0\|_\infty \le \frac{1}{2} + \frac{3}{2}l_F^{-2} \le \frac{2}{3}.$$

Therefore

$$\sup_{0\le t\le \kappa_{m_0}} \|V_t^k - z\|_\infty \le \|x_k - z\|_\infty + (2l_F)^{-1} \le \frac{2}{3} + (2l_F)^{-1} \le 1 - l_F^{-1},$$

so that on G, $\kappa_{m_0} < \sigma(z, V^1) \wedge \sigma(z, V^2)$.

By Proposition 3.23 there exist RBM($\tilde{F}_N$) U_t^k, with $U_0^k = V_{\kappa_{m_0}}^k$, such that

$$\mathbb{P}\big(T_C(U^1, U^2) < \sigma_0(z, U^1) \wedge \sigma_0(z, U^2)\big) \ge p_1^{n_1},$$

where $n_1 = d2^d(2l_F)^{m_0 d}$. Now define

$$W_t^k = \begin{cases} V_t^k & 0 \le t \le \kappa_{m_0}, \\ U_{t-\kappa_{m_0}}^k & \kappa_{m_0} \le t, \end{cases}$$

and let $W_t^x = W_t^1$, $W_t^y = W_t^y 1_{(t<T_C)} + W_t^x 1_{(t\ge T_C)}$. We have

$$\begin{aligned} &\mathbb{P}\big(T_C(W^x, W^y) < \sigma_0(x, W^x) \wedge \sigma_0(x, W^y)\big) \\ &\quad \ge \mathbb{P}(G)\mathbb{P}\big(T_C(U^1, U^2) < \sigma_0(z, U^1) \wedge \sigma_0(z, U^2) \mid G\big) \\ &\quad \ge \frac{1}{2}p_1^{n_1} = p_4. \end{aligned}$$

Since p_4 depends only on d and l_F, this proves (a).

To prove (b), we use (a) iteratively. Let m be the smallest integer divisible by 4 such that $(1-p_4)^{(m-5)/4} < \varepsilon$, and let $\delta = \frac{1}{2}l_F^{-m}$. We define inductively RBM($\tilde{F}_N$) W^x, W^y

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successively on intervals $[T_i, T_{i+1}]$, stopping times T_i, and random variables $X_i = W^x_{T_i}$, $Y_i = W^y_{T_i}$ such that $Y_i \in D_{m-4i}(X_i)$.

We take $X_0 = x, Y_0 = y$; since $\|x-y\|_\infty \le \delta \le \frac{1}{2} l_F^{-m}$, we have $y \in D_m(x)$. Suppose now W^x, W^y are defined on $[0, T_i]$, and $Y_i \in D_{m-4i}(X_i)$. Apply (a) with $n = m-4i-2$ to obtain RBM$(\tilde{F}_N)$ $V^{x,i}_t, V^{y,i}_t$ with $V^{x,i}_0 = X_i, V^{y,i}_0 = Y_i$ satisfying (3.33). Set $S_i = \sigma_{m-4i-2}(X_i, V^{x,i}) \wedge \sigma_{m-4i-2}(X_i, V^{y,i})$ and $T^i_C = T_C(V^{x,i}, V^{y,i})$, and let

$$T_{i+1} = T_i + S_i,$$

$$W^z_t = V^{z,i}_{t-T_i}, \quad T_i \le t \le T_{i+1}, \quad z = x, y,$$

$$X_{i+1} = W^x_{T_{i+1}}, \quad Y_{i+1} = W^y_{T_{i+1}}.$$

Note that as $V^{x,i}_t = V^{y,i}_t$ for $t > T^i_C$, we have $W^x_t = W^y_t$ for $t > T_C = T_C(W^x, W^y)$. If $T^i_C < S_i$ then $Y_{i+1} = X_{i+1} \in D_{m-4i-4}(X_{i+1})$. Otherwise we have, as $X_{i+1}, Y_{i+1} \in D_{m-4i-2}(X_i)$, that

$$\begin{aligned} \|X_{i+1} - X_i\|_\infty \vee \|Y_{i+1} - Y_i\|_\infty &\le \frac{3}{2} l_F^{-(m-4i-2)}, \\ \|X_{i+1} - Y_{i+1}\|_\infty \le 2 l_F^{-(m-4i-2)} &\le \frac{1}{2} 4 l_F^{-2} l_F^{-(m-4i-4)}, \end{aligned} \tag{3.35}$$

so since $4 l_F^{-2} < 1$ we have $Y_{i+1} \in D_{m-4(i+1)}(X_{i+1})$.

For each i, we have, by (a)

$$\mathbb{P}(T^i_0 < S_i \mid \mathcal{F}_{T_i}) \ge p_4.$$

Let $H = \{T^i_C < S_i \text{ for some } i \le \frac{1}{4}(m-5)\}$, so that $\mathbb{P}(H^c) \le (1-p_4)^{(m-5)/4} < \varepsilon$. On H we have, using (3.35), and writing $l = (m-5)/4$,

$$\begin{aligned} \|X_l - x\|_\infty \vee \|Y_l - y\|_\infty &\le \sum_{i=0}^{l} \frac{3}{2} l_F^{-m+4i+2} \\ &\le \frac{3}{2} l_F^{2-m} l_F^{4(l+1)} (l_F^4 - 1)^{-1} \\ &\le \frac{3}{2} l_F^{2-m} l_F^{-3} l_F^{m-5+4} = \frac{3}{2} l_F^2 < \frac{1}{6}. \end{aligned}$$

So, on H, W^x and W^y couple before leaving $D_0(x)$, and $\mathbb{P}(H) \ge 1 - \varepsilon$. ■

Remark 3.26 Much of the argument above only uses the symmetry of F_1 with respect to reflection in the sets $H_i(1/2)$, which is a strictly weaker condition than Hypothesis 2.1 (H1). We do use (H1) however in the proof of Proposition 3.5, when we reflect in the hyperplanes H_{ij}.

Remark 3.27 The arguments above use essentially only three properties of the Markov process W: its continuity, the fact that W is symmetric with respect to the symmetries of the cube, and the fact that A^*_0 is polar for W. We use this in [BB6], where we couple a "cable process" associated with the graphical pre-carpet.

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4 Uniform Harnack Inequality

In this section we use the coupling of Section 3 to prove a Harnack inequality for harmonic functions on $\tilde{F}_N$ with constants which are independent of N. The proofs use ideas of [LR] and [Ca]. For a function f and a Borel set A, define

$$\operatorname{Osc}_A f = \sup_A f - \inf_A f.$$

Proposition 4.1 *There exists $\theta \in (0,1)$ independent of n and N such that if $x_0 \in \tilde{F}_N$, and h is harmonic on $D_n(x_0) \cap \tilde{F}_N$, then*

$$\operatorname{Osc}_{D_{n+2}(x_0)} h \le \theta \operatorname{Osc}_{D_n(x_0)} h.$$

Proof There is nothing to prove if $\operatorname{Osc}_{D_n(x_0)} h$ is infinite, so we may suppose h is bounded on $D_n(x_0)$. By adding a constant to h, we may suppose $\inf_{D_n(x_0)} h = 0$. Let $x, y \in D_{n+2}(x_0) \cap \tilde{F}_N$, and let W^x, W^y be the RBM($\tilde{F}_N$) given in Theorem 3.25(a). Let $U = \sigma_n(x_0, W^x) \wedge \sigma_n(x_0, W^y)$ and $T_C = T_C(W^x, W^y)$; by (3.33) we have

$$\mathbb{P}(T_C \ge U) \le 1 - p_4,$$

where p_4 depends only on d and l_F. Set $\theta = 1 - p_4$. Since h is harmonic we have

$$\begin{aligned} h(x) &= \mathbb{E} h(W^x_{U \wedge T_C}) \\ &= \mathbb{E}\big[h(W^x_{T_C}); T_C < U\big] + \mathbb{E}\big[h(W^x_U); T_C \ge U\big] \end{aligned} \tag{4.1}$$

and similarly

$$h(y) = \mathbb{E}\big[h(W^y_{T_C}); T_C < U\big] + \mathbb{E}\big[h(W^y_U); T_C \ge U\big]. \tag{4.2}$$

Since $h(W^x_{T_C}) = h(W^y_{T_C})$, subtracting (4.2) from (4.1) gives

$$\begin{aligned} |h(x) - h(y)| &= |\mathbb{E}\big[h(W^x_U) - h(W^y_U); T_C \ge U\big]| \\ &\le \mathbb{P}(T_C \ge U) \operatorname{Osc}_{D_n(x_0)} h \\ &\le \theta \operatorname{Osc}_{D_n(x_0)} h. \end{aligned}$$

■

Theorem 4.2 *There exist M and α depending only on d and l_F such that if $x_0 \in \tilde{F}_n$ and h is harmonic in $D_n(x_0) \cap \tilde{F}_N$, then*

$$|h(x) - h(y)| \le M|x-y|^\alpha l_F^{m\alpha} \sup_{D_n(x_0)} |h|, \quad x, y \in D_{n+2}(x_0).$$

Proof By iterating Proposition 4.1,

$$\operatorname{Osc}_{D_{n+2j}(x)} h \le \theta \operatorname{Osc}_{D_{n+2(j-1)}(x)} h \le \theta^j \operatorname{Osc}_{D_n(x)} |h| \le 2\theta^j \sup_{D_n(x_0)} |h|. \tag{4.3}$$

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Let $\|x-y\|_\infty = r$, and choose j such that $y \in D_{n+2j}(x) - D_{n+2(j+1)}(x)$; by Lemma 2.3

$$\frac{3}{2} l_F^{-n-2j} \geq r \geq \frac{1}{2} l_F^{-n-2j-2}.$$

So $c_1\theta^j \geq (l_F^n r)^{\log(1/\theta)/2\log l_F} \geq c_2\theta^j$, and hence by (4.3),

$$|h(y)-h(x)| \leq \operatorname{Osc}_{D_{n+2j}(x)} h \leq c_3 l_F^{n\alpha} r^\alpha \sup_{D_n(x_0)} |h|,$$

where $\alpha = \log(1/\theta)/2\log l_F$. ■

Theorem 4.3 **(Uniform Harnack Inequality)** *There exists c_1 depending only on d and l_F such that if $x_0 \in \tilde{F}_N$ and h is nonnegative and harmonic in $D_n(x_0) \cap \tilde{F}_N$, then*

$$h(x) \leq c_1 h(y), \quad x, y \in D_{n+4}(x_0). \tag{4.4}$$

Proof Let z be the center of $D_n(x_0)$. By looking at $h+\varepsilon$ and then letting $\varepsilon \downarrow 0$, we may assume h is bounded below by a positive constant in $D_n(x_0)$. Multiplying by a constant, we may assume $\inf_{D_{n+4}(x_0)} h = 1$. Since $\tilde{F}_N$ is a Lipschitz domain, h is bounded and continuous on $A = B\big(z, l_F^{-n}(1-l_F^{-1})\big)$; we need to show that we can bound h in $D_{n+4}(x_0)$ by a constant independent of n and N.

By Corollary 3.24 we have, for $x \in A$, and $\delta = \delta(d, l_F)$,

$$\mathbb{P}^y\big(\mathcal{T}_{D_{n+j}(x)} < \sigma_n(x_0)\big) > \delta^j, \quad j \geq 3.$$

This yields an estimate on the minimum of h on $D_{n+j}(x)$; we have

$$1 = h(y) \geq \mathbb{E}^y\big[h\big(X(\mathcal{T}_{D_{n+j}(x)})\big); \mathcal{T}_{D_{n+j}(x)} < \sigma_n(x_0)\big] \geq \delta^j \inf_{D_{n+j}(x)} h,$$

so that

$$\inf_{D_{n+j}(x)} h \leq \delta^{-j}, \quad x \in A, j \geq 3. \tag{4.5}$$

Now choose $M \geq 3$ so that $\theta^{-M} \geq \delta^{-2}/(\delta^{-1}-1)$. Let $j_0 = 2M-2$. If $x \in A$, $D_{n+j-2M}(x) \subset D_n(x_0)$ for $j \geq j_0$, and so from Proposition 4.1

$$\begin{aligned} \operatorname{Osc}_{D_{n+j-2M}(x)} h &\geq \theta^{-M} \operatorname{Osc}_{D_{n+j}(x)} h \\ &\geq \frac{\delta^{-2}}{\delta^{-1}-1} \operatorname{Osc}_{D_{n+j}(x)} h, \quad j \geq j_0, x \in A. \end{aligned} \tag{4.6}$$

Let $K = 2M+3$. Suppose there exists $x' \in D_{n+4}(x_0)$ such that $h(x') \geq \delta^{-K-j_0-2}$. We use induction to construct a sequence x_j, $j \geq j_0$ such that $x_{j_0} = x'$ and

$$h(x_j) \geq \delta^{-K-j-1}, \tag{4.7}$$

$$\|x_j - z\|_\infty \leq \frac{3}{2} l_F^{-n-4} + \frac{3}{2} l_F^{-n-3} \sum_{i=j_0}^{j} l_F^{-j}. \tag{4.8}$$

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Suppose we have constructed $x_{j_0}, \dots, x_j$. Since by (4.8) $x_j \in A$, we have by (4.5) that

$$\inf_{D_{n+K+j}(x_j)} h \le \delta^{-K-j}, \tag{4.9}$$

while by (4.7)

$$h(x_j) \ge \delta^{-K-j-1}.$$

Hence

$$\mathrm{Osc}_{D_{n+K+j}(x_j)}\, h \ge (\delta^{-1} - 1)\delta^{-K-j}.$$

By (4.6),

$$\mathrm{Osc}_{D_{n+K+j-2M}(x_j)}\, h \ge \delta^{-K-j-2}.$$

Since h is nonnegative, this implies that there must exist a point $x_{j+1} \in D_{n+j+3}(x_j)$ such that $h(x_{j+1}) \ge \delta^{-K-j-2}$. As $\|x_{j+1} - x_j\|_\infty \le \frac{3}{2} l_F^{-n-j-3}$, x_{j+1} satisfies (4.7) and (4.8).

By induction, we therefore have a sequence x_j, $j \ge j_0$ in A with $h(x_j) \to \infty$, which contradicts the fact that h is bounded on A. It follows that h is bounded on $D_{n+4}(x_0)$ by δ^{-K-j_0-2}. ■

Remark 4.4 Note that the proofs of the results in this section do not use the symmetry assumption Hypothesis 2.1 (H1) directly, but only through Corollary 3.24 and Theorem 3.25. So if $\tilde{F}_N$ is a generalized Sierpinski carpet at level N, for which F_1 satisfies Hypothesis 2.1 (H2)–(H4), and in addition we know that Corollary 3.24 and Theorem 3.25 hold for $\tilde{F}_N$, then all the results in this section hold for $\tilde{F}_N$.

Remark 4.5 The proof of Theorem 1.1. is immediate from Theorem 4.3.

5 Constants and Inequalities

Constants

We define the resistance constant R_n by

$$R_n^{-1} = \inf\left\{ \int_{l_F^n F_n} |\nabla f|^2\, dx : f = 0 \text{ on } x_1 = 0,\ f = 1 \text{ on } x_1 = l_F^n \right\}. \tag{5.1}$$

Thus R_n is the resistance between two opposite faces of the set $l_F^n F_n$. In [McG] (see also [KZ] and for the case $d = 2$ [BB3]), it is shown that there exists a constant ρ_F and constants c_1, c_2 such that

$$c_1 \rho_F^n \le R_n \le c_2 \rho_F^n. \tag{5.2}$$

The proof uses a subadditivity argument, which does not yield the value of ρ_F. We call ρ_F the *resistance scale factor* of F, and define the *time scale factor* by $t_F = (m_F)(\rho_F)$. We define the fractal dimension, dimension of the walk, and spectral dimension of F by

$$\begin{aligned} d_f &= \log m_F / \log l_F, \\ d_w &= \log t_f / \log l_F, \\ d_s &= 2d_f / d_w = 2 \log m_F / \log t_F. \end{aligned} \tag{5.3}$$

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d_f is the Hausdorff dimension (and also the packing dimension) of F and $\tilde{F}$.

Using standard shorting and cutting arguments (see [DS], and also [BB3, Prop. 5.3]) we have the following estimates for ρ_F.

Proposition 5.1

(a) $t_F \geq l_F^2$.
(b) $\rho_F \leq 2^{1-d} l_F$.

Proof (a) Note that $l_F^n F_n$ consists of m_F sets each congruent to $l_F^{n-1} F_{n-1}$. For $1 \leq i \leq l_F$ let a_i be the number of sets congruent to $l_F^{n-1} F_{n-1}$ contained in $l_F^n F_n \cap \{x : (i-1) l_F^{n-1} \leq x_1 \leq i l_F^{n-1}\}$. Then $\sum a_i = m_F$, and applying shorts on each of the sets $\{x_1 = i l_F^{n-1}\}$ we have from Kirchoff's laws

$$R_n \geq R_{n-1} \sum_{i=1}^{l_F} a_i^{-1}. \tag{5.4}$$

Using (5.2) it follows that

$$\rho_F \geq \sum_{i=1}^{l_F} a_i^{-1}.$$

By Cauchy-Schwarz,

$$l_F^2 = \Big(\sum_{i=1}^{l_F} a_i^{\frac{1}{2}} a_i^{-\frac{1}{2}}\Big)^2 \leq \Big(\sum_{i=1}^{l_F} a_i\Big)\Big(\sum_{i=1}^{l_F} a_i^{-1}\Big) = m_F\Big(\sum_{i=1}^{l_F} a_i^{-1}\Big).$$

(a) is now clear.

(b) Using Hypothesis 2.1 (H4) we have that the two sides $\{x_1 = 0\}$ and $\{x_1 = l_F^n\}$ of $l_F^n F_n$ are connected by 2^{d-1} disjoint blocks, each consisting of l_F copies of $l_F^{n-1} F_{n-1}$ arranged in series. (In the case $d = 3$ we are considering the cubes that touch the four edges connecting $\{x_1 = 0\}$ and $\{x_1 = l_F^n\}$). Therefore

$$R_n \leq R_{n-1} 2^{-(d-1)} l_F, \tag{5.5}$$

from which (b) follows immediately. ■

Remark 5.2 Note that while (a) is true for any GSC satisfying Hypotheses 2.1 (H1) and (H2), (b) relies on (H4), and does not always hold for more general GSCs.

Corollary 5.3 *For GSCs which satisfy Hypotheses 2.1 (H1)–(H4) the following inequalities hold:*

$$d_w \geq 2, \qquad d_s \leq d_f < d, \tag{5.6}$$

$$d_w \leq 1 + d_f - (d-1)(\log 2/\log l_F) < 1 + d_f, \tag{5.7}$$

$$d_s > \frac{2d_f}{1+d_f} > 1. \tag{5.8}$$

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Proof (5.6) and (5.7) are immediate from Proposition 5.1 and the definitions of d_f, d_w, d_s. Hypothesis 2.1 (H2) implies that $m_F > l_F$, so $d_f > 1$, and (5.8) follows from (5.7). ■

Remarks 5.4 1. Note that the proof of Proposition 5.1(a) shows that $t_F > l_F^2$ provided that the a_i are not all equal. In fact we have a proof that strict inequality holds in Proposition 5.1(a) for any GSC. However, the proof is rather longer than that given above. As a consequence, we know that $d_w > 2$ and $d_s < d_f$ in (5.6).

2. Let $1 \le b \le a$, $d \ge 2$, and consider the GSC given by taking $l_F = a$, and $F_1 = F_0 - \big((a-b)/2a, (a+b)/2a\big)^d$. So F_1 is obtained by removing a central block of b^d cubes of side a^{-1}. We denote this GSC by SC(a, b, d). We have $m_F = a^d - b^d$, and the estimates (5.4) and (5.5) give

$$\frac{a-b}{a^{d-1}} + \frac{b}{a^{d-1} - b^{d-1}} \le \rho_F \le \frac{a}{a^{d-1} - b^{d-1}}. \tag{5.9}$$

In the particular case of the standard 3-dimensional SC, SC$(3, 1, 3)$, this implies that $25/72 \le \rho_F \le 27/72$, so that

$$2.0028 < d_w < 2.0729, \qquad 2.8614 < d_s < 2.9615.$$

This may be compared with the estimate

$$d_s \ge d_i = \frac{\log 26}{\log(26/8)} \approx 2.7642$$

obtained in [O1] using an isoperimetric inequality.

Processes

Let $\partial_a F_n = \{x \in F_n : x_1 \vee \cdots \vee x_d = 1\}$. We let W_t^n be Brownian motion on $\tilde{F}_n$ with normal reflection on $\partial\tilde{F}_n$. Let $\tau_n^W = \inf\{t : W_t^n \in \partial_a F_n\}$. Let $G_n = \{x \in F_n : x_1 \vee \cdots \vee x_d \le \frac{1}{2}\}$. Then there exist c_1, c_2 such that for all $n \ge 0$

$$c_1(t_F/l_F^2)^n \le \inf_{x \in G_n} \mathbb{E}^x \tau_n^W \le \sup_{x \in F_n} \mathbb{E}^x \tau_n^W \le c_2(t_F/l_F^2)^n; \tag{5.10}$$

this is proved just as in [BB1, Prop. 4.2] and [BB2, Prop. 2.2 and 2.3]. We make the remark that $(t_F/l_F^2)^{-n}$ is also comparable to the first eigenvalue for the Laplacian on F_n with Dirichlet boundary conditions on $\partial_a F_n$ and Neumann boundary conditions elsewhere; see [BB2, Prop. 2.2].

Let $X_t^n = W^n\big((t_F/l_F^2)^n t\big)$ and let $\mathbb{P}_n^x$ be the law of X_t^n when $X_0^n = x$. We have the following estimates on the $\mathbb{P}_n^{\cdot}$ laws of $\sigma_r(x) = \sigma_r(x, X)$.

Proposition 5.5

(a) For $0 \le r \le n$, $x \in \tilde{F}_n$,

$$c_1 t_F^{-r} \le \mathbb{E}_n^x \sigma_r(x) \le c_2 t_F^{-r}. \tag{5.11}$$

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(b) Let $n \geq 0$. If $m \leq n$ and $t \geq l_F^{n-m}/t_F^n$, then

$$\mathbb{P}_n^x(\sigma_m(x) \leq t) \leq c_3 \exp\bigl(-c_4(t_F^{-m}/t)^{1/(d_w-1)}\bigr). \tag{5.12}$$

If $m \leq n$ and $t \geq l_F^{n-m}/t_F^n$, or if $m > n$ and $t > 0$, then

$$\mathbb{P}_n^x\bigl(\sigma_m(x) \leq t\bigr) \leq c_5 \exp\bigl(-c_6(l_F^{2n}/t_F^n)(l_F^{-2m}/t)\bigr). \tag{5.13}$$

(c) For $x \in \tilde{F}_n$, $\lambda > l_F^{-n}$,

$$\mathbb{P}_n^x(\sup_{s\leq t} |X_s^n - X_0^n| \geq \lambda) \leq c_7 \exp\bigl(-c_8(\lambda^{d_w}/t)^{1/(d_w-1)}\bigr). \tag{5.14}$$

Proof As the proofs are similar to those in [BB1, Prop. 4.4] and [BB4, (3.1), Prop. 3.3 and Theorem 3.4] we only give a sketch of the argument.

Note that (5.11) follows from (5.10) using scaling, which proves (a).

We turn to the proof of (b). It is easy to see (5.11) holds if $r \leq n+2$. By scaling we may suppose $n = 0$ and $r > 2$. In this case, $D_r(x)$ is a subset of the region above a Lipschitz function. Let $\overline{v}(y,z)$ be the Green function for a RBM in $\tilde{F}_0$ killed on exiting $D_r(x)$. By Corollary 3.3 of [BH] and integration,

$$\mathbb{E}^x\sigma_r^W(x) = \int_{\tilde{F}_0} \overline{v}(x,y)\,dy \leq c_9 l_F^{-2r}.$$

As in the proof of Proposition 2.5, if $d \geq 3$, there exists K not depending on r such that $\overline{v}(y,z) \geq c_{10}|y-z|^{2-d}$ for $y,z \in D_{r+K}(x)$. An integration then gives

$$\mathbb{E}^x\sigma_r^W(x) \geq \int_{D_{r+K}(x)} \overline{v}(x,y)\,dy \geq c_{11} l_F^{-2r}.$$

The case $d = 2$ is similar.

Since X_t is a time change of W_t, we then have

$$c_{12} t_F^{-n} l_F^{-2r+2n} \leq \mathbb{E}_n^x\sigma_r^X(x) \leq c_{13} t_F^{-n} l_F^{-2r+2n}, \quad r > n+2.$$

It follows as in [BB1, Lemma 4.3] that for $t \geq 0$, $x \in \tilde{F}_n$,

$$\mathbb{P}_n^x\bigl(\sigma_r(x) \leq t\bigr) \leq c_{14} + c_{15} a_{nr} t, \tag{5.15}$$

where $c_{14} \in (0,1)$ and $a_{nr} = t_F^n$ if $r \leq n+2$, and $a_{nr} = l_F^{2r-2n} t_F^n$ if $r > n+2$. If $x(t)$ is a continuous path in $\tilde{F}_n$ we call an *r-crossing* a segment of the path $t_1 \leq s < t_2$ where $x(t_2) \in \partial D_n\bigl(x(t_1)\bigr)$ and $x(s) \in D_n\bigl(x(t_1)\bigr)$ for $t_1 \leq s < t_2$. If $m \leq r$ then any path from x to $D_m(x)^c$ must include at least $l_F^{r-m}/4$ disjoint r-crossings. So, writing $M = l_F^{r-m}/4$, $\sigma_m(x) \geq \sum_{i=1}^M V_i$, where the V_i satisfy the bounds in (5.15). By [BB1, Lemma 1.1],

$$\log \mathbb{P}_n^x\bigl(\sigma_m(x) \leq t\bigr) \leq 2(a_{nr} M t/c_{14})^{1/2} - M\log(1/c_{14}).$$

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It therefore follows that for $r \geq m$,

$$\log \mathbb{P}^x\big(\sigma_m(x) \leq t\big) \leq c_{15}\big[(t_F^r l_F^{r-m} t)^{1/2} - l_F^{r-m}\big] \text{ for } r \leq n+2, \tag{5.16}$$

and

$$\log \mathbb{P}^x\big(\sigma_m(x) \leq t\big) \leq c_{16}\big[(l_F^{2r-2n} t_F^n l_F^{r-m} t)^{1/2} - l_F^{r-m}\big] \text{ for } r > n+2. \tag{5.17}$$

The bounds in (b) then follow on choosing r to minimize the right hand sides of (5.16) and (5.17).

(c) follows easily from (b); for a similar proof see, for example, [BB4, Theorem 3.4(b)]. ■

Definition 5.6 Let A and B be Borel sets. For $A \subset \tilde{F}$ write $\partial_{\tilde{F}} A$ for the relative boundary of A in $\tilde{F}$. A local isometry of $\tilde{F}$ is a triple (φ, A, B) where $A, B \subset \tilde{F}$ and φ is an isometry between A and B and also between $\partial_{\tilde{F}} A$ and $\partial_{\tilde{F}} B$. A Markov process $Y = (\mathbb{Q}^x, Y_t)$, $x \in \tilde{F}, t > 0$, is said to be *invariant with respect to the local isometries of* $\tilde{F}$ if for every local isometry (φ, A, B) and $x \in A$,

$$\mathbb{Q}^x\big(\varphi(Y_{t\wedge\tau(A)}) \in \cdot\,, t \geq 0\big) = \mathbb{Q}^{\varphi(x)}(Y_{t\wedge\tau(B)} \in \cdot\,, t \geq 0). \tag{5.18}$$

We will also say that Y is *locally isotropic*.

As in [BB1] we have the theorem

Theorem 5.7 *There exists a strong Markov process $X = (\mathbb{P}^x, X_t)$, $x \in \tilde{F}$, $t \geq 0$, with state space $\tilde{F}$ such that X has a Feller transition semigroup which is μ-symmetric, X has continuous paths, and the process X is invariant with respect to the local isometries of $\tilde{F}$.*

As a consequence of our results on the transition densities of X_t later on, we shall see that X_t actually has a strong Feller transition semigroup.

The proof of Theorem 5.7 is the same as that in [BB1, Sect. 6]. We briefly review the outline of the proof and refer the reader to that paper for details. Define

$$U_n^\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t^n)\, dt.$$

By the Harnack inequality and the modulus of continuity estimates for harmonic functions proved in Section 4 of this paper, it is not hard to see that when f is bounded and continuous, then $\{U_n^\lambda f\}_{n=1}^\infty$ is equicontinuous on compact sets. By a diagonalization and limit argument, there exists a subsequence n' such that $U_{n'}^\lambda f$ converges uniformly on compacts, say to $U^\lambda f$, for all $\lambda > 0$ and f bounded and continuous. Since $\{\mathbb{P}_{n'}^x\}$ is tight, for each x there exist convergent subsequences. Any limit point $\mathbb{P}^x$ satisfies $\mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt = U^\lambda f(x)$ for f bounded and continuous, from which one deduces that $\mathbb{P}_{n'}^x$ converges. If one calls the limit $\mathbb{P}^x$ and lets X_t be the canonical process on F, one then can show that $(\mathbb{P}^x, X_t)$ has the strong Markov property as well as the other required properties. It is then straightforward to extend $(\mathbb{P}^x, X_t)$ to a process on $\tilde{F}$.

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Many properties of X follow almost immediately from the corresponding properties of the X^n. Thus from Proposition 5.5 we deduce

Proposition 5.8 *For $r \in \mathbb{Z}$, $t > 0$,*

$$\mathbb{P}^x\big(\sigma_r(x) \le t\big) \le c_1 \exp\big(-c_2(t_F^r t)^{-1/(d_w-1)}\big), \tag{5.19}$$

$$c_3 t_F^{-r} \le \mathbb{E}^x \sigma_r(x) \le c_4 t_F^{-r}, \tag{5.20}$$

and

$$\mathbb{P}^x(\sup_{s\le t} |X_s - X_0| \ge \lambda) \le c_5 \exp\big(-c_6(\lambda^{d_w}/t)^{1/(d_w-1)}\big), \quad \lambda > 0. \tag{5.21}$$

We also have the *weak scaling property*: the $\mathbb{P}^{l_F^r x}$ law of $l_F^{-r} X(t_F^r t)$ satisfies the estimates (5.19)–(5.21).

Let P_t be the semigroup of the process X_t (acting on $L^2(\tilde{F}, \mu)$), and $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ be the infinitesimal generator of (P_t). Since X is μ-symmetric, we have for $f, g \in L^2(\tilde{F}, \mu)$,

$$\int f(x) P_t g(x)\, \mu(dx) = \int P_t f(x) g(x)\, \mu(dx).$$

By [FOT] there exists a Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}(\mathcal{E}) \subset L^2(\tilde{F}, \mu)$ associated with the semigroup (P_t). See [FOT, Chapter 1] for details of the relations between X, P_t and $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$; here we just note that $\mathcal{D}(\mathcal{L})$ is dense in $\mathcal{D}(\mathcal{E})$ and that

$$\mathcal{E}(f, g) = -\int \mathcal{L} f(x) g(x)\, \mu(dx), \quad f \in \mathcal{D}(\mathcal{L}), \quad g \in \mathcal{D}(\mathcal{E}). \tag{5.22}$$

If $h\colon \tilde{F} \to \mathbb{R}$ we say h is *harmonic* (with respect to X) if $h \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}h = 0$, or, equivalently, if $h(X_t)$ is a local martingale.

Theorem 5.9 *Let $x_0 \in \tilde{F}$, $n \in \mathbb{Z}$, and h be nonnegative and harmonic in $D_n(x_0) \cap \tilde{F}$. There exist c_1, c_2 and α, depending only on d and l_F such that*

(a)

$$h(x) \le c_2 h(y), \quad x, y \in D_{n+4}(x_0). \tag{5.23}$$

(b)

$$|h(x) - h(y)| \le c_1 |x - y|^\alpha l_F^{m\alpha} \sup_{D_n(x_0)} |h|, \quad x, y \in D_{n+2}(x_0), \tag{5.24}$$

Proof (a) By (5.19) we have that X_t leaves a point x immediately. It follows from the symmetry of X that if $x \in \partial D_m(0)$, then $\mathbb{P}^x(\tau_{D_m(0)} > 0) > 0$, and by the Blumenthal 0-1 law this probability must be 1. This shows that if f is a continuous function on $\partial D_m(0)$, then $f(X_{\sigma_m(0)})$ is a continuous functional of the path except for a null set, so $u_n(x) = \mathbb{E}_n^x f(X_{\sigma_m(0)})$ converges (along a subsequence) to $u(x) = \mathbb{E}^x f(X_{\sigma_m(0)})$. By Theorem 4.3 the functions u_n

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satisfy (5.23) with constants independent of n, and therefore u also satisfies (5.23). This proves the result when u is non-negative and continuous on $D_n(x_0)\cap\tilde{F}$; by a limit argument we then obtain the result for all positive harmonic functions.

(b) This is standard using an argument of Moser [M]. ■

Remark 5.10 By very analogous methods, we can construct a process $\hat{X}$ on domains of the form $\tilde{F}\cap D_n(x_0)$ with normal reflection on $\partial D_n(x_0)$. We let $\widehat{W}_t^m$ be RBM$\big(\tilde{F}_0\cap D_m(x_0)\big)$, set $\hat{X}_t = \widehat{W}^m\big((t_F/l_F^2)^n t\big)$, and proceed as in the proof of Theorem 5.7.

For use in Section 6 we note the estimates,

$$\mathbb{P}^x\big(\sigma_r(x,\hat{X})\le t\big)\le c_1\exp\big(-c_2(t_F^r t)^{-1/(d_w-1)}\big),\quad t>0, r\ge n, \tag{5.25}$$

and

$$\mathbb{E}^x\sigma_r(x,\hat{X})\le c_3 t_F^{-r},\quad r\ge n. \tag{5.26}$$

Remark 5.11 Note that the process X in Theorem 5.7 is only obtained as a subsequential limit of the processes X^n. If we had $X=\lim X^n$ then it would follow easily that X was self-similar, that is, that the $\mathbb{P}^x$ law of $l_F^{-1}X_t$ would be equal to the $\mathbb{P}^{x/l_F}$ law of X_{t/t_F}. See Section 9 for some further remarks on this point.

However, a self-similar process Y_t on $\tilde{F}$ can be constructed as follows. By Proposition 5.1 we have $\rho_F m_F\ge l_F^2$, so that $\rho_F\ge l_F^d m_F^{-d} l_F^{2-d}\ge l_F^{1-d}$. So F satisfies the "good borders" hypothesis in [KZ], and we may now follow the argument of [KZ] to construct a diffusion Y_t on $\tilde{F}$ that is also self-similar. We refer the reader to [KZ] for details. Briefly, in that paper $\mathcal{E}_n$ is defined to be the Dirichlet form for a graphical approximation to Y^n, $\frac{1}{n}\sum_{j=1}^n\mathcal{E}_j$ is shown to have subsequential limit points, and if $\mathcal{E}$ is one of those subsequential limit points, it is shown that $\mathcal{E}$ is a closable Dirichlet form. The process Y_t is then the Markov process associated with $\mathcal{E}$.

6 Transition Density Estimates

Let $(\mathbb{P}^x, W_t)$ be a RBM$(\tilde{F}_0)$. We have the following estimates on the transition densities of W_t.

Proposition 6.1 *W_t has continuous transition densities $q(t,x,y)$ with respect to Lebesgue measure μ_0 on $\tilde{F}_0$, which satisfy*

$$q(t,x,y)=q(t,y,x)\le q(t,x,x)^{\frac12}q(t,y,y)^{\frac12}, \tag{6.1}$$

$q(t,x,x)$ is nonincreasing in t for each $x\in\tilde{F}_0$, and for each t_0

$$q(t,x,y)\le c_1(t_0)t^{-d/2},\quad 0\le t\le t_0,\quad x,y\in\tilde{F}_0. \tag{6.2}$$

Proof Fix $x,y\in\tilde{F}_0$ and select $m<0$ so that $x,y\in D_{m+4}(x_0)\cap\tilde{F}_0$. Let C be a small subset of $D_{m+4}(x_0)\cap\tilde{F}_0$ containing y. Let $S_1=\sigma_m(x_0)$, $U_i=\inf\{t>S_i : W_t\in D_{m+2}(x_0)\}$, and

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$S_{i+1} = \inf\{t > U_i : W_t \notin D_m(x_0)\}$. Then

$$\begin{aligned}\mathbb{P}^x(W_t \in C) &= \mathbb{P}^x(W_t \in C, t < S_1) + \mathbb{P}^x(W_t \in C, U_1 < t < S_2) \\ &\quad + \mathbb{P}^x(W_t \in C, U_2 < t < S_3) + \cdots .\end{aligned} \tag{6.3}$$

If $\overline{q}(t, x, y)$ is the transition density for W_t killed on exiting $D_m(x_0)$, the first term on the right of (6.3) is

$$\int_C \overline{q}(t, x, z)\, dz.$$

The second term is

$$\int_C \mathbb{E}^x\left[\int_0^t \overline{q}(t-s, W_{U_1}, z); U_1 \in ds\right] dz.$$

Note that this is less than

$$\sup_{w \in \partial D_{m+2}(x_0)} \sup_{z \in D_{m+4}(x_0)} \sup_{r \le t} \overline{q}(r, w, z)\mathbb{P}^x(U_1 \le t)\mu_0(C), \tag{6.4}$$

and the succeeding terms of (6.3) are bounded similarly. Provided t is small enough, (5.13) and (2.10) imply that the i-th term of (6.3) is bounded by $c_2\rho^i t^{-d/2}\mu_0(C)$, where $\rho < 1$. Therefore $\mathbb{P}^x(W_t \in dy)$ is absolutely continuous with respect to Lebesgue measure, the density $q(t, x, y)$ satisfies

$$q(t, x, z) = \overline{q}(t, x, z) + \mathbb{E}^x\left[\int_0^t \overline{q}(t-s, W_{U_1}, z); U_1 \in ds\right] + \cdots , \tag{6.5}$$

and the series is uniformly convergent in a neighborhood of y. Since for each w we know that $\overline{q}(t, w, z)$ is continuous in z, this implies that $q(t, x, z)$ is continuous in z for z in a neighborhood of y.

Taking $y = x$, we obtain from (6.5) that

$$q(t, x, x) \le c_3 t^{-d/2}, \quad t \le t_1, \quad x \in \tilde{F}_0. \tag{6.6}$$

Since $q(t, x, y) < \infty$ for t small enough, the semigroup property shows that $q(t, x, y) < \infty$ for all t. Letting $m \to -\infty$, (2.13) implies (6.1). Finally, (6.2) is implied by (6.6), (6.1), and the semigroup property. ■

Proposition 6.2 *There exist c_1 and c_2 such that $q(t, x, y) \ge c_1 t^{-d/2}$ if $|x - y| \le c_2 t^{\frac{1}{2}}$, $t \le 1$.*

Proof Let $\overline{q}(t, x, y)$ be the transition density for W killed on exiting $\tilde{F}_0 \cap D_4(x_0)$. Let D' be the region above the graph of a Lipschitz function in some coordinate system such that $D' \cap D_4(x_0) = \tilde{F}_0 \cap D_4(x_0)$. Let $q'(t, x, y)$ be the transition densities for RBM in D'. Let S_i and U_i be defined as in Proposition 6.1, and as in (6.5),

$$q'(t, x, y) = \overline{q}(t, x, y) + \mathbb{E}^x\left[\int_0^t q'(t-s, W_{U_1}, y); U_1 \in ds\right], \quad x, y \in D_8(x_0).$$

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By [BH], Theorem 3.4,

$$q'(t,x,y) \geq c_3 t^{-d/2} \exp(-c_4|x-y|^2/t), \quad x,y \in D_8(x_0),$$

while [BH], Theorem 3.1 says that

$$q'(t-s,w,y) \leq c_5 t^{-d/2} \exp\bigl(-c_6|w-y|^2/(t-s)\bigr).$$

Since $\mathbb{P}^x(U_1 \leq t) \leq c_7 \exp(-c_8/t)$ by (5.13) for t sufficiently small and $x \in D_8(x_0)$, then if we take t_0 sufficiently small

$$\overline{q}(t,x,y) \geq c_9 t^{-d/2} \exp(-c_{10}|x-y|^2/t), \quad x,y \in D_8(x_0), \quad t \leq t_0,$$

and so

$$\overline{q}(t,x,y) \geq c_{11} t^{-d/2}, \quad x,y \in D_8(x_0), \quad |x-y| \leq c_{12} t^{\frac{1}{2}}, \quad t \leq t_0.$$

The proposition now follows by repeated applications of the semigroup property. ■

Lemma 6.3 *Let $x \in \tilde{F}_0$. Then writing $\sigma_{-n}(x) = \sigma_{-n}(x,W)$,*

$$\mathbb{P}^x\bigl(\sigma_{-n}(x) \leq t\bigr) \leq c_1 \exp\bigl(-c_2(t t_F^{-n})^{-1/(d_w-1)}\bigr), \quad t \geq l_F^n, \quad n \geq 0, \tag{6.7}$$

$$\mathbb{P}^x\bigl(\sigma_{-n}(x) \leq t\bigr) \leq c_3 \exp(-c_4 l_F^{2n}/t), \quad 0 \leq t \leq l_F^n, \quad n \in \mathbb{Z}, \tag{6.8}$$

$$\mathbb{P}^y\bigl(\sigma_{-n}(x) \geq t\bigr) \leq c_5 \exp(-c_6 t t_F^{-n}), \quad t > 0, \quad x \in \tilde{F}_0, \quad y \in D_{-n}(x), \quad n \geq 0. \tag{6.9}$$

Proof (6.7) and (6.8) are just restatements of (5.12) and (5.13). To prove (6.9), note first the estimate $\mathbb{E}^y \sigma_{-n}(y) \leq c_7 t_F^n$ given in (5.11). Let $t_0 = 2c_7^{-1} t_F^2$. Then $\mathbb{P}^y\bigl(\sigma_{-n-2}(y) > t_0 t_F^n\bigr) \leq \frac{1}{2}$ for $y \in \tilde{F}_0$. If $y \in D_{-n}(x)$, then $\sigma_{-n-2}(y) \geq \sigma_{-n}(x)$, so $\mathbb{P}^y\bigl(\sigma_{-n}(x) > t_0 t_F^n\bigr) \leq \frac{1}{2}$. Iterating and using the strong Markov property we deduce that $\mathbb{P}^y\bigl(\sigma_{-n}(x) > k t_0 t_F^n\bigr) \leq 2^{-k}$ for $k \geq 1$, and (6.9) follows easily. ■

Integrating these bounds we obtain:

Lemma 6.4

(a) There exists $c_1 = c_1(p)$ such that

$$\mathbb{E}^y \sigma_{-n}(x)^p \leq c_1 t_F^{pn}, \quad y \in D_{-n}(x), \quad p > 0, \quad n \geq 0. \tag{6.10}$$

(b) There exist constants c_2, c_3, such that

$$\mathbb{E}^x e^{-\lambda \sigma_{-n}(x)} \leq c_2 \exp\bigl(-c_3(\lambda t_F^n)^{1/d_w}\bigr), \quad 0 \leq \lambda \leq 1, \quad n \geq 0. \tag{6.11}$$

Lemma 6.5 *There exists c_1 such that if $x_0 \in \tilde{F}_0$ and $A \subset D_6(x_0)$ then*

$$\mathbb{E}^y \int_0^{\sigma_{-n}(x_0)} 1_A(W_s)\, ds \leq c_1 \mu_0(A) t_F^{n(1-d_s/2)}, \quad n \geq 0,$$
$$y \in \bigl(D_{-n}(x_0) - D_{-n+2}(x_0)\bigr) \cap \tilde{F}_0.$$

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Proof Let n, x_0 be fixed, let $\overline{q}_n(t,x,y)$ be the transition density of W killed on exiting $D_{-n}(x_0)$, and let $\overline{v}(x,y) = \int \overline{q}_n(t,x,y)\,dt$. So

$$\mathbb{E}^y \int_0^{\sigma_{-n}(x_0)} 1_A(W_s)\,ds = \int_A \overline{v}(y,z)\,\mu_0(dz). \tag{6.12}$$

Let $z \in A$. Then $\overline{v}(\cdot,z)$ is harmonic on $\big(D_{-n}(x_0) - \{z\}\big) \cap \tilde{F}_0$, and is zero on the boundary of $D_{-n}(x_0)$. Thus it is sufficient to bound (6.12) in the case $y \in \partial D_{-n+2}(x_0)$. So let $y_0 \in \partial D_{-n+2}(x_0) \cap \tilde{F}_0$. Note that if $m = -n+3$ then $D_m(y_0) \subset D_{-n}(x_0)$, and $D_m(y_0) \cap D_{-n+4}(x_0) = \varnothing$. So $\overline{v}(\cdot,z)$ is harmonic on $D_m(y_0)$ and applying Theorem 4.3 in $D_m(y_0)$ we have

$$\overline{v}(y_0,z) \le c_2 \overline{v}(y,z), \quad y \in D_{m+2}(y_0).$$

Therefore

$$\begin{aligned}
\overline{v}(y_0,z) &\le c_2 \mu_0\big(D_{m+2}(y_0)\big)^{-1} \int_{D_{m+2}(y_0)} \overline{v}(y,z)\,\mu_0(dy) \\
&\le c_3 m_F^{-n} \int_{D_{-n}(x_0)} \overline{v}(y,z)\,\mu_0(dy) \\
&= c_3 t_F^{-nd_s/2} \mathbb{E}^z \sigma_{-n}(x_0) \le c_4 t_F^{-nd_s/2} t_F^n.
\end{aligned}$$

As this estimate holds for any $z \in A$, using (6.12) proves the lemma. ■

Lemma 6.6 *Let $p > 0$. There exists $c_1 = c_1(p)$ such that if $x_0 \in \tilde{F}_0$ and $A \subset D_6(x_0)$ then*

$$\mathbb{E}^y \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_t)\,dt \le c_1 \mu_0(A) t_F^{n(1+p-d_s/2)}, \quad n \ge 0, \quad y \in \partial D_{-n+1}(x_0).$$

Proof Fix x_0 and A, and let $\overline{q}_n(t,x,y)$ be the transition density of W killed on exiting $D_{-n}(x_0)$. Set

$$w(x,y) = \int_0^\infty \overline{q}_n(t,x,y) t^p\,dt;$$

note that $w(x,y) = w(y,x)$. Set

$$g(y) = \mathbb{E}^y \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_t)\,dt = \int_A w(y,z)\,\mu_0(dz).$$

Let $y_0 \in \partial D_{-n+2}(x_0)$. Then $D_{-n+4}(y_0) \cap A = \varnothing$, and so if $y \in D_{-n+4}(y_0)$, then writing $T = \sigma_{-n+4}(y_0)$, and using Lemmas 6.4 and 6.5,

$$\begin{aligned}
g(y) &= \mathbb{E}^y \int_T^{\sigma_{-n}(x_0)} t^p 1_A(W_t)\,dt \\
&\le c_2 \mathbb{E}^y \left(T^p \mathbb{E}^{W_T} \int_0^{\sigma_{-n}(x_0)} 1_A(W_t)\,dt \right) + c_3 \mathbb{E}^y \left(\mathbb{E}^{W_T} \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_t)\,dt \right) \\
&\le c_4 \mu_0(A) (t_F^n)^{p+1-d_s/2} + c_3 \mathbb{E}^y g(W_T).
\end{aligned} \tag{6.13}$$

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Now let

$$h(y) = \mathbb{E}^y g(W_T).$$

h is harmonic on $D_{-n+4}(y_0)$, so applying Theorem 4.3 in $D_{-n+4}(y_0)$, we have

$$h(y_0) \le c_5 h(y), \quad y \in D_{-n+6}(y_0).$$

Using the definitions of g and h and the strong Markov property,

$$\begin{aligned} h(y) &= \mathbb{E}^y g(W_T) \\ &= \mathbb{E}^y \mathbb{E}^{W_T} \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_T)\, dt \\ &= \mathbb{E}^y \int_T^{\sigma_{-n}(x_0)} (t-T)^p 1_A(W_t)\, dt \\ &\le \mathbb{E}^y \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_T)\, dt = g(y). \end{aligned}$$

We deduce that

$$\begin{aligned} h(y_0) &\le c_5 \mu_0\big(D_{-n+6}(y_0)\big)^{-1} \int_{D_{-n+6}(y_0)} g(y)\, \mu_0(dy) \\ &\le c_6 m_F^{-n} \int_{D_{-n}(y_0)} \int_A w(y,z)\, \mu_0(dz)\, \mu_0(dy) \\ &= c_6 m_F^{-n} \int_A \mu_0(dz) \int_{D_{-n}(y_0)} w(z,y)\, \mu_0(dy). \end{aligned}$$

However,

$$\int_{D_{-n}(y_0)} w(z,y)\, \mu_0(dy) = \mathbb{E}^z \int_0^{\sigma_{-n}(x_0)} t^p\, dt = c_7 \mathbb{E}^z \sigma_{-n}(x_0)^{p+1} \le c_8 (t_F^n)^{1+p}.$$

Therefore

$$h(y_0) \le c_9 \mu_0(A) (t_F^n)^{1+p-d_s/2},$$

and using (6.13) it follows that

$$\mathbb{E}^{y_0} \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_t)\, dt \le c_{10} \mu_0(A) (t_F^n)^{1+p-d_s/2} \tag{6.14}$$

for $y_0 \in \partial D_{-n+2}(x_0) \cap \tilde{F}_0$. Now if $y_1 \in \partial D_{-n+1}(x_0)$, then applying (6.14) in the case $n+1$ we have

$$\begin{aligned} \mathbb{E}^{y_1} \int_0^{\sigma_{-n}(x_0)} t^p 1_A(W_t)\, dt &\le \mathbb{E}^{y_1} \int_0^{\sigma_{-n-1}(x_0)} t^p 1_A(W_t)\, dt \\ &\le c_{11} \mu_0(A) (t_F^n)^{1+p-d_s/2}, \end{aligned}$$

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proving the lemma. ■

Choose $p > d > d_s$, and for $\lambda > 0$ write

$$g_{\lambda,p}(x,y) = \int_0^\infty q(t,x,y)t^p e^{-\lambda t}\,dt.$$

Proposition 6.7 *There exists $c_1 = c_1(p)$ such that*

$$g_{\lambda,p}(x,x) \le c_1\lambda^{-p-1+d_s/2}, \quad x \in \tilde{F}_0, \lambda \le 1.$$

Proof (In this proof, we suppress the dependence of the constants c_i on p.) Fix $x \in \tilde{F}_0$, and write $\sigma_{-n} = \sigma_{-n}(x)$. Let $A \subset D_6(x)$, and set

$$\begin{aligned} V(A) &= \mathbb{E}^x \int_0^\infty 1_A(W_s)s^p e^{-\lambda s}\,ds \\ &= \mathbb{E}^x \int_0^{\sigma_0} 1_A(W_t)t^p e^{-\lambda t}\,dt + \sum_{n=0}^\infty \mathbb{E}^x \int_{\sigma_{-n}}^{\sigma_{-n-1}} 1_A(W_t)t^p e^{-\lambda t}\,dt. \end{aligned} \tag{6.15}$$

Write $\overline{q}(t,x,y)$ for the transition density of W killed at $\sigma_0(x)$. Then by (2.12)

$$\overline{q}(t,x,y) \le c_2 t^{-d/2}e^{-c_3 t}, \quad t > 0.$$

Therefore

$$\begin{aligned} \mathbb{E}^x \int_0^{\sigma_0(x)} 1_A(W_t)t^p e^{-\lambda t}\,dt &= \int_A \int_0^\infty \overline{q}(t,x,y)t^p e^{-\lambda t}\,dt\,\mu_0(dy) \\ &\le c_2 \int_A \mu_0(dy) \int_0^\infty t^{p-d/2}e^{-c_3 t}\,dt = c_4\mu_0(A). \end{aligned}$$

Write J_n for the n-th term of the sum in (6.15), and note that

$$\begin{aligned} J_n &= \mathbb{E}^x\left(e^{-\lambda\sigma_{-n}}\mathbb{E}^{W_{\sigma_{-n}}} \int_0^{\sigma_{-n-1}} 1_A(W_t)(\sigma_{-n}+t)^p e^{-\lambda t}\,dt\right) \\ &\le c_5\left(\mathbb{E}^x e^{-\lambda\sigma_{-n}}\sigma_{-n}^p \mathbb{E}^{W_{\sigma_{-n}}} \int_0^{\sigma_{-n-1}} 1_A(W_t)\,dt\right) \\ &\quad + c_5\mathbb{E}^x\left(e^{-\lambda\sigma_{-n}}\mathbb{E}^{W_{\sigma_{-n}}} \int_0^{\sigma_{-n-1}} t^p 1_A(W_t)\,dt\right). \end{aligned}$$

Using the bounds in Lemmas 6.4–6.6, we therefore have

$$\begin{aligned} J_n\mu_0(A)^{-1} &\le c_5\mathbb{E}^x(\sigma_{-n}^p e^{-\lambda\sigma_{-n}})t_F^{n(1-d_s/2)} + c_5\mathbb{E}^x(e^{-\lambda\sigma_{-n}})t_F^{n(1+p-d_s/2)} \\ &\le c_5 t_F^{n(1-d_s/2)}\Big(\big(\mathbb{E}^x(\sigma_{-n}^{2p})\big)^{1/2}\big(E^x(e^{-\lambda\sigma_{-n}})\big)^{1/2} + t_F^{np}E^x e^{-\lambda\sigma_{-n}}\Big) \\ &\le c_6 t_F^{n(1+p-d_s/2)}\exp\big(-c_7(\lambda t_F^n)^{1/d_w}\big). \end{aligned}$$

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Writing $\beta = 1 + p - d_s/2$, we deduce from the calculations above that

$$V(A) \le c_8 \mu_0(A)\Big(1 + \sum_{n=0}^{\infty} (t_F^{\beta})^n \exp\big(-c_9(\lambda t_F^n)^{1/d_w}\big)\Big).$$

Let m_0 be such that $t_F^{m_0} \ge 1/\lambda > t_F^{m_0-1}$. Then, as $\beta > 0$, the sum above is bounded by $c_{10}\mu_0(A) t_F^{m_0\beta}$. So

$$V(A) = \int_A g_{\lambda,p}(x,y)\,\mu_0(dy) \le c_{11}\mu_0(A)\lambda^{-1-p+d_s/2},$$

and as $g_{\lambda,p}(x,y)$ is continuous, this proves the proposition. ■

Theorem 6.8 *For $x \in \tilde{F}_0$,*

$$q(t,x,y) \le \begin{cases} c_1 t^{-d/2}, & 0 < t \le 1, \\ c_1 t^{-d_s/2}, & 1 \le t < \infty. \end{cases}$$

Proof The bound for $t \in (0,1]$ is given by (6.2). If $t \ge 1$, let $\lambda = t^{-1}$. Since $q(t,x,x)$ is non-increasing in t,

$$\begin{aligned} c_2 t^{1+p-d_s/2} &\ge g_{1/t,p}(x,x) \\ &\ge \int_{t/2}^{t} q(s,x,x) s^p e^{-s/t}\,ds \\ &\ge q(t,x,x)\int_{t/2}^{t} s^p e^{-s/t}\,ds = c_3 t^{1+p} q(t,x,x). \end{aligned}$$

This proves the theorem if $x = y$, and the general case now follows from (6.1). ■

Theorem 6.9 *There exist $c_1, \dots, c_8 \in (0,\infty)$ such that if $x, y \in \tilde{F}_0$ and*

(a) $t \ge 1$, $|x-y| \le t$, then

$$\begin{aligned} &c_1 t^{-d_s/2} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ &\qquad \le q(t,x,y) \le c_3 t^{-d_s/2} \exp\left(-c_4\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right); \end{aligned} \tag{6.16}$$

(b) if $t \ge 1$, $|x-y| \ge t$, or if $t \le 1$, then writing $D(t) = d$, $t \le 1$, $D(t) = d_s$ for $t > 1$,

$$c_5 t^{-D(t)/2} \exp(-c_6|x-y|^2/t) \le q(t,x,y) \le c_7 t^{-D(t)/2} \exp(-c_8|x-y|^2/t). \tag{6.17}$$

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Proof The argument for the upper bounds in (a) and (b) follows that of [BB4] quite closely.

Fix x, y, t, and choose small neighborhoods C_x, C_y of x and y. Let $A_x = \{z \in \tilde{F}_0 : |x-z| \le |x-y|\}$, and define A_y similarly. Let $\nu_x = \mu_0|_{C_x}$, $\nu_y = \mu_0|_{C_y}$, and let

$$S = \inf\{t \ge 0 : |W_t - W_0| > \frac{1}{3}|x-y|\}.$$

As $\mu_0(A_x \cap A_y) = 0$,

$$\begin{aligned}\int_{C_y}\int_{C_x} q(t,x',y')\,\mu_0(dx')\,\mu_0(dy') &= \mathbb{P}^{\nu_x}(W_t \in C_y) \\ &= \mathbb{P}^{\nu_x}(W_t \in C_y, W_{t/2} \in A_x) \\ &\quad + \mathbb{P}^{\nu_x}(W_t \in C_y, W_{t/2} \in A_y).\end{aligned} \tag{6.18}$$

We begin with the second term in (6.18).

$$\begin{aligned}\mathbb{P}^{\nu_x}(W_t \in C_y, W_{t/2} \in A_y) &= \mathbb{P}^{\nu_x}(S < t/2, W_t \in C_y, W_{t/2} \in A_y) \\ &\le \mathbb{P}^{\nu_x}(S < t/2) \sup_{y' \in A_y} \mathbb{P}^{y'}(W_{t/2} \in C_y) \\ &\le \mu_0(C_x) \sup_{x' \in C_x} \mathbb{P}^{x'}(S < t/2) \sup_{y' \in A_y} \mathbb{P}^{y'}(W_{t/2} \in C_y).\end{aligned} \tag{6.19}$$

For the first term in (6.18) we have, using the symmetry of W,

$$\mathbb{P}^{\nu_x}(W_t \in C_y, W_{t/2} \in A_x) = \mathbb{P}^{\nu_y}(W_t \in C_x, W_{t/2} \in A_x),$$

which can be bounded in exactly the same way as (6.19). Therefore, as q is continuous it follows from (6.18), (6.19) and (6.1) that

$$q(t,x,y) \le 2 \sup_{x'} \mathbb{P}^{x'}(S < t/2) \sup_{z} q(t/2, z, z). \tag{6.20}$$

The upper bounds in (a) and (b) now follow immediately from (6.20) on using the bounds given in Lemma 6.3 and Theorem 6.8.

We now turn to the lower bounds in (a) and (b). Using a standard chaining argument such as in the proof of Theorem 7.4 of [BB4], these can be proved once we establish the estimates

$$q(t,x,y) \ge c_9 t^{-d_s/2}, \quad |x-y| \le c_{10} t^{1/d_w}, \quad t \ge 1, \tag{6.21}$$

$$q(t,x,y) \ge c_{11} t^{-d/2}, \quad |x-y| \le c_{12} t^{1/2}, \quad t \le 1. \tag{6.22}$$

(6.22) is proved in Proposition 6.2, so we just prove (6.21).

We deduce from the upper bound (6.16) that there exists c_{13} such that

$$q(t,x,x) \ge c_{13} t^{-d_s/2}, \quad x \in \tilde{F}_0, \quad t \ge 1; \tag{6.23}$$

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the proof is as follows (*cf.* Lemma 7.1 in [BB4]). By (5.19)

$$\mathbb{P}^x\big(\sigma_r(x) \le t\big) \le c_{14} \exp\big(-c_{15}(t_F^r t)^{-1/(d_w-1)}\big).$$

Pick a such that $c_{14} \exp(-c_{15} a^{-1/(d_w-1)}) \le \frac{1}{2}$ and let $r = [\log(2a/t)/\log t_F]$. Then

$$\mathbb{P}^x\big(W_s \in D_r(x)\big) \ge \mathbb{P}^x\big(\sigma_r(x) > t/2\big) \ge \frac{1}{2}$$

and

$$\mu_0\big(D_r(x)\big) \le c_{16} m_F^{-r} \le c_{17} t^{d_s/2}.$$

By Cauchy-Schwarz,

$$\begin{aligned} 1/4 \le \big[\mathbb{P}^x\big(W_{t/2} \in D_r(x)\big)\big]^2 &= \left(\int_{D_r(x)} q(t/2,x,y)\,\mu_0(dy)\right)^2 \\ &\le \mu_0\big(D_r(x)\big) \int_{D_r(x)} q(t/2,x,y)^2\,\mu_0(dy) \\ &\le \mu_0\big(D_r(x)\big) q(t,x,x). \end{aligned}$$

(6.23) now follows immediately.

By (6.16) there exists c_{18} such that

$$q(t,x,y) \le c_{18} t^{-d_s/2}, \quad x,y \in \tilde{F}_0.$$

Now fix $t \ge 1$ and $x \in \tilde{F}_0$. Take $c_{19} = (c_{13} \wedge c_{18})/2$ and let $A = \{y : q(t/2,x,y) > c_{19} t^{-d_s/2}\}$. Then

$$\begin{aligned} c_{13} t^{-d_s/2} \le q(t,x,x) &= \int q(t/2,x,y) q(t/2,x,y)\,\mu_0(dy) \\ &\le \int_A c_{18} t^{-d_s/2} q(t/2,x,y)\,\mu_0(dy) + \int_{A^c} c_{19} t^{-d_s/2} q(t/2,x,y)\,\mu_0(dy). \end{aligned}$$

Therefore

$$\begin{aligned} c_{13} &\le c_{18} \mathbb{P}^x(W_{t/2} \in A) + c_{19} \mathbb{P}^x(W_{t/2} \in A^c) \\ &= c_{19} + (c_{18} - c_{19}) \mathbb{P}^x(W_{t/2} \in A). \end{aligned}$$

So, if $c_{20} = (c_{13} - c_{19})/(c_{18} - c_{19})$, then $\mathbb{P}^x(W_{t/2} \in A) \ge c_{20}$.

Let $\varepsilon = c_{20}/4$, and choose m such that

$$\mathbb{P}^y\big(\sigma_m(x) \ge t/2\big) < \varepsilon, \quad y \in D_m(x).$$

By (6.9) this will hold if m is chosen so that $t t_F^m \ge c_{21} \ge t t_F^{m-1}$, for a suitable c_{21}. Write $D = D_m(x)$. As ε depends only on d and l_F, by Theorem 3.25(b) there exists $\delta > 0$,

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depending only on d and l_F, such that if $|x-y| \le \delta l_F^{-m}$, then there exist RBM($\tilde{F}_0$), W^x, W^y, starting at x, y, respectively, such that

$$\mathbb{P}\big(T_C(W^x, W^y) < \mathcal{T}_D(W^x) \wedge \mathcal{T}_D(W^y)\big) > 1-\varepsilon.$$

Therefore, writing $T_C = T_C(W^x, W^y)$,

$$\begin{aligned} 1-\varepsilon &< \mathbb{P}\big(T_C < \mathcal{T}_D(W^x) \wedge \mathcal{T}_D(W^y)\big) \\ &\le \mathbb{P}(T_C \le t/2) + P\big(\mathcal{T}_D(W^x) \ge t/2\big) + P\big(\mathcal{T}_D(W^y) \ge t/2\big), \end{aligned}$$

so that

$$\mathbb{P}(T_C \le t/2) \ge 1 - 3\varepsilon.$$

Thus

$$\begin{aligned} \mathbb{P}(W^y_{t/2} \in A) &\ge \mathbb{P}(W^x_{t/2} \in A, T_C \le t/2) \\ &\ge \mathbb{P}(W^x_{t/2} \in A) - \mathbb{P}(T_C > t/2) \\ &\ge c_{20} - 3\varepsilon = c_{20}/4. \end{aligned}$$

So if $|x-y| \le \delta l_F^{-m}$, then

$$\begin{aligned} q(t,x,y) &\ge \int_A q(t/2,x,z)q(t/2,z,y)\,\mu_0(dz) \\ &\ge c_{19}t^{-d_s/2}\mathbb{P}^y(W_{t/2} \in A) \ge c_{19}c_{20}t^{-d_s/2}/4. \end{aligned}$$

Since $l_F^{-m} \le c_{21}t^{1/d_w}$ and δ depends only on d and l_F, the estimate (6.21) follows. ■

Observe that Theorem 6.9 is the same as Theorem 1.4.

For $\lambda \ge 0$ let

$$v^\lambda(x,y) = \int_0^\infty q(t,x,y)e^{-\lambda t}\,dt$$

be the λ-potential kernel density for W. Let $v = v^0$; v is the Green function for W. Integrating the bounds in Theorem 6.9 we have:

Corollary 6.10 *Let $d_s > 2$. Then*

$$\begin{aligned} c_1|x-y|^{d-2} \le v(x,y) \le c_2|x-y|^{d-2} \quad &\text{if } |x-y| \le 1, \\ c_3|x-y|^{d_w-d_f} \le v(x,y) \le c_4|x-y|^{d_w-d_f} \quad &\text{if } |x-y| \ge 1. \end{aligned}$$

Recall that for $n \ge 0$ the process X^n on $\tilde{F}_n$ is given by

$$X^n_t = l_F^{-n} W_{t(t^n_F)}, \quad t \ge 0. \tag{6.24}$$

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Let $p_n(t,x,y)$ be the transition density of X^n with respect to μ_n. Then from (6.24) we have the scaling relation

$$p_n(t,x,y) = m_F^n q(t_F^n t, l_F^n x, l_F^n y). \tag{6.25}$$

Theorem 6.9(a) therefore implies:

Corollary 6.11 *There exist $c_1, c_2, c_3, c_4 \in (0,\infty)$ such that if $n \geq 0$, $x, y \in \tilde{F}_n$ and $t \geq t_F^{-n}$, $|x-y| \leq l_F^{n(d_w-1)} t$, then*

$$\begin{aligned} c_1 t^{-d_s/2} \exp\left(-c_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ \leq p_n(t,x,y) \leq c_3 t^{-d_s/2} \exp\left(-c_4 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right). \end{aligned} \tag{6.26}$$

Remark 6.12 Taking a limit along a subsequence shows that the density $p(t,x,y)$ of X_t with respect to μ exists and gives the bounds in Theorem 1.3.

Define the λ-resolvent of X by $U^\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda s} f(X_s)\, ds$ for $\lambda \geq 0$ and write U for U^0. Set also

$$u^\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)\, dt,$$

and write $u(x,y) = u^0(x,y)$. Then u^λ is the density of the U^λ, and u is a Green function for X. The following bounds for $u^\lambda(x,y)$ are obtained by integrating those for $p(t,x,y)$; in (b) and (c) below one uses Laplace's method.

Corollary 6.13

(a) If $d_s > 2$ then

$$c_1|x-y|^{d_w-d_f} \leq u(x,y) \leq c_2|x-y|^{d_w-d_f}, \quad x, y \in \tilde{F}. \tag{6.27}$$

(b) If $d_s < 2$ then for $\lambda > 0$, $x, y \in \tilde{F}$,

$$c_3 \lambda^{d_s/2-1} \exp(-c_4 \lambda^{1/d_w}|x-y|) \leq u^\lambda(x,y) \leq c_5 \lambda^{d_s/2-1} \exp(-c_6 \lambda^{1/d_w}|x-y|).$$

(c) If $d_s = 2$ then writing $r = \lambda^{1/d_w}|x-y|$,

$$c_7\big(\log^+(1/r) + e^{-c_8 r}\big) \leq u^\lambda(x,y) \leq c_9\big(\log^+(1/r) + e^{-c_{10} r}\big).$$

Since there are only countably many GSCs, and (as far as we know) there is no simple expression for d_s, it is unlikely that any GSC actually satisfies case (c) above. However, we include it for completeness.

We now show that $p(t,x,y)$ is C^∞ in t and each of the partial derivatives is Hölder continuous in x and y. Let $m < -2$ be fixed and let $\overline{P}_t$ be defined on $D_m(0)$ by

$$\overline{P}_t f(x) = \mathbb{E}^x\big[f(X_t); t < \sigma_m(0)\big].$$

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Let $\overline{p}(t,x,y)$ be the corresponding transition density for X_t killed on exiting $D_m(0)$. Since $\overline{p}(t,x,y) \le p(t,x,y)$, we have

$$\overline{p}(t,x,y) \le c_1 t^{-d_s/2}, \quad x,y \in D_m(0) \cap \tilde{F}, \quad t \ge 0. \tag{6.28}$$

Let $\overline{u}^\lambda(x,y) = \int_0^\infty e^{-\lambda t}\overline{p}(t,x,y)\,dt$ and let $\overline{U}^\lambda$ be the corresponding resolvent operator. Write $\overline{U}$ for $\overline{U}^0$.

Proposition 6.14

(a) There exist c_1 and β not depending on m such that if f is bounded, then

$$|\overline{U}^\lambda f(x) - \overline{U}^\lambda f(y)| \le c_1(1+\lambda^{-1})|x-y|^\beta \|f\|_\infty, \quad x,y \in D_{m+2}(0) \cap \tilde{F}.$$

(b) $\overline{U}^\lambda f$ is continuous in $D_m(0) \cap \tilde{F}$ with a modulus of continuity that depends only on λ, m and $\|f\|_\infty$.

Proof (a) Let $\overline{U}_{D_n(x_0)}$ be the Green operator for X_t killed on exiting $D_n(x_0)$. If $x_0 \in D_m(0)$ and $D_n(x_0) \cap \tilde{F} \subset D_m(0) \cap \tilde{F}$, then for bounded h

$$\overline{U}h(x) = \overline{U}_{D_n(x_0)}h(x) + \mathbb{E}^x \overline{U}h(X_{\sigma_n(x_0)}), \quad x \in D_n(x_0). \tag{6.29}$$

The second term is harmonic in $D_{n+2}(x_0)$, and so by Theorem 5.9 there exist c_2 and α (independent of m) such that

$$|\mathbb{E}^x \overline{U}h(X_{\sigma_n(x_0)}) - \mathbb{E}^y \overline{U}h(X_{\sigma_n(x_0)})| \le c_2|x-y|^\alpha l_F^{m\alpha}\|\overline{U}h\|_\infty$$

if $x,y \in D_{n+2}(x_0)$. On the other hand, by (5.20),

$$\|\overline{U}_{D_n(x_0)}h\|_\infty \le c_3 t_F^{-n}\|h\|_\infty.$$

Taking $x = x_0$ we therefore deduce that if n, y satisfy

$$D_n(x) \cap \tilde{F} \subset D_m(0) \cap \tilde{F}, \quad y \in D_{n+2}(x) \tag{6.30}$$

then

$$\begin{aligned} |\overline{U}h(x) - \overline{U}h(y)| &\le 2c_3 t_F^{-n}\|h\|_\infty + c_2|x-y|^\alpha l_F^{m\alpha}\|\overline{U}h\|_\infty, \\ &\le c_4(\|h\|_\infty + \|\overline{U}h\|_\infty)(t_F^{-n} + l_F^{-n}|x-y|^\alpha). \end{aligned} \tag{6.31}$$

Now suppose $x \in D_{m+2}(0)$ and $|x-y| \le l_F^{-2}/2$. Then if $n \ge 0$, (6.30) is satisfied. Choosing n be the greatest integer less than $\alpha \log|x-y|^{-1}/\big((\alpha+d_w)\log l_F\big)$ it follows from (6.31) that

$$|\overline{U}h(x) - \overline{U}h(y)| \le c_5(\|h\|_\infty + \|\overline{U}h\|_\infty)|x-y|^\beta,$$

where $\beta = \alpha d_w(\alpha + d_w)^{-1}$.

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By the resolvent identity,

$$\overline{U}^\lambda f = \overline{U}(f - \lambda \overline{U}^\lambda f).$$

Set $h = f - \lambda \overline{U}^\lambda f$; as $\|\overline{U}^\lambda f\|_\infty \le \lambda^{-1}\|f\|_\infty$, we deduce $\|h\|_\infty \le 2\|f\|_\infty$. As $\overline{U}h = \overline{U}^\lambda f$, $\|\overline{U}h\|_\infty \le \lambda^{-1}\|f\|_\infty$, and so $\|h\|_\infty + \|\overline{U}h\|_\infty \le (2+\lambda^{-1})\|f\|_\infty$. Therefore

$$|\overline{U}^\lambda f(x) - \overline{U}^\lambda f(y)| \le c_6(1+\lambda^{-1})|x-y|^\beta \|f\|_\infty.$$

Finally, if $|x-y| \le 1$ then we can find a chain $x = x_0, x_1 \cdots x_k = y$ with $|x_{i-1} - x_i| \le l_F^{-2}/2$ and $k \le c_7(d, l_F)$. So, using the triangle inequality (a) follows.

(b) Let $A = \partial_r D_m(0)$ be the relative boundary of $D_m(0) \cap \tilde{F}$ in $\tilde{F}$. Write $D = D_m(0) \cap \tilde{F} - A$. If $x \in D$ then there exists $n_0 \ge 2$ such that $D_{n_0}(x) \cap \tilde{F} \subset D$. If $y \in D_{n_0+2}(x) \cap \tilde{F}$ and $y \ne x$ choose n so that $y \in D_{2n}(x) - D_{2n+2}(x)$. Then (6.30) holds, and so by (6.31)

$$\begin{aligned} |\overline{U}h(x) - \overline{U}h(y)| &\le c_8(\|h\|_\infty + \|\overline{U}h\|_\infty)(t_F^{-n} + l_F^{-n\alpha}) \\ &\le c_9(\|h\|_\infty + \|\overline{U}h\|_\infty)|x-y|^\gamma \end{aligned}$$

for some $\gamma > 0$. Thus $\overline{U}h$ is continuous on D.

We now show $\overline{U}h$ vanishes continuously on A. For $n \ge 1$ set $B_n = \bigcup_{z\in A} D_n(z) \cap \tilde{F}$, $b_n = \sup_{x\in B_n} |\overline{U}h(x)|$. Note that by (5.20) $b_1 \le c_3 t_F^{-m}\|h\|_\infty$.

By Corollary 3.24, there exists δ depending only on l_F, d such that

$$\mathbb{P}^x\big(X_{\sigma_n(x)} \notin D_m(0)\big) \ge \delta, \quad x \in B_{n+2}.$$

Since $\overline{U}h = 0$ outside $D_m(0)$,

$$\begin{aligned} |\overline{U}h(x)| &\le c_3 t_F^{-n}\|h\|_\infty + |\mathbb{E}^x \overline{U}h(X_{\sigma_n(x)})| \\ &\le c_3 t_F^{-n}\|h\|_\infty + (1-\delta)b_{n-2}, \quad x \in B_{n+2}. \end{aligned}$$

Thus

$$b_{n+2} \le c_3 t_F^{-n}\|h\|_\infty + (1-\delta)b_n.$$

Thus $b_n \to 0$ as $n \to \infty$. We therefore have that $\overline{U}h$ is continuous on $D_m(0) \cap \tilde{F}$ with a modulus of continuity depending on m and $\|h\|_\infty$. Using the resolvent identity as in (a), we obtain (b). ■

Proposition 6.15 *There exist reals $0 \le \lambda_1 \le \lambda_2 \le \cdots$ and bounded continuous functions φ_i on $D_m(x_0)$ such that*

(a) the only subsequential limit point of $\{\lambda_i\}$ is ∞;
(b) the φ_i form a complete orthonormal system for $L^2\big(D_m(x_0), \mu\big)$;
(c) we have the expansion

$$\overline{p}(t,x,y) = \sum_{i=1}^\infty e^{-\lambda_i t}\varphi_i(x)\varphi_i(y) \tag{6.32}$$

where the convergence is absolute and uniform for each $t > 0$.

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Proof We first show that if $\overline{P}_t f = 0$ a.e. for $f \in L^2$, then $f = 0$ a.e. By [FOT], Lemma 1.4.3 the semigroup $(\overline{P}_t)$ is strongly continuous on $L^2(F, \mu)$, so that $\|\overline{P}_t f - f\|_2 \to 0$ as $t \downarrow 0$. If $\overline{P}_t f = 0$, then

$$0 = \int f(\overline{P}_t f) = \int (\overline{P}_{t/2} f)(\overline{P}_{t/2} f),$$

or $\overline{P}_{t/2} f = 0$ a.e. By induction $\overline{P}_{t/2^n} f = 0$ a.e., and so $f = 0$ a.e.

$\overline{p}(t, x, y)$ is symmetric and bounded. Since $\mu\big(D_m(x_0)\big) < \infty$, then

$$\int_{D_m(x_0)} \int_{D_m(x_0)} \overline{p}(t, x, y)^2 \, \mu(dx) \, \mu(dy) < \infty.$$

By [RS-N], $\overline{P}_t$ is a self-adjoint completely continuous operator on L^2, and by the proofs in [Bas2, Sect. II.4], there exist $\tilde{\varphi}_i$ that form a complete orthonormal system of functions in L^2 with

$$\overline{p}(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \tilde{\varphi}_i(x) \tilde{\varphi}_i(y)$$

for μ^2 a.e. pair (x, y). Moreover the convergence is absolute and takes place in $L^\infty\big(D_m(x_0) \times D_m(x_0)\big)$, and (a) holds.

Next note $\overline{P}_t \tilde{\varphi}_i = e^{-\lambda_i t} \tilde{\varphi}_i$ a.e., so $\overline{U}^\lambda \tilde{\varphi}_i = (\lambda + \lambda_i)^{-1} \tilde{\varphi}_i$ a.e. Setting $\varphi_i = (\lambda + \lambda_i) \overline{U}^\lambda \tilde{\varphi}_i$, we have $\varphi_i = \tilde{\varphi}_i$ a.e., while by Proposition 6.14 φ_i is continuous.

If $s < t$,

$$e^{\lambda_i t} \overline{P}_t \tilde{\varphi}_i(x) = e^{\lambda_i s} \overline{P}_s (e^{\lambda_i (t-s)} \overline{P}_{t-s} \tilde{\varphi}_i)(x) = e^{\lambda_i s} \overline{P}_s \tilde{\varphi}_i(x)$$

because $e^{\lambda_i (t-s)} \overline{P}_{t-s} \tilde{\varphi}_i = \tilde{\varphi}_i$ a.e. and $\overline{P}_s$ has a density. It follows that for each x and i, $e^{\lambda_i t} \overline{P}_t \tilde{\varphi}_i(x)$ is constant in t. Writing $w_i(x)$ for $e^{\lambda_i t} \overline{P}_t \tilde{\varphi}_i(x)$,

$$\overline{U}^\lambda \tilde{\varphi}_i(x) = \int_0^t e^{-\lambda t} \overline{P}_t \tilde{\varphi}_i(x) \, dt = \int_0^\infty e^{-\lambda t} e^{-\lambda_i t} w_i(x) \, dt = (\lambda + \lambda_i)^{-1} w_i(x).$$

Hence

$$e^{\lambda_i t} \overline{P}_t \tilde{\varphi}_i(x) = w_i(x) = \varphi_i(x)$$

for all t and x. Since $\int \tilde{\varphi}_i^2(x) \, \mu(dx) = 1$, from Cauchy-Schwarz and Remark 6.12,

$$\varphi_i(x) = e^{\lambda_i t} \overline{P}_t \tilde{\varphi}_i(x) \le e^{\lambda_i t} \left(\int \tilde{\varphi}_i^2(y) \, \mu(dy) \right)^{\frac{1}{2}} \left(\int_{D_m(x_0)} \overline{p}(t, x, y)^2 \, \mu(dy) \right)^{\frac{1}{2}},$$

and so φ_i is bounded in x.

Let

$$r(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y).$$

As the φ_i are orthonormal, then

$$\int r(t/2, x, z) r(t/2, z, y) \, \mu(dz) = \sum_{i=1}^{\infty} e^{-\lambda_i t/2} e^{-\lambda_i t/2} \varphi_i(x) \varphi_i(y) = r(t, x, y). \tag{6.33}$$

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If $f = \sum a_i \varphi_i \in L^2$,

$$\int f(y) r(t,x,y)\,\mu(dy) = \sum e^{-\lambda_i t} a_i \varphi_i(x),$$

while

$$\overline{P}_t f(x) = \sum a_i \overline{P}_t \tilde{\varphi}_i(x) = \sum a_i e^{-\lambda_i t} \varphi_i(x).$$

Thus $r(t,x,\cdot)$ is a transition density for $\overline{P}_t$ and $r(t,x,\cdot) = \overline{p}(t,x,\cdot)$ a.e. We then have by (6.33) that

$$\begin{aligned}\overline{p}(t,x,y) &= \int \overline{p}(t/2,x,z)\overline{p}(t/2,z,y)\,\mu(dz)\\ &= \int r(t/2,x,z) r(t/2,z,y)\,\mu(dy)\\ &= r(t,x,y).\end{aligned}$$

From [Bas2, Sect. II.4], the sum in (6.33) is absolutely and uniformly convergent; consequently $\overline{p}(t,x,y) = r(t,x,y)$ is jointly continuous in x and y. ■

Fix $t_0 > 0$, write $\lambda = t_0^{-1}$, and let

$$s_k(x,y) = \sum_{i=1}^{\infty} (\lambda + \lambda_i)(-\lambda_i)^k e^{-\lambda_i t_0} \varphi_i(x)\varphi_i(y), \quad k = 0,1,2,\ldots,$$

$$S_k f(x) = \int f(y) s_k(x,y)\,\mu(dy).$$

Proposition 6.16 *There exist constants $c_1(k)$ independent of m such that if $f \in L^1$ then*

$$\|S_k f\|_\infty \le c_1(k) t_0^{-k-1-d_s/2} \|f\|_1.$$

Proof Note first that $\sup_{r\ge 0}(\lambda + r) r^k e^{-rt_0/2} = c_1(k) t_0^{-k-1}$ for some function $c_1(k)$. So, using Cauchy-Schwarz,

$$\begin{aligned}|s_k(x,y)| &\le \sum_{i=1}^{\infty} (\lambda+\lambda_i)\lambda_i^k e^{-\lambda_i t_0} |\varphi_i(x)|\,|\varphi_i(y)|,\\ &\le \Big(\sum_{i=1}^{\infty} (\lambda+\lambda_i)\lambda_i^k e^{-\lambda_i t_0} \varphi_i(x)^2\Big)^{1/2} \Big(\sum_{i=1}^{\infty} (\lambda+\lambda_i)\lambda_i^k e^{-\lambda_i t_0} \varphi_i(y)^2\Big)^{1/2}\\ &\le \sup_x \Big(\sup_i (\lambda+\lambda_i)\lambda_i^k e^{-\lambda_i t_0/2} \sum_{i=1}^{\infty} e^{-\lambda_i t_0/2} \varphi_i(x)^2\Big)\\ &\le c_1(k) t_0^{-k-1} \sup_x \overline{p}(t_0/2,x,x)\\ &\le c_2(k) t_0^{-k-1-d_s/2}.\end{aligned}$$

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The result is now immediate. ■

Let ∂_t be an abbreviation for $\partial/\partial t$ and ∂_t^k an abbreviation for $\partial^k/\partial t^k$.

Proposition 6.17 *For each $k \geq 0$, $t > 0$, $\partial_t^k \overline{p}(t, x, y)$ is continuous in x and y on $D_m(0) \cap \tilde{F}$. Further there exist constants $c_1(k)$, $c_2(k)$, β, not depending on m, such that*

$$|\partial_t^k \overline{p}(t, x, y)| \leq c_1(k) t^{-k-d_s/2}, \quad x, y \in D_m(0) \cap \tilde{F}, \tag{6.34}$$

$$|\partial_t^k p(t, x, y) - \partial_t^k p(t, x', y)| \leq c_2(k)(1+t)|x - x'|^\beta t^{-k-1-d_s/2} \tag{6.35}$$
$$\textit{for } x, x', \quad y \in D_m(0) \cap \tilde{F}, \quad |x - x'| \leq 1.$$

Proof Let $f(z) = \overline{p}(s, z, y)$. Then

$$\overline{P}_{t_0} f(x) = \int \overline{p}(t_0, x, z) \overline{p}(s, z, y)\, \mu(dz) = \overline{p}(t_0 + s, x, y).$$

Therefore, using the eigenvalue expansion,

$$\partial_t^k \overline{p}(t_0 + s, x, y) = \partial_t^k \overline{P}_{t_0} f(x) = \overline{U}^\lambda S_k f(x).$$

Thus $\partial_t^k \overline{p}(t_0 + s, x, y)$ is continuous on $D_m(0)$ by Proposition 6.14(b). As $\|f\|_1 = 1$ we have, using Proposition 6.16,

$$\begin{aligned} |\partial_t^k \overline{p}(t_0 + s, x, y)| &\leq \|\overline{U}^\lambda S_k f\|_\infty \\ &\leq \lambda^{-1} \|S_k f\|_\infty \\ &\leq \lambda^{-1} c_3(k) t_0^{-1-k-d_s/2}. \end{aligned}$$

If $x, x' \in D_{m+2}(0) \cap \tilde{F}$ and $|x - x'| \leq 1$ then by Proposition 6.14(a)

$$\begin{aligned} |\partial_t^k \overline{p}(t_0 + s, x, y) - \partial_t^k \overline{p}(t_0 + s, x', y)| &\leq |\overline{U}^\lambda S_k f(x) - \overline{U}^\lambda S_k f(x')| \\ &\leq c_4(1 + \lambda^{-1})|x - x'|^\beta \|S_k f\|_\infty \\ &\leq c_5(k)(1 + t_0)|x - x'|^\beta t_0^{-1-k-d_s/2}. \end{aligned}$$

The result follows immediately from these bounds. ■

Theorem 6.18 *There exist constants $c_1(k), c_2(k)$, depending only on k, and $\beta > 0$ such that for $x, x', y \in \tilde{F}$, $t > 0$, $k \geq 0$*

$$|\partial_t^k p(t, x, y)| \leq c_1(k) t^{-k-d_s/2}. \tag{6.36}$$

Writing $R = |x - x'| t^{-1/d_w}$,

$$|\partial_t^k p(t, x, y) - \partial_t^k p(t, x', y)| \leq c_2(k) \max(R^\beta, R^{d_w}) t^{-k-d_s/2}. \tag{6.37}$$

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Proof As the bounds in Proposition 6.17 are independent of m it follows by the argument of [BB4, Sect. 4], that $\partial_t^k p(t,x,y)$ is continuous and satisfies the bounds (6.34) and (6.35). This proves (6.36).

To prove (6.37) we use weak scaling. Let $X'_t = l_F^n X(t/t_F^n)$, and let $p'(t,x,y)$ be the transition density of X'. Then p' also satisfies (6.34) and (6.35), and

$$p(t,x,y) = m_F^n p'(t_F^n t, l_F^n x, l_F^n y).$$

Write $\theta = l_F^n$; if $\theta|x-x'| \le 1$, then using (6.35) it follows that

$$|\partial_t^k p(t,x,y) - \partial_t^k p(t,x',y)| \le c_3(k)(\theta^{-d_w}+t)\theta^\beta |x-x'|^\beta t^{-k-1-d_s/2}.$$

Optimizing over $\theta \in \{l_F^n, n \in \mathbb{Z}\}$ we obtain (6.37). ■

Proof of Theorem 1.3 The continuity and symmetry of $p(t,x,y)$ follow from Propositions 6.15 and 6.17, while the bounds in (1.4) follow from Corollary 6.11. ■

Proof of Theorem 1.2 Combine Theorem 1.3 and Theorem 5.7. ■

Remark 6.19 We can use the spectral theorem to obtain an expansion of $p(t,\cdot,\cdot)$ on $\tilde{F}\times\tilde{F}$.

Proposition 6.20 *Suppose $\overline{p}(t,x,y) = \overline{p}_{D_0(x_0)}(t,x,y)$ is the transition density for X_t killed on exiting $D_0(x_0)$. Let $t_0 \in [1, l_F^d]$ and $c_1 > 0$. Then there exists c_2 such that $\overline{p}(t,x,y) \ge c_2 t_0^{-d_s/2}$ if $|x-y| \le c_1 t_0^{1/d_w}$ and $x, y \in D_2(x_0)$.*

Proof We have $\overline{p}(t,x,y) \le p(t,x,y) \le c_3 t^{-d_s/2}$ by Theorem 1.3. Just as in the proof of (6.23) there exists $r > 2$ such that

$$\overline{p}(t,x,x) \ge c_4 t^{-d_s/2}$$

provided $x \in D_r(x_0)$. Just as in the proof of (6.21), we have

$$\overline{p}(t,x,y) \ge c_5 t^{-d_s/2}, \quad |x-y| \le c_6 t^{1/d_w}, \quad x, y \in D_{r+2}(x_0).$$

Finally we obtain our proposition by using the chaining argument in the proof of Theorem 7.4 of [BB4]. ■

Proposition 6.21 *Suppose $\widehat{p}(t,x,y) = \widehat{p}_{D_0(x_0)}(t,x,y)$ is the transition density for X_t with reflection on $\partial D_0(x_0)$. There exists c_1 such that*

$$\widehat{p}(t,x,y) \ge c_1 t^{-d_s/2}, \quad t \ge 1.$$

Proof The proof is very similar to that for the unreflected process. In view of (5.26), if $x \in D_{r+2}(y)$, then $D_{r+4}(y) \subseteq D_r(x)$ and

$$\mathbb{E}^x \sigma_{r+4}(y) \le \mathbb{E}^x \sigma_r(x) \le c_2 t_F^{-r}.$$

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If $x \notin D_{r+2}(y)$, then $\mathbb{E}^x \sigma_{r+4}(y) = 0$. Therefore,

$$E^x \sigma_r(y) \le c_2 t_F^{-(r-4)} \le c_3 t_F^{-r}, \quad x, y \in F. \tag{6.38}$$

$\sigma_r(y)$ is an additive functional for X_t and by (6.38) and subadditivity (see [BK]),

$$\mathbb{E}^x \sigma_r(y)^p \le c_4(p) t_F^{-rp}, \quad p \ge 1, \quad x, y \in F. \tag{6.39}$$

Proceeding just as in the proofs of Lemma 6.8, Proposition 6.7, and Theorem 6.8, we deduce

$$\widehat{p}(t, x, y) \le c_5 t^{-d_s/2}.$$

Just as in the proof of (6.23),

$$\widehat{p}(t, x, x) \ge c_6 t^{-d_s/2}, \quad x \in F,$$

and as in the proof of (6.21),

$$\widehat{p}(t, x, y) \ge c_7 t^{-d_s/2}$$

if $|x - y| \le c_8 t^{1/d_w}$. An application of the chaining argument of Theorem 7.4 in [BB4] completes the proof. ■

7 Inequalities

In this section we apply the estimates obtained in Section 6 to deduce a number of Sobolev and related inequalities for the processes X and W. Let U be the potential operator for X_t, and $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ be the Dirichlet form defined in Section 5. From [CKS] and the upper bound $p(t, x, y) \le c_1 t^{-d_s/2}$ given in Theorem 1.3, we deduce immediately

Theorem 7.1 **(Nash Inequality)** *There exists c_1 such that*

$$\|f\|_2^{2+4/d_s} \le c_1 \mathcal{E}(f, f) \|f\|_1^{4/d_s}, \quad f \in \mathcal{D}(\mathcal{E}).$$

Theorem 7.2 **(Sobolev Inequality)** *Suppose $d_s > 2$. Let $q = 2d_s/(d_s - 2)$. Then there exists c_1 such that*

$$\|f\|_q \le c_1 \mathcal{E}(f, f)^{\frac{1}{2}}, \quad f \in \mathcal{D}(\mathcal{E}).$$

Thus Theorem 1.5 is proved.

Recall from Remark 5.11 the construction of Brownian motion on $D_n(x_0) \cap \tilde{F}$ with reflection on the boundaries of $D_n(x_0)$. Let $\widehat{p}(t, x, y)$ be the transition densities for this reflecting Brownian motion. Normalized Lebesgue measure on $D_n(x_0) \cap F_m$ is an invariant measure for the approximating processes X_t^m, and a limit argument shows that $\mu|_{D_n(x_0)}$ is an invariant measure for reflecting Brownian motion. We write $\mathcal{E}_{D_n(x_0)}(f, f)$ for the Dirichlet form of this process, and let $\mathcal{D}_{n,x_0}$ be its domain. For $A \in \mathcal{B}(\tilde{F})$ let $f_A = \mu(A)^{-1} \int_A f \, d\mu$.

We now give a proof of the Poincaré inequality, modeled after the proof in [SC].

Theorem 7.3 **(Poincaré Inequality)** *There exists c_1 such that*

$$\int_{D_n(x_0)} |f - f_{D_n(x_0)}|^2 \le c_1 l_F^{-nd_w} \mathcal{E}_{D_n(x_0)}(f, f), \quad f \in \mathcal{D}_{n,x_0}.$$

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Proof Let $t = l_F^{-nd_w}$. By Proposition 6.21 and weak scaling,

$$\widehat{p}(t,x,y) = \widehat{p}_{D_n(x_0)}(t,x,y) \geq c_2 l_F^{nd_f}. \tag{7.1}$$

Let $\widehat{P}_t f(x) = \int f(y)\widehat{p}(t,x,y)\,\mu(dy)$. Since $\mu|_{D_n(x_0)}$ is an invariant measure for $\widehat{p}(t,x,y)$, then

$$\int_{D_n(x_0)} \widehat{P}_t f(x)\,\mu(dx) = \int\int \widehat{p}(t,x,y) f(y) 1_{D_n(x_0)}(x)\,\mu(dy)\,\mu(dx) = \int_{D_n(x_0)} f(y)\,\mu(dy). \tag{7.2}$$

Now from (7.1), for $x \in D_{n+1}(x_0)$,

$$\widehat{P}_t(f - \widehat{P}_t f)^2(x) \geq c_2 l_F^{nd_f} \int_{D_{n+1}(x_0)} |f - \widehat{P}_t f|^2 \geq c_2 l_F^{nd_f} \int_{D_{n+1}(x_0)} |f - f_{D_{n+1}(x_0)}|^2.$$

Integrating both sides over $D_{n+1}(x_0)$,

$$\int_{D_{n+1}(x_0)} \widehat{P}_t(f - \widehat{P}_t f)^2(x)\,\mu(dx) \geq c_3 \int_{D_{n+1}(x_0)} |f - f_{D_{n+1}(x_0)}|^2. \tag{7.3}$$

The left-hand side of (7.3) is equal to

$$\begin{aligned}\int_{D_n(x_0)} \widehat{P}_t(f^2) - \int_{D_n(x_0)} (\widehat{P}_t f)^2 &= \int_{D_n(x_0)} f^2 - \int_{D_n(x_0)} (\widehat{P}_t f)^2 \\ &= -\int_0^t \int_{D_n(x_0)} \partial_s (\widehat{P}_s f)^2\,\mu(dy)\,ds \\ &= c_4 t \mathcal{E}_{D_n(x_0)}(f,f).\end{aligned}$$

The conclusion of Theorem 7.3 now follows by applying the techniques of Jerison [Je]. ■

Suppose $d_s > 2$. For $A \subset \tilde{F}$ define $C(A)$, the capacity of A, by

$$C(A) = \sup\{\nu(A) : U\nu \leq 1\}.$$

It is standard [FOT] that $C(A)$ can also be defined as

$$C(A) = \inf\{\mathcal{E}(f,f) : f = 1 \text{ on } A,\ f(x) \to 0 \text{ as } x \to \infty\}$$

if all points of $\partial A \cap \tilde{F}$ are regular for A^c.

Theorem 7.4 **(Mass-Capacity Inequality)** *Let $d_s > 2$. Then for $A \in \mathcal{B}(\tilde{F})$*

$$\mu(A) \leq c_1 C(A)^{d_s/(d_s-2)}.$$

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Proof Choose $n \in \mathbb{Z}$ such that $m_F^{-n} \le \mu(A) \le m_F^{-n+1}$. Write $\mu_A = \mu|_A$. Then

$$U(\mu_A)(x) = \int_A u(x,y)\,\mu(dy) = \int_{D_n(x)} u(x,y)\,\mu(dy) + \int_{A-D_n(x)} u(x,y)\,\mu(dy). \tag{7.4}$$

Now $\mu\big(D_m(x) - D_{m+1}(x)\big) \le c_2 m_F^{-m}$, while by Corollary 6.13 we have

$$u(x,y) \le c_3 (l_F^{-m})^{d_w - d_f} = c_3 t_F^{-m} m_F^m, \quad y \in D_n(x)^c. \tag{7.5}$$

Therefore

$$\int_{D_m(x)-D_{m+1}(x)} u(x,y)\,\mu(dy) \le c_4 t_F^{-m}.$$

Summing over $m \ge n$ to bound the first term in (7.4) and using the estimate (7.5) in the second, we deduce that

$$\begin{aligned} U(\mu_A)(x) &\le c_5 t_F^{-n} + c_3 t_F^{-n} m_F^n \mu_A(A) \\ &\le c_6 t_F^{-n} \le c_7 \mu(A)^{2/d_s}. \end{aligned}$$

So if $\nu = \big(c_7 \mu(A)^{2/d_s}\big)^{-1} \mu_A$, then $U\nu \le 1$, and therefore $C(A) \ge \nu(A) = c_7^{-1} \mu(A)^{1-2/d_s}$. ∎

We now proceed to obtain the Sobolev, mass-capacity, and Poincaré inequalities for the pre-carpet. We start with the mass-capacity inequality. We let $C_0(A)$ denote the capacity of A with respect to $\mathrm{RBM}(\tilde{F}_0)$. Recall that μ_0 is Lebesgue measure restricted to $\tilde{F}_0$.

Theorem 7.5 *Suppose $d_s > 2$. There exists c_1 such that if $A \in \mathcal{B}(\tilde{F}_0)$ and $\mu_0(A) \ge 1$, then*

$$C_0(A) \ge c_1 \mu_0(A)^{(d_s-2)/d_s}.$$

Proof This is very similar to the proof of Theorem 7.4, except that we use Corollary 6.10 instead of Corollary 6.13. Let $A \subset \tilde{F}_0$, with $\mu_0(A) \ge 1$. Then choosing n such that $m_F^{-n} \le \mu_0(A) \le m_F^{-n+1}$, we have $n \le 1$. Corollary 6.10 implies that

$$\int_{D_0(x)} u(x,y)\,\mu(dy) \le c_2,$$

and the remainder of the proof follows that of Theorem 7.4. ∎

Remark 7.6 Write

$$\mathcal{E}_0(f,f) = \frac{1}{2} \int_{\tilde{F}_0} |\nabla f|^2\,dx$$

for the Dirichlet form associated with W. By [FOT, Sect. 2.2], the capacity of A is equal to

$$\inf\{\mathcal{E}_0(f,f) : f \ge 1 \text{ on } A,\ f \to 0 \text{ as } \|x\|_\infty \to \infty\}.$$

We now proceed to the Sobolev inequality.

Theorem 7.7 *Suppose $d_s > 2$. Let $q = d_s/(d_s - 2)$.*

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(a) There exists c_1 such that if f is nonnegative and r is such that $\mu_0\big(\{x : f(x) > r\}\big) \geq 1$, then

$$\|f\|_{2q} \leq c_1 \mathcal{E}_0(f,f)^{1/2} + \|(f-r)^+\|_{2q}.$$

(b) There exists c_2 such that if r and s are such that $\mu_0\big(\{x : f^+(x) > r\}\big) \geq 1$ and $\mu_0\big(\{x : f^-(x) > s\}\big) \geq 1$, then

$$\|f\|_{2q} \leq c_2 \mathcal{E}_0(f,f)^{1/2} + \|(f^+-r)^+\|_{2q} + \|(f^- - s)^+\|_{2q}.$$

Proof We have

$$\|f\|_{2q} \leq \|f \wedge r\|_{2q} + \|(f-r)^+\|_{2q}.$$

So if we show

$$\|f \wedge r\|_{2q} \leq c_1 \mathcal{E}_0(f \wedge r, \wedge r)^{1/2}, \tag{7.6}$$

(a) will follow since $\mathcal{E}_0(f \wedge r, f \wedge r) \leq \mathcal{E}_0(f,f)$.

We follow the proof of [Ma, pp. 110–112]. Let $\Phi(x,\xi) = |\xi|$ and $p = 2$; then the $(p-\Phi)$-capacity of Maz'ja is, using Remark 7.6, the same as our capacity. Let $\mathcal{N}_t = \{x : f(x) \wedge r > t\}$. By our assumption on r, $\mathcal{N}_t$ is either empty or $\mu_0(\mathcal{N}_t) \geq 1$. So using Theorem 7.5, $\mu_0(\mathcal{N}_t) \leq c_3 C_0(A)^q$. We now obtain (a) by the proof in [Ma].

To prove (b), we write

$$\|f\|_{2q} \leq \|f^+\|_{2q} + \|f^-\|_{2q}$$

and apply (a) to f^+ and f^-, observing that $f^+ = -(-f \wedge 0)$, so

$$\mathcal{E}_0(f^+,f^+) = \mathcal{E}_0(-f \wedge 0, -f \wedge 0) \leq \mathcal{E}_0(-f,-f) = \mathcal{E}_0(f,f)$$

and similarly for $\mathcal{E}_0(f^-,f^-)$. ■

Remark 7.8 The Sobolev inequality for $\mathcal{E}_0$ is more complicated than that for $\mathcal{E}$, as $q(t,x,x)$ behaves differently at 0 and ∞. We give here two other Sobolev inequalities for $\mathcal{E}_0$, both of which follow from the bounds in Theorem 6.9.

First, from [V2] we have

$$\|f\|_{2q} \leq c_1\big(\mathcal{E}_0(f,f) + \|f\|_\infty\big). \tag{7.7}$$

We also have, from [Co],

$$\|P_1 f\|_{2q} \leq c_2 \mathcal{E}_0(f,f). \tag{7.8}$$

Here P_1 can be replaced by other "regularizing" operators which smooth out short range fluctuations in f.

From [CKS, Theorem 2.9] we have the following conditional Nash inequality for $\mathcal{E}_0$.

Theorem 7.9 *For each $c_1 > 0$ there exists c_2 such that*

$$\|f\|_2^{2+4/d_s} \leq c_2 \mathcal{E}_0(f,f)\|f\|_1^{4/d_s} \quad \textit{whenever} \quad \mathcal{E}_0(f,f) \leq c_1\|f\|_1^2.$$

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Finally we give a Poincaré inequality for $\tilde{F}_0$. Write

$$\mathcal{E}_{0,D_n(x_0)}(f,f) = \int_{D_n(x_0)\cap\tilde{F}_0} |\nabla f(x)|^2\,dx.$$

Theorem 7.10 *There exists c_1 such that if $x_0 \in \tilde{F}$, $n \le 0$, then*

$$\int_{D_n(x_0)} |f - f_{D_n(x_0)}|^2 \le c_1 t_F^{-n} \mathcal{E}_{0,D_n(x_0)}(f,f).$$

The proof is very similar to that of Theorem 7.3.

As remarked in the introduction, the bounds in Theorem 6.9 are sufficient to show that, while (by Theorem 4.3) an elliptic Harnack inequality holds for $\tilde{F}_0$, the usual type of parabolic Harnack inequality fails for any GSC for which $d_w > 2$. (See Remark 5.4.)

Let $B \subset \mathbb{R}^d$ be open, and $0 < t_1 < t_2 \le \infty$. A function $u = u(t,x)$, $t \in (t_1,t_2)$, $x \in B \cap \tilde{F}_0$, is *parabolic* on $(t_1,t_2) \times (B \cap \tilde{F}_0)$ if

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u \quad \text{on} \quad (t_1,t_2) \times (B \cap \tilde{F}_0),$$

and $\partial u/\partial n = 0$ almost everywhere on $B \cap \partial\tilde{F}_0$. Let $x \in \tilde{F}_0$, $r > 0$, $t > 4r^2$, and set

$$Q_- = [t-4r^2, t-3r^2] \times \big(B(x,2r) \cap \tilde{F}_0\big)$$
$$Q_+ = [t-r^2, t] \times \big(B(x,r) \cap \tilde{F}_0\big).$$

The standard parabolic Harnack inequality (see for example [FS]) would assert that if u is parabolic and non-negative then

$$\sup_{Q_-} u \le c_1 \inf_{Q_+} u, \tag{7.9}$$

where the constant c_1 is independent of x, r, t and u.

Proposition 7.11 *Suppose the* GSC *F is such that $d_w > 2$. Then the parabolic Harnack inequality (7.9) fails for $\tilde{F}_0$.*

Proof Fix $x \in \tilde{F}_0$, and let $r > 1$. Let $y \in \tilde{F}_0$ be chosen so that $|x-y| = r$, and let $u(s,z) = q(s,y,z)$. Take $t = 5r^2$. Then by (6.16)

$$\sup_{Q_-} u \ge q(r^2,y,y) \ge c_1 r^{-d_s},$$

while (since $|x-y| = r < r^2 < t$)

$$\begin{aligned}\inf_{Q_+} u &\le q(5r^2,x,y)\\ &\le c_2 r^{-d_s} \exp\big(-c_3 (r^{d_w}/5r^2)^{1/(d_w-1)}\big)\\ &= c_2 r^{-d_s} \exp(-c_4 r^{(d_w-2)/(d_w-1)}).\end{aligned}$$

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Taking r sufficiently large, (7.9) fails. ■

We can, however, give a different form for a parabolic Harnack inequality for $\tilde{F}_0$. Let $x \in \tilde{F}_0$, $r > 1$, $t > 4r^{d_w}$, let n be such that $l_F^{-n} \le r < l_F^{-n+1}$, and set

$$R_- = [t - 4r^{d_w}, t - 3r^{d_w}] \times \big(D_n(x) \cap \tilde{F}_0\big),$$
$$R_+ = [t - r^{d_w}, t] \times \big(D_{n+2}(x) \cap \tilde{F}_0\big).$$

Let $0 \le t_1 < t - 4r^{d_w} < t < t_2$, and $B \subset \mathbb{R}^d$ be an open set with $D_n(x) \subset B$. Let u be non-negative and parabolic on $(t_1, t_2) \times (B \cap \tilde{F}_0)$.

Theorem 7.12 *There exists a constant c_1, independent of x, r, t, u such that*

$$\sup_{R_-} u \le c_1 \inf_{R_+} u. \tag{7.10}$$

The proof follows from the estimates on $q(t, x, y)$ in Theorem 6.9 by the same argument as that used in [FS].

We now discuss the functions in $\mathcal{D}(\mathcal{E})$ in a little more detail. If $f \in \mathcal{D}(\mathcal{E})$ then by [FOT, Lemma 1.3.4]

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \mathcal{E}_t(f, f),$$

where

$$\mathcal{E}_t(f, f) = t^{-1}\big((I - P_t)f, f\big).$$

Set

$$\Gamma_t(f)(x) = (2t)^{-1} \int p_t(x, y)\big(f(x) - f(y)\big)^2 \mu(dy).$$

Then

$$\begin{aligned}\int \Gamma_t(f)(x)\, \mu(dx) &= \frac{1}{2} t^{-1} \int\int p_t(x, y)\big(f(x) - f(y)\big)^2 \mu(dy)\, \mu(dx) \\ &= t^{-1} \int\int f(x) p_t(x, y)\big(f(x) - f(y)\big)\, \mu(dy)\, \mu(dx) \\ &= \mathcal{E}_t(f, f).\end{aligned}$$

Proposition 7.13 *Suppose the* GSC *F is such that $d_w > 2$. Let $f \in C^1(\mathbb{R}^d)$, and suppose $g = f|_{\tilde{F}} \in \mathcal{D}(\mathcal{E})$. Then g is constant.*

Proof Suppose g is non-constant. Then we can find $x_0 \in \tilde{F}$ such that f is approximately linear (with non-zero gradient) in a neighbourhood of x_0. It follows that we can find sufficiently small constants a_0, a_1, a_2, a_3 such that whenever $x \in B_0 = B(x_0, a_0)$ and $0 < t < a_1$, there exists $y_0 \in B(x, t^{1/d_w})$ such that

$$|f(y) - f(x)| \ge a_2 t^{1/d_w} \quad \text{for } y \in B(y_0, a_3 t^{1/d_w}).$$

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So, writing $B_1 = B(y_0, a_3 t^{1/d_w})$, and c_1, c_2 for positive constants depending on the a_i,

$$\begin{aligned}\Gamma_t(f)(x) &\geq (2t)^{-1} \int_{B_1} p_t(x,y)\big(f(x) - f(y)\big)^2 \, \mu(dy) \\ &\geq c_1 t^{-1} \mu(B_1) t^{-d_f/d_w} t^{2/d_w} \geq c_2 t^{(2-d_w)/d_w}.\end{aligned}$$

Hence

$$\mathcal{E}_t(f,f) \geq \int_{B_0} \Gamma_t(f)(x)\, \mu(dx) \geq c_2 \mu(B_0) t^{(2-d_w)/d_w}.$$

Hence $\lim_{t \downarrow 0} \mathcal{E}_t(f,f) = \infty$, so $f \notin \mathcal{D}(\mathcal{E})$. ■

Remark 7.14 For the Sierpinski gasket, see [BP, Corollary 9.2] (or [BST, Theorem 4.4] for an analytic proof) for an analogous result for $\mathcal{D}(\mathcal{L})$.

Remark 7.15 There are several recent papers on Sobolev inequalities and capacity on general metric spaces (including fractals)—see, for example, [H1] and [KM]. Specialized to a GSC $\tilde{F}$, Hajłasz [H1] defines the space $L^{1,2}(\tilde{F}, \mu)$ to be the set of $f\colon \tilde{F} \to \mathbb{R}$ such that there exists $E \subset \tilde{F}$ and $g \in L^2(\tilde{F}, \mu)$ such that for $x, y \in \tilde{F} - E$,

$$|f(x) - f(y)| \leq |x - y|\big(g(x) + g(y)\big).$$

The norm $\|f\|_{L^{1,2}}$ is defined to be the infimum of $\|g\|_2$ for g satisfying the above.

As an example of the kind of results that can be obtained, Hajłasz proves [H1, Lemma 2] the Poincaré inequality

$$\int_B |f(x) - f_B|^2 \leq c r^2 \|f\|_{L^{1,2}}, \tag{7.11}$$

where $B = B(x,r) \cap \tilde{F}$. Thus this Poincaré inequality has the usual r^2 scaling, rather than the r^{d_w} scaling of Theorem 7.3.

While the relation between these two approaches is not clear to us in general, we can remark that Hajłasz's space $L^{1,2}(\tilde{F}, \mu)$ is very different from $\mathcal{D}(\mathcal{E})$. For example, if $f \in C_K^1(\mathbb{R}^d)$ and $f_0 = f|_{\tilde{F}}$ then $f_0 \in L^{1,2}(\tilde{F}, \mu)$, while, by Proposition 7.13, $f_0 \in \mathcal{D}(\mathcal{E})$ only if f_0 is constant.

8 Properties of the Process

In this section we list a number of properties of the processes X and W.

(a) Spectral Dimension

Let x_0 be fixed and let $\overline{p}(t,x,y)$ be the transition densities of X_t killed on exiting $D_0(x_0)$. By Proposition 6.15 we may write

$$\overline{p}(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$$

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for suitable eigenvalues λ_i and eigenfunctions φ_i (*cf.* [Bas2, Section II.4]). Just as in [BP, pp. 618–619],

$$\lim_{\lambda\to\infty} \frac{\log \#\{\lambda_j : \lambda_j \le \lambda\}}{\log \lambda} = d_s,$$

which says that d_s agrees with what is known in the mathematical physics literature as the spectral dimension.

(b) Transience, Recurrence and Regularity of Points

Let $T_x = \inf\{t > 0 : X_t = x\}$.

Theorem 8.1 *X is transient if and only if $d_s > 2$. Points are regular for themselves if and only if $d_s < 2$. More precisely, if $x, y \in \tilde{F}$ then:*

(a) if $d_s < 2$, then $\mathbb{P}^x(T_y < \infty) = 1$ and $\mathbb{P}^x(T_x = 0) = 1$,
(b) if $d_s = 2$, then $\mathbb{P}^x(T_y < \infty) = 0$, $\mathbb{P}^x(T_x = 0) = 0$, and $\mathbb{P}^x(\mathcal{T}_{D_n(y)} < \infty) = 1$ for all n,
(c) if $d_s > 2$, then $\mathbb{P}^x(T_y < \infty) = 0$, $\mathbb{P}^x(T_x = 0) = 0$, and $\mathbb{P}^x(\lim_{t\to\infty} |X_t - y| = \infty) = 1$.

Proof (a) may be proved as in [BB4, Theorem 8.1].

(b) Let $\lambda > 0$. By Corollary 6.13(c) $u^\lambda(x, y) < \infty$ for $x \ne y$, while $u^\lambda(x, x) = \infty$ for all x. Since $u^\lambda(X_t, y)$ is a non-negative $\mathbb{P}^x$-supermartingale, it follows that X_t cannot hit y.

Since X_t has a density, $\mathbb{E}^x \int_0^{\sigma_m(x_0)} 1_{\{x\}}(X_s)\, ds \le \int_0^\infty p(s, x, x)\, ds = 0$, or X_t leaves $\{x\}$ immediately. By the preceding paragraph and the strong Markov property, X_t never returns to x.

Let $\overline{u}(x, y)$ be the Green function for X_t killed on exiting $D_0(y)$. We observe that there exist c_1 and c_2 such that

$$\overline{u}(x, y) = \int_0^\infty \overline{p}(t, x, y)\, dt \ge c_1 \int_{c_2|x-y|^{d_w}}^1 t^{-d_s/2}\, dt \ge -c_3 \log|x - y|, \quad x, y \in D_2(y).$$

From (2.12), Theorem 6.9, and integration we obtain $\overline{u}(x, y) \le -c_4 \log|x - y|$. Since $\overline{u}(X_{t\wedge\sigma_{n+m}(y)}, y)$ is a martingale and $\overline{u}(x, y) = 0$ if $x \in \partial D_0(y)$, this martingale is a time-change of a one-dimensional Brownian motion. Since $\sigma_0(y)$ has finite expectation, it is finite almost surely, and we conclude that

$$\mathbb{P}^{I_F^{m-n}x}\big(\mathcal{T}_{D_{n+m}(y)} < \sigma_0(y)\big) \to 1 \quad \text{as} \quad m \to -\infty.$$

By weak scaling, $\mathbb{P}^x\big(\mathcal{T}_{D_n(y)} < \sigma_m(y)\big) \to 1$ as $m \to -\infty$, which completes the proof of (b).

(c) The first two assertions are proved as in (b). $u(X_t, y)$ is a nonnegative supermartingale, hence it converges a.s. It is clear that the only possible limit value is 0. Therefore $|X_t| \to \infty$, a.s. ■

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(c) Modulus of Continuity

There exists c_1 such that

$$\lim_{\delta\to 0}\sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} \frac{|X_t - X_s|}{|s-t|^{1/d_w}\big(\log(1/|s-t|)\big)^{(d_w-1)/d_w}} \le c_1, \quad \text{a.s.} \tag{8.1}$$

The proof is the same as that of the corresponding result for the Sierpinski gasket in [BP].

(d) Occupation Times

Proposition 8.2 *Suppose $d_s > 2$ and let*

$$A_t(r) = \sup_{x\in F_0}\int_0^t 1_{B(x,r)}(X_s)\,ds.$$

Then there exists $c_1 < \infty$ such that

$$\limsup_{r\to 0}\frac{A_\infty(r)}{r^{d_w}\log(1/r)} \le c_1.$$

Proof Let $C_t(x,r) = \int_0^t 1_{B(x,r)}(X_s)\,ds$. By Corollary 6.10

$$\sup_y \mathbb{E}^y C_\infty(x,r) \le \sup_y \int_{B(x,r)} |z-y|^{d_w-d_f}\,\mu(dz) \le c_2 r^{d_w}.$$

Since $C_t(x,r)$ is an additive functional, by subadditivity (see [BK]), there exist c_3 and c_4 such that

$$\sup_y \mathbb{E}^y \exp\big(c_3 C_\infty(x,r)/r^{d_w}\big) \le c_4.$$

So

$$\mathbb{P}^y\big(C_\infty(x,r) > \lambda r^{d_w}\big) \le c_5 e^{-c_6\lambda}.$$

There exist $M = c_7 r^{-d}$ balls of radius $4r$ such that if $x \in [0,1]^d$, then $B(x,r)$ is contained in one of these M balls. Then

$$\mathbb{P}^y\big(A_\infty(r) > c_8\log(1/r)r^{d_w}\big) \le c_5 c_7 r^{-d} e^{-c_6 c_8 \log(1/r)}.$$

If we choose c_8 sufficiently large (but depending only on c_6 and d), then

$$\mathbb{P}^y\big(A_\infty(r) \ge c_8\log(1/r)r^{d_w}\big) \le r^{-2}.$$

We apply this estimate for $r_j = 2^{-j}$, $j = 1, 2, \ldots$. By the Borel-Cantelli lemma,

$$\mathbb{P}^y\big(A_\infty(r_j) \ge c_8\log(1/r_j)r_j^{d_w} \text{ i.o.}\big) = 0.$$

Since $A_\infty(r)$ increases in r as does $\log(1/r)r^{d_w}$ for r small, while $\log(1/2r)(2r)^{d_w}$ is bounded by a constant times $\log(1/r)r^{d_w}$, the proposition follows. ■

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(e) Local Times

If $d_s < 2$, the argument of [BB4] shows X_t has a jointly continuous local time. If $d_s \geq 2$, Theorem 8.1 shows that a point y is never hit, so a local time at y cannot exist.

(f) Hausdorff Dimension of the Range

Proposition 8.3 *Suppose $d_s > 2$. If $R(\omega) = \{X_t(\omega) : 0 \leq t \leq 1\}$, then with probability one the Hausdorff dimension of $R(\omega)$ is d_w.*

This provides some justification for calling d_w the "dimension" of the walk. (See [AO], [RT].) If $d_s < 2$, then the continuity of the local times of X implies that the range of X_t includes an open subset of F, and hence the Hausdorff dimension of the range is d_f.

Proof We first show the dimension of $R(\omega)$ is less than or equal to d_w. Let $\varepsilon > 0$. By the modulus of continuity result in (c), there exists M (depending on ω) such that

$$|X_t - X_s| \leq M|s-t|^{1/(d_w+\varepsilon)}$$

if $0 \leq s, t \leq 1$. Let j be an integer, $\delta = 1/(j+1)$, and $t_i = i\delta$, $i = 0, 1, \ldots, j$. Then the set of balls $B\big(X_{t_i}(\omega), M\delta^{1/(d_w+\varepsilon)}\big)$ covers $R(\omega)$. We also have

$$\sum_{i=0}^{j} (M\delta^{1/(d_w+\varepsilon)})^{d_w+\varepsilon} = M^{d_w+\varepsilon} \sum_{i=0}^{j} \delta = M^{d_w+\varepsilon} < \infty$$

for all j. This implies that the dimension of $R(\omega)$ does not exceed $d_w + \varepsilon$, and since ε is arbitrary, $\dim R(\omega) \leq d_w$.

Next we show that $\dim R(\omega)$ is at least d_w. By Proposition 8.2 there exists M' (depending on ω) such that $A_1(r)(\omega) \leq M' r^{d_w-\varepsilon}$ if $r \leq 1$. Suppose there exists a sequence of balls $B(x_1, r_1), B(x_2, r_2), \ldots$ that cover $R(\omega)$ with $\sup_i r_i \leq \frac{1}{2}$ and $\sum_i r_i^{d_w-2\varepsilon} \leq 1/M'$. We have

$$\int_0^1 1_{B(x_i,r_i)}(X_s)\, ds \leq A_1(r_i) \leq M' r_i^{d_w-\varepsilon}.$$

Summing over i,

$$1 = \int_0^1 1_{R(\omega)}(X_s)\, ds \leq M' \sum_i r_i^{d_w-\varepsilon} \leq (\frac{1}{2})^{\varepsilon} M' \sum_i r_i^{d_w-2\varepsilon} \leq (\frac{1}{2})^{\varepsilon},$$

a contradiction. Therefore there cannot exist such a sequence of balls, and so $\dim R(\omega) \geq d_w - 2\varepsilon$. However, ε is arbitrary. ■

(g) Self-Intersections

By [Ro] and [FiS], X_t intersects itself if and only if

$$\int_{D_1(x)} \big(u^1(x,y)\big)^2 \mu(dy) < \infty,$$

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where u^1 is the 1-resolvent for X_t. Note that

$$\int_{D_1(x)} \left(u^1(x,y)\right)^2 \mu(dy) = \sum_{i=1}^{\infty} \int_{D_i(x)-D_{i+1}(x)} \left(u^1(x,y)\right)^2 \mu(dy).$$

$\mu\big(D_i(x)\big) \le c_1 l_F^{-id_f}$ and it is easy to see that $\mu\big(D_i(x) - D_{i+1}(x)\big) \ge c_2 l_F^{-id_f}$. Using the estimates for $u^1(x,y)$, which are obtained similarly to those for $u(x,y)$, it follows that X_t has self-intersections if and only if

$$\sum_{i=1}^{\infty} \left[(l_F^{-i})^{d_w - d_f}\right]^2 l_F^{-id_f} < \infty,$$

that is, if $2(d_w - d_f) + d_f > 0$, or if $d_s = 2d_f/d_w < 4$.

(h) Zero-One Law

An event A is a tail event if it is in $\sigma(X_u; u \ge t)$ for all $t > 0$. We prove that the tail σ-field is trivial.

Theorem 8.4 *Suppose A is a tail event. Then either $\mathbb{P}^x(A)$ is 0 for all x or else it is 1 for all x.*

Proof Let $\varepsilon > 0$ and fix $x_0 \in \tilde{F}$. By the martingale convergence theorem, $\mathbb{E}^{x_0}[1_A \mid \mathcal{F}_t] \to 1_A$ almost surely as $t \to \infty$. Choose t_0 large enough so that

$$\mathbb{E}^{x_0}|\mathbb{E}^{x_0}[1_A \mid \mathcal{F}_{t_0}] - 1_A| < \varepsilon. \tag{8.2}$$

Write Y for $\mathbb{E}^{x_0}[1_A \mid \mathcal{F}_{t_0}]$. Using‘(5.21), choose M large so that

$$\mathbb{P}^{x_0}(\sup_{s \le t_0} |X_s - x_0| > M t_0^{1/d_w}) < \varepsilon. \tag{8.3}$$

For each t, by Theorem 6.18 we have the continuity of $P_t f(x)$ in x with a modulus depending only on t and $\|f\|_\infty$. We use scaling and choose t_1 large so that

$$|P_{t_1} f(x) - P_{t_1} f(x_0)| < \varepsilon \|f\|_\infty, \quad |x - x_0| \le M t_0^{1/d_w}. \tag{8.4}$$

We note

$$|\mathbb{P}^{x_0}(A) - \mathbb{E}^{x_0}(Y; A)| = |\mathbb{E}^{x_0}(1_A; A) - \mathbb{E}^{x_0}(Y; A)| < \varepsilon. \tag{8.5}$$

Since A is a tail event, there exists C such that $A = C \circ \theta_{t_0+t_1}$. Let $f(z) = \mathbb{P}^z(C)$. By the Markov property at time t_1,

$$\mathbb{E}^w(1_C \circ \theta_{t_1}) = \mathbb{E}^w \mathbb{E}^{X(t_1)} 1_C = \mathbb{E}^w f(X_{t_1}) = P_{t_1} f(w). \tag{8.6}$$

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By the Markov property at time t_0 and (8.6),

$$\mathbb{E}^{x_0}(Y;A) = \mathbb{E}^{x_0}\left[Y\mathbb{E}^{X(t_0)}(1_C \circ \theta_{t_1})\right] = \mathbb{E}^{x_0}\left[YP_{t_1}f(X_{t_0})\right], \tag{8.7}$$

while

$$\mathbb{P}^{x_0}(A) = \mathbb{E}^{x_0}1_A = \mathbb{E}^{x_0}\mathbb{E}^{X(t_0)}(1_C \circ \theta_{t_1}) = \mathbb{E}^{x_0}\left[P_{t_1}f(X_{t_0})\right]. \tag{8.8}$$

If $|X_{t_0} - x_0| \le Mt_0^{1/d_w}$, then $|P_{t_1}f(X_{t_0}) - P_{t_1}f(x_0)| < \varepsilon$ by (8.4). Since

$$\begin{aligned}\mathbb{E}^{x_0}\left[YP_{t_1}f(X_{t_0})\right] &= \mathbb{E}^{x_0}\left[YP_{t_1}f(X_{t_0}); |X_{t_0} - x_0| \le Mt_0^{1/d_w}\right] \\ &\quad + \mathbb{E}^{x_0}\left[YP_{t_1}f(X_{t_0}); |X_{t_0} - x_0| > Mt_0^{1/d_w}\right],\end{aligned}$$

$$|\mathbb{E}^{x_0}\left[YP_{t_1}f(X_{t_0}); |X_{t_0} - x_0| \le Mt_0^{1/d_w}\right] - P_{t_1}f(x_0)\mathbb{E}^{x_0}[Y; |X_{t_0} - x_0| \le Mt_0^{1/d_w}]| \le \varepsilon,$$

and

$$\mathbb{E}^{x_0}[Y; |X_{t_0} - x_0| \le Mt_0^{1/d_w}] = \mathbb{E}^{x_0}Y - \mathbb{E}^{x_0}[Y; |X_{t_0} - x_0| > Mt_0^{1/d_w}],$$

then

$$|\mathbb{E}^{x_0}\left[YP_{t_1}f(X_{t_0})\right] - P_{t_1}f(x_0)\mathbb{E}^{x_0}Y| \le 3\varepsilon. \tag{8.9}$$

Similarly

$$|\mathbb{E}^{x_0}P_{t_1}f(X_{t_0}) - P_{t_1}f(x_0)| \le 3\varepsilon. \tag{8.10}$$

Combining (8.5), (8.7), (8.8), (8.9), and (8.10),

$$|\mathbb{P}^{x_0}(A) - \mathbb{P}^{x_0}(A)\mathbb{E}^{x_0}Y| \le 7\varepsilon.$$

Using this and (8.2),

$$|\mathbb{P}^{x_0}(A) - \mathbb{P}^{x_0}(A)\mathbb{P}^{x_0}(A)| \le 10\varepsilon.$$

Since ε is arbitrary, we deduce $\mathbb{P}^{x_0}(A) = [\mathbb{P}^{x_0}(A)]^2$, or $\mathbb{P}^{x_0}(A)$ is 0 or 1. Since $\mathbb{P}^x(A) = \mathbb{E}^xP_{t_1}f(X_{t_0}) = P_{t_0}(P_{t_1}f)(x)$ is continuous in x and $\tilde{F}$ is connected, then $\mathbb{P}^x(A)$ is either identically 0 or identically 1. ■

Remarks 8.5 1. An event A is invariant if $A \circ \theta_t = A$ for all A. In particular, invariant events are in the tail σ-field, hence trivial. It follows that there are no nonconstant bounded harmonic functions on $\tilde{F}$.

2. Since we have similar estimates for the transition densities of Brownian motion on the nested fractals, the same proof shows there is a zero-one law for (unbounded) nested fractals as well.

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(i) Rate of Escape

Proposition 8.6 *Suppose $d_s > 2$. Then*

$$\liminf_{t\to\infty} \frac{|X_t - x|}{t^{1/d_w}(\log t)^{\gamma}}$$

equals 0 *with $\mathbb{P}^x$ probability one if $\gamma > 1/(d_w - d_f)$ and equals ∞ with $\mathbb{P}^x$ probability one if $\gamma < 1/(d_w - d_f)$.*

Proof By Corollary 6.13(a) and the fact that $u(X_t, x)$ is a local martingale, there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1\big(1 \wedge (r/|y-x|)^{d_w - d_f}\big) \le \mathbb{P}^y\big(X_t \text{ ever hits } B(x,r)\big) \le c_2\big(1 \wedge (r/|y-x|)^{d_w - d_f}\big).$$

Using this, for any $q > 1$ and any $z \in \tilde{F}$,

$$\begin{aligned}
&\mathbb{P}^z(|X_t - x| \le r_n \text{ for some } t > q^n) \\
&\quad = \mathbb{E}^z \mathbb{P}^{X_{q^n}}(|X_t - x| \le r_n \text{ for some } t \ge 0) \\
&\quad = \int \mathbb{P}^y(|X_t - x| \le r_n \text{ for some } t \ge 0)\mathbb{P}^z\big(X_{q^n} \in \mu(dy)\big) \\
&\quad \le c_2 \int \left(\frac{r_n}{|y-x|} \wedge 1\right)^{d_w - d_f} p(q^n, z, y)\, \mu(dy).
\end{aligned}$$

Set $r_n = c_3 q^{n/d_w}(\log q^n)^{-\gamma}$. If

$$A_n = \{|X_t - x| \le r_n \text{ for some } t \ge q^n\},$$

then

$$\mathbb{P}^z(A_n) \le c_4(\log q^n)^{-\gamma(d_w - d_f)}. \tag{8.11}$$

By the Borel-Cantelli lemma we see that if $\gamma > 1/(d_w - d_f)$, then $\mathbb{P}^x(A_n \text{ i.o.}) = 0$, no matter what c_3 is.

For the other direction

$$\begin{aligned}
P^x(|X_t - x| \le r_n \text{ for some } t \in [q^n, q^{n+1}]) &= \mathbb{E}^x \mathbb{P}^{X_{q^n}}(|X_t - x| \le r_n \text{ for some } t \ge 0) \\
&\quad - \mathbb{E}^x \mathbb{P}^{X_{q^{n+1}}}(|X_t - x| \le r_n \text{ for some } t \ge 0).
\end{aligned}$$

Let $r_n = c_5 q^{n/d_w}(\log q^n)^{-\gamma}$. As above, the second term on the right is bounded by

$$c_6(\log q^n)^{-\gamma(d_w - d_f)} q^{-1/d_w}.$$

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On the other hand, the first term on the right is bounded below by

$$\int \mathbb{P}^y(|X_t - x| \le r_n \text{ for some } t \ge 0) p(q^n, x, y)\, \mu(dy) \ge c_7 (\log q^n)^{-\gamma(d_w - d_f)}.$$

Set $B_n = \{|X_t - x| \le r_n \text{ for some } t \in [q^n, q^{n+1}]\}$. Then provided q is sufficiently large,

$$\mathbb{P}^x(B_n) \ge c_8 (\log q^n)^{-\gamma(d_w - d_f)}. \tag{8.12}$$

If $\gamma < 1/(d_w - d_f)$, then $\sum \mathbb{P}^x(B_n) = \infty$.

Let $\varepsilon > 0$. If $j \ge i + 2$, then

$$\begin{aligned}\mathbb{P}^x(B_i \cap B_j) &= \mathbb{E}^x\big[\mathbb{P}^{X(q^{i+1})}(|X_t - x| \le r_j \text{ for some } t \in [q^j - q^{i+1}, q^{j+1} - q^{i+1}]); B_i\big] \\ &\le \mathbb{E}^x\big[\sup_y \mathbb{P}^y(|X_t - x| \le r_j \text{ for some } t \ge q^j - q^{i+1}); B_i\big].\end{aligned}$$

As in the proof of (8.11), the right hand side is at most $c_9 \mathbb{P}^x(B_i)(\log q^j)^{-\gamma(d_w - d_f)}$. Using (8.12), this in turn is bounded by $c_{10}\mathbb{P}^x(B_i)\mathbb{P}^x(B_j)$, so

$$\mathbb{P}^x(B_i \cap B_j) \le c_{10}\mathbb{P}^x(B_i)\mathbb{P}^x(B_j).$$

By a Borel-Cantelli lemma for dependent events (see, *e.g.*, [Bas2, Prop. II.5.14]), $\mathbb{P}^x(B_n \text{ i.o.}) > 0$. The result now follows from Theorem 8.4, the zero-one law. ■

(j) Process on the Pre-Carpet

The properties of X above which relate to large scale behavior of the process have analogues for the process W on $\tilde{F}_0$, with almost identical proofs. We summarize some of them in the following theorem.

Theorem 8.7

(a) If $d_s \le 2$ then $\mathbb{P}^x(\mathcal{T}^W_{D_n(y)} < \infty) = 1$ for all $x, y \in \tilde{F}_0$, so that W is neighborhood-recurrent.

(b) If $d_s > 2$ then W is transient, and

$$\liminf_{t\to\infty} \frac{|W_t - x|}{t^{1/d_w}(\log t)^\gamma}$$

equals 0 with probability one if $\gamma > 1/(d_w - d_f)$ and equals ∞ with probability one if $\gamma < 1/(d_w - d_f)$.

(c) The tail σ-field for W is trivial.

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9 Examples and Open Problems

Examples

We have already, in Example 5.4, defined the the carpet $\mathrm{SC}(a,b,d)$, where a,b,d are integers, $d \geq 2$, and $1 \leq b < a$. Note that the symmetry assumption Hypothesis 2.1 (H1) implies that $a = b \pmod 2$, and that therefore $b \leq a-2$. Recall that for $\mathrm{SC}(a,b,d)$, $m_F = a^d - b^d$, and that

$$\frac{a-b}{a^{d-1}} + \frac{b}{a^{d-1}-b^{d-1}} \leq \rho_F \leq \frac{a}{a^{d-1}-b^{d-1}}. \tag{9.1}$$

Since $b^{d-1} \leq (a-2)^{d-1} < (a-2)a^{d-2} = a^{d-1} - 2a^{d-2}$, we have $\rho_F < \frac{1}{2}a^{3-d}$. In particular $\rho_F < 1$ whenever $d > 2$, so that $\mathrm{SC}(a,b,d)$ is transient for all $d \geq 3$. (This generalizes a result of Zhou [Z]). This result is not surprising, since all these spaces contain a copy of $\mathbb{R}^2_+$. $\mathrm{SC}(a,b,2)$ is recurrent by Corollary 5.3, since $d_s \leq d_f < 2$.

To obtain a GSC in dimension 3 for which the processes W or X are recurrent, we need to ensure that $\tilde{F}$ does not contain a copy of $\mathbb{R}^2_+$. Take $d = 3$, let $1 \leq b \leq a-2$, write $I_{ab} = \big((a-b)/2a, (a+b)/2a\big)$, let $\alpha(x) = \sum_{i=1}^{3} 1_{I_{ab}}(x_i)$, and let

$$F_1 = \{x \in F_0 : \alpha(x) \leq 1\}.$$

Call the resulting GSC a Menger Sponge (see [Man, p. 134]) or $MS(a,b)$. For the basic $MS(3,1)$, F_1 consists of F_0 with the central cube and the 6 adjacent cubes removed. (See [Man, p. 145] for an excellent picture.) Note that $m_F = a^3 - \big(b^3 + 3b^2(a-b)\big)$. Using shorts and cuts, as in Proposition 5.1, we can easily estimate ρ_F for these sets: we obtain

$$\frac{a-b}{a^2-b^2} + \frac{b}{(a-b)^2} \leq \rho_F \leq \frac{a}{(a-b)^2}. \tag{9.2}$$

In particular, for $MS(3,1)$ we have $\frac{1}{2} \leq \rho_F \leq 3/4$, so that $MS(3,1)$ is transient. However, if $a = 6$ and $b = 4$ then we have $11/10 \leq \rho_F \leq 3/2$, so that $MS(6,4)$ is recurrent. On the other hand, as $m_F = 56$, we have $d_f = \log 56/\log 6 \approx 2.25 > 2$, so the family of Menger Sponges includes sets with Hausdorff dimension greater than 2, but spectral dimension less than 2.

Open Problems

We conclude this paper with some open questions concerning these processes. (See also the problems in [Bas3].)

1. Spectral Dimension Of the three "dimensions" mentioned in this paper, d_f, d_w, and d_s, the first is given explicitly in terms of l_F and m_F. The other two are defined in terms of the resistance exponent ρ_F, which we only know how to define as a limit:

$$\log \rho_F = \lim_{n\to\infty} \frac{\log R_n}{n}.$$

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Is there a more explicit equation which defines ρ_F? S. R. S. Varadhan has asked if ρ_F could be defined by some kind of variational equation.

2. Uniqueness There are two different kinds of uniqueness we can consider. The first relates to the construction of X and its law $\mathbb{P}^x$ given here. In Section 5 we stated that the set of laws of the approximating processes $\{\mathbb{P}^x_n, n \geq 0\}$ was weakly compact, and so has limit points. Can one show that P^x_n converges weakly to $\mathbb{P}^x$, that is, that the limit is unique? If so, then one would immediately have that the resulting process was self-similar.

A second kind of uniqueness is given by our conjecture that the process constructed in this paper is (up to a deterministic time change) the unique symmetric diffusion on X which is locally isotropic. (For an analogous theorem for the Sierpinski gasket see [BP].) If we knew this, then any two limit points of $\mathbb{P}^x_n$ could differ only by a deterministic time change, and also, more significantly, we would know that any other construction of a symmetric locally isotropic diffusion on $\tilde{F}$ (such as the one given in [KZ] using random walks) would lead to essentially the same process.

We remark here that Osada [O2] has recently defined a diffusion $Z = (\mathbb{Q}^x, Z_t)$ on F which is not equal in law to X and does not satisfy (5.18), but which does satisfy a more limited kind of invariance under local isometries. Let $S_1, S_2 \in \mathcal{S}_n$, write $A_i = \mathrm{int}(S_i)$, and suppose that φ is an isometry between $S_1 \cap F$ and $S_2 \cap F$. Then for $x \in A_1 \cap F$ one has

$$\mathbb{Q}^x\big(\varphi(Z_{t\wedge\tau(A_1)}) \in \cdot\,, t \geq 0\big) = \mathbb{Q}^{\varphi(x)}(Z_{t\wedge\tau(A_2)} \in \cdot\,, t \geq 0). \tag{9.3}$$

The key difference between (9.3) and (5.18) is that in (9.3) the processes are stopped on hitting the boundary of the cubes S_i.

This example shows that the definition of "locally isotropic" is quite sensitive.

See [Sa] and [Me] for some uniqueness results in the finitely ramified case.

3. Continuity of Harmonic Functions Let $x_0 \in \tilde{F}$ and set $D = D_n(x_0) \cap \tilde{F}$, $D' = D_{n+2}(x_0)$. Let h be non-negative and harmonic in D. Then we know by Theorem 4.2 that there exists $\alpha > 0$ such that h is Hölder continuous of order α in D'. What is the correct Hölder exponent for h? The constant α given in Theorem 4.2 comes from the Harnack inequality, so would not be a good estimate. If $d_s < 2$ then the estimates of Section 4 of [BB3] imply that h is Hölder continuous of order $d_w - d_f$. Even here, though, we do not know if that is best possible.

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