Compositio Mathematica **109:** 195–250, 1997.

Levi–Tanaka algebras and homogeneous CR manifolds

C. MEDORI[1] and M. NACINOVICH[2]
[1]*S.I.S.S.A., Trieste, Italy*
[2]*Dipartimento di Matematica, Università di Pisa, via F. Buonarroti 2, 56127 Pisa, Italy*
e-mail: nacinovich@dm.unipi.it

Recieved 29 May 1996; accepted in final form 11 July 1996

Abstract. In this paper we take up the problem of discussing CR manifolds of arbitrary CR codimension. We closely follow the general method of N. Tanaka, while concentrating our attention to the case of manifolds endowed with partial complex structures. This study required a deeper understanding of the structure of the Levi–Tanaka algebras, which are the canonical prolongation of pseudocomplex fundamental graded Lie algebras. These algebras enjoy special properties, the understanding of which provided also a way to build up several different examples and points to a rich field of investigations. Here we restrained further our consideration to the homogeneous models.

Mathematics Subject Classifications (1991): 32F25, 53C30

Key words: CR manifold, Levi–Tanaka algebra

Introduction

In a series of papers ([12], [13], [14]) N. Tanaka developed a general method for the study of geometrical structures associated to the datum of a vector distribution on a differentiable manifold. One important application of its research was the study of the automorphism group of a CR hypersurface. Similar results were later obtained by Chern and Moser in [4]. The main difficulty in the study of this problem is that, although it has a quite natural formulation in terms of G-structures, the classical methods do not apply because in general the group of infinitesimal CR automorphisms does not have a faithful representation into the group of infinitesimal automorphisms of the frame bundle. By considering the prolongation of a graded Lie algebra associated to the vector distribution and building up a principal bundle canonically associated to it, Tanaka succeeded in showing that in several cases the group of CR automorphisms is a finite dimensional Lie group, whose dimension does not exceed that of the prolongation.

In this paper we take up the problem of discussing CR manifolds of arbitrary CR codimension. We closely follow the general method of N. Tanaka, while concentrating our attention to the case of manifolds endowed with partial complex structures. This study required a deeper understanding of the structure of the Levi–Tanaka algebras, which are the canonical prolongations of pseudocomplex fundamental graded Lie algebras. These algebras enjoy special properties, the understanding of which provides also a way to build up several different examples and points to a rich field of investigations. Here we restrained further our consideration to the homogeneous models, which are interesting for their relationship to simpler objects already considered in quantum mechanics.

1. Preliminaries

1.1. Partial complex structures and CR manifolds

Let M be a smooth real manifold of dimension m, countable at infinity. Let $n,\ k$ be nonnegative integers with $2n + k = m$. A *partial almost complex structure of type* (n, k) on M is the pair consisting of a real vector subbundle HM of rank $2n$ of the tangent bundle TM and a smooth fiber preserving bundle isomorphism $J\colon HM \to HM$, with

$$J^2 = -\mathrm{Id}\colon HM \to HM$$

such that

$$[X, Y] - [JX, JY] \in \Gamma(M, HM) \quad \forall X, Y \in \Gamma(M, HM). \tag{1}$$

Here we use Γ to indicate smooth sections of a fiber bundle.

The triple $\mathbf{M} = (M, HM, J)$, where (HM, J) is a partial almost complex structure of type (n, k) on M, is then called an *almost CR manifold of type* (n, k).

We say that the almost partial complex structure (HM, J) on M is a *partial complex structure* if it is *formally integrable,* i.e. if

$$\mathcal{N}(X, Y) := [JX, Y] + [X, JY] - J([X, Y] - [JX, JY]) = 0 \tag{2}$$

for every $X, Y \in \Gamma(M, HM)$. When (HM, J) is a partial complex structure of type (n, k), we say that the triple $\mathbf{M} = (M, HM, J)$ is a *CR manifold of type* (n, k).

The integrability conditions (1) and (2) can be expressed in another equivalent formulation. Let

$$\begin{aligned} T^{1,0}M &= \{X - iJX \mid X \in HM\} \quad \text{and} \\ T^{0,1}M &= \{X + iJX \mid X \in HM\} \end{aligned}$$

be the complex vector subbundles of the complexification $\mathbb{C}HM$ of HM, corresponding to the eigenvalues i and $-i$ of J. Then (1) and (2) are equivalent to each of the following

$$\begin{aligned} [\Gamma(M, T^{1,0}M), \Gamma(M, T^{1,0}M)] &\subset \Gamma(M, T^{1,0}M), \\ [\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] &\subset \Gamma(M, T^{0,1}M). \end{aligned}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

1.2. CR MAPS

Let $\mathbf{M}_1 = (M_1, HM_1, J_1)$ and $\mathbf{M}_2 = (M_2, HM_2, J_2)$ be two almost CR manifolds, of type (n_1, k_1) and (n_2, k_2) respectively. A differentiable map $f: M_1 \to M_2$ is a *CR map* if

(1) $f_*(HM_1) \subset HM_2$ and
(2) $f_*(J_1 X_x) = J_2 f_*(X_x) \quad \forall x \in M_1\,, \forall X_x \in H_x M_1.$

When $\mathbf{M}_2$ is $\mathbb{C}$ with the standard complex structure of a CR manifold of type $(1,0)$, a CR map from M_1 to $\mathbb{C}$ is called a *CR function*.

A differomorphism $f: M_1 \to M_2$ is called a *CR diffeomorphism* if f and $f^{-1}: M_2 \to M_1$ are both CR maps. Two CR diffeomorphic almost CR manifolds have necessarily the same type.

1.3. THE FORM OF LEVI–TANAKA

We begin by an easy proposition from linear algebra, that will be usefull in the sequel.

PROPOSITION 1.1. *Let V be a real vector space, of even dimension $2n$, on which a complex structure $J \in \hom_{\mathbb{R}}(V,V)$, with $J^2 = -\mathrm{Id}$, is given. Then:*

(1) *For every alternating $\mathbb{R}$-bilinear form $\mathfrak{a}: V \times V \to \mathbb{R}^k$ such that $\mathfrak{a}(Jv, Jw) = \mathfrak{a}(v,w)$ for every $v, w \in V$ there is a unique Hermitian form $\mathfrak{f}: V \times V \to \mathbb{C}^k$ such that*

$$\Im\mathfrak{f}(v,w) = \mathfrak{a}(v,w) \quad \forall v, w \in V.$$

It is given by

$$\mathfrak{f}(v,w) = \mathfrak{a}(Jv,w) + \sqrt{-1}\,\mathfrak{a}(v,w) \quad \forall v, w \in V$$

and is Hermitian symmetric for the real form $\mathbb{R}^k$ of $\mathbb{C}^k$.

(2) *If $\mathfrak{a}$ and $\mathfrak{f}$ are as in (1), $A \in \hom_{\mathbb{C}}(V,V)$ and $B \in \hom_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^k)$, the following are equivalent*

$$\mathfrak{a}(Av, Aw) = B\mathfrak{a}(v,w) \quad \forall v, w \in V, \tag{i}$$

$$\mathfrak{f}(Av, Av) = B\mathfrak{f}(v,v) \quad \forall v \in V. \tag{ii}$$

(3) *If $\mathfrak{a}$ and $\mathfrak{f}$ are as in (1), $A \in \hom_{\mathbb{C}}(V,V)$, $B \in \hom_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^k)$, the following are equivalent*

$$\mathfrak{a}(Av, w) + \mathfrak{a}(v, Aw) = B\mathfrak{a}(v,w) \quad \forall v, w \in V, \tag{iii}$$

$$\mathfrak{f}(Av, v) + \mathfrak{f}(v, Av) = B\mathfrak{f}(v,v) \quad \forall v \in V. \tag{iv}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

We note that (ii) and (iv) are respectively equivalent to

$$\mathfrak{f}(Av, Aw) = B\mathfrak{f}(v, w) \quad \forall v, w \in V, \tag{ii$'$}$$

$$\mathfrak{f}(Av, w) + \mathfrak{f}(v, Aw) = B\mathfrak{f}(v, w) \quad \forall v, w \in V, \tag{iv$'$}$$

for the complexification, still denoted by B, of the real linear map B.

Let now **M** $= (M, HM, J)$ be an almost CR manifold of type (n, k), denote by QM the quotient bundle TM/HM and let $\pi: TM \to QM$ be the projection onto the quotient. Given two sections $X, Y \in \Gamma(M, HM)$ and a point $x \in M$, the value $\pi([X, Y]_x) \in Q_xM$ only depends on the values X_x, Y_x at x of X and Y. Thus we obtain an alternating bilinear form

$$\mathfrak{l}_x: H_xM \times H_xM \ni (X_x, Y_x) \to \pi([X, Y]_x) \in Q_xM,$$

which is called the *Levi–Tanaka form* of **M** at x. Clearly the assignement $M \ni x \to \mathfrak{l}_x \in \Lambda^2(HM, QM)$ is smooth.

By condition (1) this form is J-invariant

$$\mathfrak{l}_x(JX_x, JY_x) = \mathfrak{l}_x(X_x, Y_x) \quad \forall x \in M, \quad \forall X_x, Y_x \in H_xM.$$

By applying the proposition above, we obtain for every $x \in M$ a unique Hermitian symmetric form $\mathfrak{f}_x$ for the complex structure of H_xM such that

$$\mathfrak{l}_x(X_x, Y_x) = \Im\mathfrak{f}_x(X_x, Y_x) \quad \forall X_x, Y_x \in H_xM.$$

It is given by

$$\mathfrak{f}_x(X_x, Y_x) = \mathfrak{l}_x(JX_x, Y_x) + \sqrt{-1}\,\mathfrak{l}_x(X_x, Y_x)$$

and therefore smoothly depends on x. The corresponding Hermitian quadratic form

$$H_xM \ni X_x \to \mathfrak{f}_x(X_x, X_x) \in Q_xM$$

is often referred to as the *(vector valued) Levi form.*

1.4. Pseudoconvexity and Pseudoconcavity

Let **M** $= (M, HM, J)$ be an almost CR manifold of type (n, k). We define the *characteristic bundle* of **M** as the smooth linear subbundle H^0M of the cotangent bundle T^*M of M whose fiber H^0_xM at the point $x \in M$ is the annihilator of $H_xM \subset T_xM$

$$H^0_xM = \{\xi_x \in T^*_xM \mid \langle X_x, \xi_x\rangle = 0 \quad \forall X_x \in H_xM\}.$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

We define the *(scalar) Levi form* at $\xi_x \in H^0_x M$ by

$$\mathcal{L}(\xi_x, X_x) = \langle \mathfrak{f}_x(X_x, X_x),\, \xi_x \rangle \quad \text{for} \quad X_x \in H_x M.$$

This is a real valued hermitian form for the complex structure of $H_x M$.

We say that **M** is *q-pseudoconvex* at $x \in M$ if we can find $\xi_x \in H^0_x M$ such that the hermitian form $\mathcal{L}(\xi_x, \cdot\,)$ has at least $n \Leftrightarrow q$ positive eigenvalues.

We say that **M** is *q-pseudoconcave* at $x \in M$ if for every $\xi_x \in H^0_x M$ with $\xi_x \neq 0$ the Hermitian form $\mathcal{L}(\xi_x, \cdot\,)$ has at least q negative eigenvalues.

Pseudoconvexity and pseudoconcavity are related to the local properties of the CR complexes (see for instance [10] and [8]).

2. Prolongations of fundamental graded Lie algebras

2.1. Graded Lie algebras

A *graduation* of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a decomposition of $\mathfrak{g}$ into a direct sum of $\mathbb{K}$-linear subspaces $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that

$$\begin{cases} \dim_{\mathbb{K}} \mathfrak{g}_p < \infty & \forall p \in \mathbb{Z}, \\ [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} & \forall p, q \in \mathbb{Z}. \end{cases}$$

We say that $\mathfrak{g}$ is of *finite kind* μ, for a nonnegative integer μ, if $\mathfrak{g}_p = 0$ for $p < \Leftrightarrow \mu$ and $\mathfrak{g}_{-\mu} \neq 0$. In this case we call the dimension k of $\oplus_{p<-1} \mathfrak{g}_p$ the *codimension* of $\mathfrak{g}$.

We note that $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$, $\mathfrak{g}_+ = \bigoplus_{p>0} \mathfrak{g}_p$ and $\mathfrak{g}_0$ are Lie subalgebras of $\mathfrak{g}$. Moreover, for every $p \in \mathbb{Z}$ the map

$$\rho_p \colon \mathfrak{g}_0 \to \hom_{\mathbb{K}}(\mathfrak{g}_p, \mathfrak{g}_p)$$

defined by

$$\rho_p(X_0)(X_p) = [X_0, X_p] \quad \text{for} \quad X_0 \in \mathfrak{g}_0, X_p \in \mathfrak{g}_p \tag{3}$$

is a linear representation of the Lie algebra $\mathfrak{g}_0$ in $\mathfrak{g}_p$.

A graded Lie algebra $\mathfrak{g}$ is said to be:

(1) *fundamental* if $\mathfrak{g}_p = 0$ for $p \geqslant 0$ and $[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ for $p < 0$, i.e. $\mathfrak{g}_{-1}$ generates $\mathfrak{g}$;
(2) *nondegenerate* if $[X, \mathfrak{g}_{-1}] \neq 0$ when $0 \neq X \in \mathfrak{g}_{-1}$;
(3) *irreducible* (respectively *completely reducible*) if the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is irreducible (resp. completely reducible);
(4) *transitive* if $[X, \mathfrak{g}_{-1}] \neq 0$ when $p \geqslant 0$ and $0 \neq X \in \mathfrak{g}_p$. In this case the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is faithful.

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

A graded Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is said to be *pseudocomplex* if a complex structure

$$J\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, \quad J^2 = -\mathrm{Id}_{\mathfrak{g}_{-1}}$$

is given on $\mathfrak{g}_{-1}$ in such a way that

$$[X, Y] = [JX, JY] \quad \forall X, Y \in \mathfrak{g}_{-1}. \tag{4}$$

2.2. Fundamental graded Lie algebras associated to vector distributions

Graded Lie algebras were considered by Tanaka in [12] in order to investigate canonical forms of vector distributions and CR manifolds. We rehearse here the relevant construction.

Let $D \subset TM$ be a rank r linear subbundle of the tangent bundle of a smooth differentiable manifold M of dimension m. We set

$$\mathcal{D}_{-1} = \Gamma(M, D)$$

and define by recurrence

$$\mathcal{D}_p = [\mathcal{D}_{p+1}, \mathcal{D}_{-1}] + \mathcal{D}_{p+1} \quad \text{for} \quad p < -1.$$

Then we have an increasing sequence of $\mathcal{E}(M)$-modules of vector fields

$$\mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \cdots \subset \Gamma(M, TM).$$

For every $x \in M$ and $p < 0$ we set

$$(\mathcal{D}_p)_x = \{X_x \in T_xM \mid X \in \mathcal{D}_p\}.$$

Note that:

(i) $[\mathcal{D}_p, \mathcal{D}_q] \subset \mathcal{D}_{p+q} \quad \forall p, q < 0$;

(ii) if $p, q < 0, X \in \mathcal{D}_p, Y \in \mathcal{D}_q, f, g \in \mathcal{E}(M)$, then

$$[fX, gY] - fg\,[X, Y] \in \mathcal{D}_{p+q+1}.$$

Let us define then, for every fixed $x \in M$,

$$\begin{cases} \mathfrak{g}_{-1}(x) = (\mathcal{D}_{-1})_x, \\ \mathfrak{g}_p(x) = \dfrac{(\mathcal{D}_p)_x}{(\mathcal{D}_{p+1})_x} \quad \text{for} \quad p < -1. \end{cases}$$

By conditions (i) and (ii), the commutator of vector fields in $\mathcal{D}_p$ and $\mathcal{D}_q$, composed with the projection onto the quotient $(\mathcal{D}_{p+q})_x \to \mathfrak{g}_{p+q}(x)$, defines on

$$\mathfrak{g}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x),$$

the structure of a real fundamental graded Lie algebra.

We say that D is *regular* if, for every $p < 0$, $\mathcal{D}_p$ is a vector distribution of constant rank in M, i.e if

$$\dim_{\mathbb{R}} \mathfrak{g}_p(x) = \dim_{\mathbb{R}} \mathfrak{g}_p(y) \quad \forall p < 0, \quad \forall x, y \in M.$$

In this case there is a smallest positive integer μ such that

$$\mathcal{D}_p = \mathcal{D}_{-\mu} \quad \forall p < \Leftrightarrow \mu$$

and $\mathcal{D}_{-\mu}$ is the smallest formally integrable vector distribution in M containing $\mathcal{D}_{-1} = \Gamma(M, D)$. By the classical Frobenius Theorem M is locally foliated by integral leaves of $\mathcal{D}_{-\mu}$.

In particular we can apply the construction above to the linear vector subbundle HM of TM for a given almost CR manifold $\mathbf{M} = (M, HM, J)$. We say that $\mathbf{M}$ is *contact regular* if HM is regular.

We shall denote by $\mathfrak{m}(x)$ the fundamental graded Lie algebra associated to HM at the point $x \in M$. It is pseudocomplex with respect to the complex structure J on $H_xM = \mathfrak{m}_{-1}(x)$. We note that $\mathfrak{m}(x)$ is nondegenerate if and only if the Levi form is nondegenerate at x.

A CR diffeomorphism induces isomorphisms of the pseudocomplex fundamental graded Lie algebras associated to the partial almost complex structures at the corresponding points. In particular the algebras $\mathfrak{m}(x)$ are pseudoconformal invariants of the CR manifolds.

The fundamental graded Lie algebra $\mathfrak{m}(x)$ takes into account also the higher order Levi forms (see [11]). However, for the study of the local CR invariants of $\mathbf{M}$, it is convenient to extend $\mathfrak{m}(x)$ to a graded Lie algebra $\mathfrak{g}(x)$, via a canonical prolongation. This $\mathfrak{g}(x)$ will be called the *Levi–Tanaka algebra* of $\mathbf{M}$ at x.

2.3. Canonical prolongations of fundamental graded Lie algebras

Given a finite dimensional graded Lie algebra $\mathfrak{a} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{a}_p$, we say that a graded Lie algebra $\mathfrak{b} = \oplus_{-\mu \leqslant p} \mathfrak{b}_p$ is a *prolongation* of $\mathfrak{a}$ if there is a monomorphism of graded Lie algebras $\mathfrak{a} \to \mathfrak{b}$ inducing an isomorphism of $\mathfrak{a}$ onto $\mathfrak{b}_{\leqslant \nu} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{b}_p$.

In [13] the following theorem is proved:

THEOREM 2.1. *Let $\mathfrak{m} = \bigoplus_{-\mu \leqslant p < 0} \mathfrak{m}_p$ be a fundamental graded Lie algebra over $\mathbb{R}$. Then we can construct a graded Lie algebra $\mathfrak{g} = \bigoplus_{p \geqslant -\mu} \mathfrak{g}_p$, unique up to*

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

isomorphisms, which is maximal between the transitive graded Lie algebras $\mathfrak{g}$ *for which there is a graded Lie algebras isomorphism*

$$\mathfrak{g}_- = \bigoplus_{-\mu\leqslant p<0} \mathfrak{g}_p \to \mathfrak{m}.$$

Such a transitive graded Lie algebra $\mathfrak{g}$ will be called the *canonical prolongation* of $\mathfrak{m}$.

Let $\mathfrak{m}$ be a fundamental graded Lie algebra of kind μ and let $\tilde{\mathfrak{m}}$ be its canonical prolongation, given by the theorem above. We fix a Lie subalgebra $\mathfrak{g}_0$ of the algebra $\tilde{\mathfrak{m}}_0$ of all derivations of degree 0 of $\mathfrak{m}$. Then we define the *canonical prolongation of* $\mathfrak{m} \oplus \mathfrak{g}_0$ setting by recurrence

$$\mathfrak{g}_p = \{X_p \in \tilde{\mathfrak{m}}_p \mid [X_p, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{p-1}\}.$$

This is a graded Lie subalgebra of $\tilde{\mathfrak{m}}$ and hence a transitive graded Lie algebra, maximal between the graded Lie algebras $\mathfrak{a}$ which are transitive and satisfy

$$\mathfrak{m} \oplus \mathfrak{g}_0 \simeq \mathfrak{a}_{\leqslant 0} = \bigoplus_{p\leqslant 0} \mathfrak{a}_p \quad \text{as graded Lie algebras.}$$

When $\mathfrak{m}$ is a pseudocomplex fundamental graded Lie algebra, we say that a prolongation $\mathfrak{a} = \oplus_{p\geqslant -\mu}\mathfrak{a}_p$ of $\mathfrak{m}$ is *pseudocomplex* if the elements of $\mathfrak{a}_0$ define derivations of degree 0 of $\mathfrak{m}$ which are $\mathbb{C}$-linear on $\mathfrak{m}_{-1}$ for the complex structure induced by J.

If we define $\mathfrak{g}_0$ to be the space of all degree 0 derivations of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m}$ which are $\mathbb{C}$-linear on $\mathfrak{m}_{-1}$, we call the canonical prolongation of $\mathfrak{m} \oplus \mathfrak{g}_0$ the *canonical pseudocomplex prolongation of* $\mathfrak{m}$.

A graded Lie algebra $\mathfrak{g} = \oplus_{p\geqslant -\mu}\mathfrak{g}_p$ such that:

(i) $\mathfrak{m} = \oplus_{-\mu\leqslant p<0}\mathfrak{g}_p$ is a fundamental pseudocomplex Lie algebra;
(ii) $\mathfrak{g}$ is the canonical pseudocomplex prolongation of $\mathfrak{m}$;

will be called a *Levi–Tanaka algebra*.

In particular, when $\mathfrak{m} = \mathfrak{m}(x)$ is the pseudocomplex fundamental graded Lie algebra associated to a point $x \in M$ of an almost CR manifold $\mathbf{M} = (M, HM, J)$, its canonical pseudocomplex prolongation $\mathfrak{g}(x)$ is called the *Levi–Tanaka algebra* of $\mathbf{M}$ at x.

We note that CR diffeomorphisms induce isomorphisms of the Levi–Tanaka algebras at corrisponding points. In particular the Levi–Tanaka algebras, modulo isomorphisms, are pseudoconformal invariants.

2.4. Finiteness of the Canonical Prolongation

A useful criterion for the finiteness of transitive prolongations was given by Serre (cf. [5] and [13]).

THEOREM 2.2. *Let* $\mathfrak{g} = \oplus_{p\geqslant -\mu}\mathfrak{g}_p$ *be a transitive prolongation of a fundamental graded Lie algebra of kind* μ *and let*

$$H(\mathfrak{g}) = \{X \in \mathfrak{g} \,|\, [X,Y] = 0 \quad \forall Y \in \oplus_{p<-1}\mathfrak{g}_p\}.$$

Then $\mathfrak{g}$ *is finite dimensional if and only if* $H(\mathfrak{g})$ *is finite dimensional.*

3. Levi–Tanaka algebras

3.1. Canonical pseudocomplex prolongations

The finiteness criterion given in the previous section yields in the pseudocomplex case:

THEOREM 3.1. *Let* $\mathfrak{m} = \oplus_{-\mu\leqslant p\leqslant -1}\mathfrak{m}_p$ *be a pseudocomplex fundamental graded Lie algebra. The canonical pseudocomplex prolongation* $\mathfrak{g} = \oplus_{p\geqslant -\mu}\mathfrak{g}_p$ *of* $\mathfrak{m}$ *is finite dimensional if and only if* $\mathfrak{m}$ *is nondegenerate, i.e.*

$$\{X \in \mathfrak{g}_{-1} \,|\, [X, \mathfrak{g}_{-1}] = 0\} = 0.$$

A necessary and sufficient condition in order that $\mathfrak{g}$ *be finite dimensional is that*

$$\{X \in \mathfrak{g}_1 \,|\, [X,Y] = 0 \quad \forall Y \in \oplus_{p<-1}\mathfrak{g}_p\} = 0.$$

Proof. Let $\mathfrak{n} = \oplus_{p<-1}\mathfrak{g}_p$ and let $\mathfrak{h}$ denote the graded Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{h} = \{X \in \mathfrak{g} \,|\, [X,\mathfrak{n}] = 0\}$.

The condition is necessary: assume that there is $0 \neq X \in \mathfrak{g}_{-1}$ such that $[X,Y] = 0$ for every $Y \in \mathfrak{g}_{-1}$. Let $\mathfrak{g}'_{-1}$ be a J-invariant subspace of $\mathfrak{g}_{-1}$ complementary to the subspace $\mathfrak{g}''_{-1}$ generated by X and JX. Then we define $Y_0 \in \mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0 \subset \mathfrak{g}_0$ by

$$\begin{cases} [Y_0, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n}, \\ [Y_0, Z] = Z \quad \text{for} \quad Z \in \mathfrak{g}''_{-1}. \end{cases}$$

We note that $Y_0 \in \mathfrak{h}_0$ and that also the element $\tilde{Y}_0$, defined by

$$\begin{cases} [\tilde{Y}_0, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n}, \\ [\tilde{Y}_0, Z] = JZ \quad \text{for} \quad Z \in \mathfrak{g}''_{-1} \end{cases}$$

belongs to $\mathfrak{h}_0 \subset \mathfrak{g}_0$. By recurrence we can define sequences $\{Y_p\}_{p\geqslant 0}, \{\tilde{Y}_p\}_{p\geqslant 0}$, with $0 \neq Y_p, \tilde{Y}_p \in \mathfrak{h}_p \subset \mathfrak{g}_p$ by setting, for $p \geqslant 1$,

$$\begin{cases} [Y_p, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n}, \\ [Y_p, X] = Y_{p-1}, \\ [Y_p, JX] = \tilde{Y}_{p-1}, \end{cases}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

$$\begin{cases} [\tilde{Y}_p, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n}, \\ [\tilde{Y}_p, X] = \tilde{Y}_{p-1}, \\ [\tilde{Y}_p, JX] = -Y_{p-1}. \end{cases}$$

This shows that $\mathfrak{g}$ is infinite dimensional.

Conversely, when $\mathfrak{m}$ is nondegenerate, $\mathfrak{h}_1 = 0$ and the criterion applies. Indeed, let us consider the Hermitian symmetric $\mathbb{C} \otimes \mathfrak{g}_2$-valued form

$$(X|Y) = [JX, Y] + \sqrt{-1}[X, Y] \quad \text{for} \quad X, Y \in \mathfrak{g}_{-1}.$$

Then $\mathfrak{h}_0$ is contained in the space of $A \in \hom_{\mathbb{C}}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ such that

$$(AX|Y) + (X|AY) = 0 \quad \forall X, Y \in \mathfrak{g}_{-1}.$$

Let $\xi \in \mathfrak{h}_1$ and denote by $B\colon \mathfrak{g}_{-1} \to \mathfrak{g}_0$ the corresponding $\mathbb{R}$-linear map. Then we have

$$\begin{cases} B(X)Y = B(Y)X \quad \forall X, Y \in \mathfrak{g}_{-1}, \\ (B(X)Y|Z) + (Y|B(X)Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}_{-1}. \end{cases}$$

Since $B(X) \in \mathfrak{h}_0$ for $X \in \mathfrak{g}_{-1}$, we obtain

$$\begin{aligned} (B(X)Y|Z) &= (B(Y)X|Z) = -(X|B(Y)Z) = -(X|B(Z)Y) \\ &= (B(Z)X|Y) = (B(X)Z|Y) = -(Z|B(X)Y) \\ &= -\overline{(B(X)Y|Z)} \quad \forall X, Y, Z \in \mathfrak{g}_{-1}. \end{aligned}$$

This shows that

$$\Re(B(X)Y|Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}_{-1}$$

and hence $B = 0$, which gives $\xi = 0$. The proof is complete.

The subspaces $\mathfrak{g}_p = \mathfrak{g}_p(\mathfrak{m})$ of a canonical pseudocomplex prolongation $\mathfrak{g}(\mathfrak{m}) = \oplus_{-\mu \leqslant p} \mathfrak{g}_p(\mathfrak{m})$ of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m} = \oplus_{-\mu \leqslant p < 0} \mathfrak{m}_p$ can be defined inductively by

$$\mathfrak{g}_p(\mathfrak{m}) = \begin{cases} \mathfrak{m}_p & \text{if } p < 0, \\ \{A \in \mathrm{Der}(\mathfrak{m}, \mathfrak{m}) \mid A(\mathfrak{m}_j) \subset \mathfrak{m}_j \quad \forall j < 0; & \\ A(JX) = JA(X) \quad \forall X \in \mathfrak{m}_{-1}\} & \text{if } p = 0, \\ \{A \in \mathrm{Der}(\mathfrak{m}, \oplus_{h<p} \mathfrak{g}_h(\mathfrak{m})) \mid & \\ A(\mathfrak{m}_j) \subset \mathfrak{g}_{p+j}(\mathfrak{m}) \quad \forall j < 0\} & \text{if } p > 0, \end{cases} \tag{5}$$

where $\mathrm{Der}(\mathfrak{m}, V)$ indicates the space of derivations of the Lie algebra $\mathfrak{m}$ which take values in a left $\mathfrak{m}$-module V. This is indeed the characterization of the canonical prolongation given in [12]. In the following we will use these identifications without mentioning whenever it will simplify our arguments.

PROPOSITION 3.2. *Assume that $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{n}$ is the semidirect sum of a graded pseudocomplex ideal $\mathfrak{h}$ and of a graded pseudocomplex subalgebra $\mathfrak{n}$. Then*

(i) *if $\mathfrak{m}$ is fundamental, then $\mathfrak{n}$ is also fundamental;*
(ii) *if $\mathfrak{m}$ is fundamental and pseudocomplex, then there is a natural pseudocomplex graded Lie algebras homomorphism $\mathfrak{g}(\mathfrak{m}) \to \mathfrak{g}(\mathfrak{n})$ from the canonical pseudocomplex prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ into the canonical pseudocomplex prolongation $\mathfrak{g}(\mathfrak{n})$ of $\mathfrak{n}$ which makes the diagram*

$$\begin{tikzcd} \mathfrak{m} \arrow[r] \arrow[d] & \mathfrak{n} \arrow[d] \\ \mathfrak{g}(\mathfrak{m}) \arrow[r] & \mathfrak{g}(\mathfrak{n}) \end{tikzcd}$$

in which the top horizontal arrow is the projection associated to the direct sum decomposition $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{n}$ and the vertical arrows are the inclusion maps, commute.

Proof. Statement (i) is a consequence of the fact that $\mathfrak{h}$ is an ideal. We use formula (5) to define the subspaces of the canonical prolongations of $\mathfrak{n}$ and $\mathfrak{m}$. Then, if $\pi: \mathfrak{m} \to \mathfrak{n}$ is the pseudocomplex graded Lie algebra homomorphism associated to the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$, we define inductively $\pi_p: \mathfrak{g}_p(\mathfrak{m}) \to \mathfrak{g}_p(\mathfrak{n})$ by setting

$$\begin{array}{lll} \pi_p(X) = \pi(X) & \forall X \in \mathfrak{m}_p = \mathfrak{g}_p(\mathfrak{m}) & \text{if } p < 0; \\ \pi_p(A)(X) = \pi_{p+j}([A, X]) & \forall X \in \mathfrak{n}_j,\ j < 0,\ \forall A \in \mathfrak{g}_p(\mathfrak{m}) & \text{if } p \geqslant 0. \end{array}$$

The direct sum of the π_p's yields the desired homomorphism.

We also have:

PROPOSITION 3.3. *Assume that the pseudocomplex graded Lie algebra $\mathfrak{m} = \oplus_{-\mu \leqslant p < 0} \mathfrak{m}_p$ is the direct sum of two pseudocomplex graded ideals $\mathfrak{a} = \oplus_{-\mu \leqslant p < 0} \mathfrak{a}_p$ and $\mathfrak{b} = \oplus_{-\mu \leqslant p < 0} \mathfrak{b}_p$. Then:*

(i) *$\mathfrak{m}$ is fundamental if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are both fundamental;*
(ii) *$\mathfrak{m}$ is nondegenerate if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are both nondegenerate;*
(iii) *if $\mathfrak{m}$ is fundamental and nondegenerate, its canonical pseudocomplex prolongation $\mathfrak{g}(\mathfrak{m})$ is isomorphic to the direct sum of the canonical pseudocomplex prolongations $\mathfrak{g}(\mathfrak{a})$ and $\mathfrak{g}(\mathfrak{b})$ of $\mathfrak{a}$ and $\mathfrak{b}$ respectively.*

Proof. Statements (i) and (ii) are trivial, as $[\mathfrak{a}, \mathfrak{b}] = 0$. To prove (iii), we note first that we have an inclusion map: $\mathfrak{g}(\mathfrak{a}) \oplus \mathfrak{g}(\mathfrak{b}) \hookrightarrow \mathfrak{g}(\mathfrak{m})$. To prove that this map is an isomorphism, we argue by contradiction. If it was not the case, there is a smallest integer p such that $\mathfrak{g}_p(\mathfrak{a}) \oplus \mathfrak{g}_p(\mathfrak{b}) \neq \mathfrak{g}_p(\mathfrak{m})$. Clearly $p \geqslant 0$. Denote by $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{b}}$ the projections of $\oplus_{h<p}\mathfrak{g}_h(\mathfrak{m})$ onto $\oplus_{h<p}\mathfrak{g}_h(\mathfrak{a})$ and $\oplus_{h<p}\mathfrak{g}_h(\mathfrak{b})$ respectively. By (5) each element X of $\mathfrak{g}_p(\mathfrak{m})$ is defined by the restriction of $\mathrm{ad}(X)$ to $\mathfrak{m}$. This map is the sum $X_{\mathfrak{a}} + X_{\mathfrak{b}} + Z$ of $X_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mathrm{ad}(X) \circ \pi_{\mathfrak{a}}$, $X_{\mathfrak{b}} = \pi_{\mathfrak{b}} \circ \mathrm{ad}(X) \circ \pi_{\mathfrak{b}}$, and $Z = \pi_{\mathfrak{a}} \circ \mathrm{ad}(X) \circ \pi_{\mathfrak{b}} + \pi_{\mathfrak{b}} \circ \mathrm{ad}(X) \circ \pi_{\mathfrak{a}}$. Again by (5) we obtain $X_{\mathfrak{a}} \in \mathfrak{g}_p(\mathfrak{a})$ and $X_{\mathfrak{b}} \in \mathfrak{g}_p(\mathfrak{b})$. It suffices therefore to show that $Z = 0$. This follows because $[Z, Y] \in \mathfrak{g}_{p-1}(\mathfrak{b})$ if $Y \in \mathfrak{a}_{-1}$ and $[Z, Y] \in \mathfrak{g}_{p-1}(\mathfrak{a})$ if $Y \in \mathfrak{b}_{-1}$. Indeed, assuming $Y \in \mathfrak{a}_{-1}$ we have for every $U \in \mathfrak{m}_{-1}$

$$\begin{aligned}
\mathfrak{g}_{p-2}(\mathfrak{b}) \ni\ & [[Z, Y], U] = [[Z, Y], \pi_{\mathfrak{b}}(U)] \\
=\ & [Z, [Y, \pi_{\mathfrak{b}}(U)]] + [[Z, \pi_{\mathfrak{b}}(U)], Y] \\
=\ & [[Z, \pi_{\mathfrak{b}}(U)], Y] \in \mathfrak{g}_{p-2}(\mathfrak{a}).
\end{aligned}$$

Hence we have $[[Z, Y], U] = 0$ for all $U \in \mathfrak{m}_{-1}$. Because $p \geqslant 0$, $\mathfrak{g}(\mathfrak{m})$ is transitive and $\mathfrak{m}$ is nondegenerate, we obtain that $[Z, Y] = 0$ for all $Y \in \mathfrak{a}_{-1}$. In the same way we prove that $[Z, Y] = 0$ for all $Y \in \mathfrak{b}_{-1}$ and this implies that $Z = 0$, completing the proof of the proposition.

We note that in (iii) the assumption that $\mathfrak{m}$ is nondegenerate is essential, as the trivial example of an Abelian $\mathfrak{m} = \mathfrak{m}_{-1}$ of complex dimension larger than one shows.

Given a pseudocomplex graded Lie algebra $\mathfrak{g} = \oplus_{p\geqslant -\mu}\mathfrak{g}_p$ with complex structure $J\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, we consider its complexification $\mathfrak{g}^{\mathbb{C}} = \oplus_{p\geqslant -\mu}\mathfrak{g}_p^{\mathbb{C}}$. The complexification of the partial complex structure J is a partial complex structure $\hat{J} = \mathrm{id} \otimes J\colon \mathfrak{g}_{-1}^{\mathbb{C}} \to \mathfrak{g}_{-1}^{\mathbb{C}}$. In this way we obtain a new pseudocomplex graded Lie algebra $\hat{\mathfrak{g}}$ by considering $\mathfrak{g}^{\mathbb{C}}$ as a graded real Lie algebra endowed with the pseudocomplex structure $\hat{J}$. We have the following:

Remark 3.4. A necessary and sufficient condition in order that $\hat{\mathfrak{g}}$ be a Levi–Tanaka algebra is that $\mathfrak{g}$ is a Levi–Tanaka algebra.

Proof. First we note that $\hat{\mathfrak{m}} = \oplus_{p<0}\hat{\mathfrak{g}}_p$ is fundamental (nondegenerate) if and only if $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental (nondegenerate) and that $\hat{\mathfrak{g}}$ is transitive if and only if $\mathfrak{g}$ is transitive. Next we consider the canonical pseudocomplex prolongation $\mathfrak{a}$ of $\hat{\mathfrak{m}}$ and we prove by recurrence that the anti-$\mathbb{C}$-linear part of the map $\mathfrak{g}_{-1}^{\mathbb{C}} \to \mathfrak{g}_{p-1}^{\mathbb{C}}$ defined by any $X \in \mathfrak{a}_p$ (for $p \geqslant 0$) is 0. This implies our contention.

3.2. Properties of Levi–Tanaka algebras

LEMMA 3.5. *Let $\mathfrak{m} = \oplus_{-\mu\leqslant p<0}\mathfrak{g}_p$ be a pseudocomplex fundamental graded Lie algebra and let $\mathfrak{g} = \oplus_{p\geqslant -\mu}\mathfrak{g}_p$ be its canonical pseudocomplex prolongation, i.e.*

$\mathfrak{g}$ *is a Levi–Tanaka algebra. Assume that* $\mathfrak{m}$ *is nondegenerate, so that* $\mathfrak{g}$ *is finite dimensional. Then:*

(i) $\kappa_{\mathfrak{g}}(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ *if* $p + q \neq 0$ *where* $\kappa_{\mathfrak{g}}$ *is the Killing form of* $\mathfrak{g}$*;*
(ii) *there is a unique element* $E \in \mathfrak{g}_0$ *such that*

$$[E, X_p] = pX_p \quad \forall p \in \mathbb{Z}, \forall X_p \in \mathfrak{g}_p.$$

Proof. (i) Indeed, if $X_p \in \mathfrak{g}_p$ and $Y_q \in \mathfrak{g}_q$, then $\mathrm{ad}_{\mathfrak{g}}(X_p) \circ \mathrm{ad}_{\mathfrak{g}}(Y_q)(\mathfrak{g}_h) \subset \mathfrak{g}_{h+p+q}$ and hence is nilpotent when $p + q \neq 0$. Therefore

$$\kappa_{\mathfrak{g}}(X_p, X_q) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(X_p) \circ \mathrm{ad}_{\mathfrak{g}}(X_q)) = 0.$$

(ii) The $\mathbb{R}$-linear map $\tilde{E} : \mathfrak{m} \to \mathfrak{m}$ defined by

$$\tilde{E}(X_p) = pX_p \qquad \text{for} \quad p < 0 \quad \text{and} \quad X_p \in \mathfrak{g}_p$$

is a derivation of order zero of $\mathfrak{m}$, which commutes with J on $\mathfrak{g}_{-1}$ and therefore defines an element $E \in \mathfrak{g}_0$. We have to show that $[E, X_p] = pX_p$ when $p \geqslant 0$ and $X_p \in \mathfrak{g}_p$. This is certainly true when $p = 0$, because $\rho_{-1}(E)$ commutes with all endomorphisms in $\hom_{\mathbb{R}}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$. Assuming it is true for some $p \geqslant 0$, we have for $X_{p+1} \in \mathfrak{g}_{p+1}$ and $Y_{-1} \in \mathfrak{g}_{-1}$

$$\begin{aligned} [[E, X_{p+1}], Y_{-1}] &= [E, [X_{p+1}, Y_{-1}]] + [X_{p+1}, [Y_{-1}, E]] \\ &= (p+1)\,[X_{p+1}, Y_{-1}]. \end{aligned}$$

Since $\mathfrak{g}$ is transitive, this implies that $[E, X_{p+1}] = (p+1)X_{p+1}$.

If $\mathfrak{l}$ is any subset of a Levi–Tanaka algebra $\mathfrak{g}$, we will use in the following the notation $\mathfrak{l}_p$ for the set $\mathfrak{l} \cap \mathfrak{g}_p$ of its elements that are homogeneous of degree p. We say that $\mathfrak{l}$ is *graded* if $\mathfrak{l} = \oplus_{p\in\mathbb{Z}}\mathfrak{l}_p$.

COROLLARY 3.6. *Every ideal of a finite dimensional Levi–Tanaka algebra is graded.*

Proof. Let $X = X_{-\mu} + X_{1-\mu} + \cdots + X_\nu$ be an element of an ideal $\mathfrak{i}$ of $\mathfrak{g} = \oplus_{-\mu \leqslant p \leqslant \nu}\mathfrak{g}_p$, decomposed as a sum of its homogeneous components. Then $\mathfrak{i}$ contains all elements $\mathrm{ad}_{\mathfrak{g}}(E)^\ell(X)$, where E is the element of $\mathfrak{g}$ defined in the previous lemma and ℓ is any positive integer. Therefore $\mathfrak{i}$ contains

$$\begin{array}{ccccccc} X_{-\mu} & + & X_{1-\mu} & + & \cdots & + & X_\nu \\ \Leftrightarrow\mu X_{-\mu} & + & (1 \Leftrightarrow\mu)X_{1-\mu} & + & \cdots & + & \nu X_\nu \\ \cdots & & & & & & \\ (\Leftrightarrow\mu)^\ell X_{-\mu} & + & (1 \Leftrightarrow\mu)^\ell X_{1-\mu} & + & \cdots & + & \nu^\ell X_\nu \end{array}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

from which it follows that the ideal $\mathfrak{i}$ contains all the homogeneous components of X.

COROLLARY 3.7. *If $\mathfrak{g}$ is a finite dimensional Levi–Tanaka algebra, then the adjoint representation $\mathfrak{g} \ni X \to \mathrm{ad}_{\mathfrak{g}}(X) \in \mathfrak{gl}(\mathfrak{g})$ is faithful. In particular, all finite dimensional Levi–Tanaka algebras have a trivial center.*

Proof. Let $X = X_{-\mu} + \cdots + X_{\nu}$ be an element of $\mathfrak{g}$, decomposed into the sum of its homogeneous components, such that $\mathrm{ad}_{\mathfrak{g}}(X) = 0$. Let E be the element of $\mathfrak{g}_0$ defined in Lemma 3.5. From

$$\mathrm{ad}_{\mathfrak{g}}(X)(E) = \Leftrightarrow \sum_{-\mu \leqslant j \leqslant p} j\, X_j = 0$$

we deduce that $X_j = 0$ for every $j \neq 0$. Therefore $X = X_0 \in \mathfrak{g}_0$ and

$$\mathrm{ad}_{\mathfrak{g}}(X_0)(Y) = \rho_{-1}(X_0)(Y) = 0 \quad \forall Y \in \mathfrak{g}_{-1}$$

implies that $X_0 = 0$.

Using this corollary, we will often identify $\mathfrak{g}$ with the Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ which is the image of $\mathfrak{g}$ by the adjoint representation. We will call an element X of $\mathfrak{g}$ *nilpotent* (resp. *semisimple*) if $\mathrm{ad}_{\mathfrak{g}}(A)$ is nilpotent (resp. semisimple) as an element of $\mathfrak{gl}(\mathfrak{g})$.

Let V be a vector space over a field $\mathbb{K}$ of characteristic 0 and $\mathfrak{a}$ a Lie subalgebra of $\mathfrak{gl}_{\mathbb{K}}(V)$. We say that $\mathfrak{a}$ is *splittable* if it contains the semisimple and nilpotent component of each of its elements.

LEMMA 3.8. *The Lie subalgebra $\mathfrak{g}_0$ of the Levi–Tanaka algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$, considered as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$, is splittable, i.e. contains the semisimple and nilpotent component of each of its elements.*

Assume that $\mathfrak{m}$ is nondegenerate, so that $\mathfrak{g}$ is finite dimensional. Then:

(i) *if S and $N \in \mathfrak{g}_0$ are the semisimple and nilpotent components of $A \in \mathfrak{g}_0$ as endomorphisms of $\mathfrak{m}$, then $\mathrm{ad}_{\mathfrak{g}}(S)$ and $\mathrm{ad}_{\mathfrak{g}}(N)$ are respectively semisimple and nilpotent in $\mathfrak{gl}(\mathfrak{g})$;*

(ii) *the algebra $\mathfrak{g}$ is splittable as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.*

Proof. Every element $A \in \mathfrak{g}_0$ defines a derivation of the fundamental Lie algebra $\mathfrak{m}$. The semisimple component S and the nilpotent component N of A in $\mathfrak{gl}(\mathfrak{m})$ are still derivations of $\mathfrak{m}$ (cf. [3] Ch. 7 Section 1 Proposition 4(ii)). Moreover, since S and N are polynomials of A, we have $S(\mathfrak{g}_p) \subset \mathfrak{g}_p$ and $N(\mathfrak{g}_p) \subset \mathfrak{g}_p$ for all $p < 0$ and S and N define $\mathbb{C}$-linear endomorphisms of $\mathfrak{g}_{-1}$. This shows that $S, N \in \mathfrak{g}_0$.

Let us assume now that $\mathfrak{g}$ be finite dimensional. First we note that the elements of $\mathfrak{g}_0$ are splittable as endomorphisms of $\mathfrak{g}$. This follows by the same argument given above: if $A \in \mathfrak{g}_0$, then $\mathrm{ad}_{\mathfrak{g}}(A)$ is a 0-degree derivation of $\mathfrak{g}$ which defines a $\mathbb{C}$-linear endomorphism of $\mathfrak{g}_{-1}$. Then the semisimple and nilpotent components $\tilde{S}$

and $\tilde{N}$ of $\mathrm{ad}_{\mathfrak{g}}(A)$ in $\mathfrak{gl}(\mathfrak{g})$ are 0-degree derivations of $\mathfrak{g}$ which define $\mathbb{C}$-linear endomorphisms of $\mathfrak{g}_{-1}$. Their restrictions to $\mathfrak{m}$ are commuting semisimple and nilpotent endomorphisms of $\mathfrak{m}$ and thus are the semisimple and nilpotent components S and N of the representation of A in $\mathfrak{gl}(\mathfrak{m})$. This shows that $\mathrm{ad}_{\mathfrak{g}}(S)$ and $\mathrm{ad}_{\mathfrak{g}}(N)$ are still semisimple and nilpotent respectively. Indeed they coincide with $\tilde{S}$ and $\tilde{N}$ because by the construction of the canonical prolongation 0-degree derivations of $\mathfrak{m}$ uniquely extend to 0-degree derivations of $\mathfrak{g}$. This proves (i).

To complete the proof, we observe that when $\mathfrak{g}$ is finite dimensional the elements of $\cup_{p\neq 0}\mathfrak{g}_p$ are all nilpotent. It follows from (i) that $\mathfrak{g}$, considered as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$, is generated by its semisimple and nilpotent elements. This implies that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is splittable (see [3] Ch. VII Section 5 Theorem 1).

LEMMA 3.9. *Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra. Then we can find a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ containing the element E of Lemma 3.5 and contained in $\mathfrak{g}_0$. Every Cartan subalgebra of the Lie algebra $\mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}$ and therefore $\mathfrak{g}_0$ contains regular elements of $\mathfrak{g}$.*

Proof. Let $\mathcal{S}$ denote the set of semisimple elements of $\mathfrak{g}$ and $\mathcal{T}$ the set of all commutative Lie subalgebras of $\mathfrak{g}$ contained in $\mathcal{S}$. Let $\mathcal{T}_1$ denote the set of maximal (with respect to $\subset$) elements of $\mathcal{T}$. Because $\mathfrak{g}$ is splittable by Lemma 3.8, for every $\mathfrak{t} \in \mathcal{T}_1$ its centralizer $C_{\mathfrak{g}}(\mathfrak{t}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{t}] = 0\}$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ (see [3] Ch. VII Section 5 Proposition 6). The element E defined in Lemma 3.5 is semisimple. Therefore it can be included in a Lie subalgebra $\mathfrak{t} \in \mathcal{T}_1$. Let $\mathfrak{a}$ denote its centralizer in $\mathfrak{g}$. It is a Cartan subalgebra of $\mathfrak{g}$ and, if $X \in \mathfrak{a}$ and $X = \Sigma_p X_p$ is its decomposition into the sum of its homogeneous components, we have

$$0 = [E, X] = \sum_{p\neq 0} p\, X_p,$$

and hence $X = X_0 \in \mathfrak{g}_0$.

The last statement follows from [3] Ch. VII Section 3 Proposition 3.

LEMMA 3.10. *Let $\mathfrak{m}$ be a pseudocomplex fundamental graded Lie algebra of kind 2 and let $\mathfrak{g} = \oplus_{p\geqslant -2}\mathfrak{g}_p$ be its pseudocomplex canonical prolongation. Then there is a unique element $\tilde{J} \in \mathfrak{g}_0$ such that*

$$[\tilde{J}, X] = JX \quad \forall X \in \mathfrak{g}_{-1}. \tag{6}$$

Proof. When $\mathfrak{m}$ is of kind 2, the elements of $\mathfrak{g}_0$ can be identified to the space of $\mathbb{C}$-linear maps $A\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ for which there is an $\mathbb{R}$-linear map $B\colon \mathfrak{g}_{-2} \to \mathfrak{g}_{-2}$ such that

$$[AX, Y] + [X, AY] = B([X, Y]) \quad \forall X, Y \in \mathfrak{g}_{-1}.$$

From the definition of a pseudocomplex graded Lie algebra, this relation holds true for $A = J$ and $B = 0$.

We will see below that the existence of such an element $\tilde{J}$ is not guaranteed when the kind μ of $\mathfrak{m}$ is greater than 2. We say in general that a pseudocomplex graded Lie algebra $\mathfrak{g} = \oplus_{p \geqslant -\mu} \mathfrak{g}_p$ *has the (J) property* if there is an element $\tilde{J} \in \mathfrak{g}_0$ for which (6) holds true. In this case we denote by J_p the representation $\rho_p(\tilde{J})$ of $\tilde{J}$ in $\mathfrak{g}_p$. Note that $J_{-1} = J$ is the complex structure of $\mathfrak{g}_{-1}$.

LEMMA 3.11. *Let $\mathfrak{g} = \oplus_{p \geqslant -\mu} \mathfrak{g}_p$ be a canonical pseudocomplex prolongation of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m}$ of kind $\mu \geqslant 2$. If $\mathfrak{g}$ has the (J) property, then*:

(i) *J_p defines a complex structure on $\mathfrak{g}_p$ for $p = -3, -1, 1$;*
(ii) *$J_p = 0$ for $p = -2, 0$.*

When $\mu = 2$, and $\mathfrak{m}$ is nondegenerate, J_p is a complex structure in $\mathfrak{g}_p$ for p odd and 0 for p even.

Proof. The statement is certainly true when $p = -2, -1, 0$.

Let us consider the case $p = -3$. The elements $[X, T]$, for $X \in \mathfrak{g}_{-1}$ and $T \in \mathfrak{g}_{-2}$ are a set of generators of $\mathfrak{g}_{-3}$ because $\mathfrak{m}$ is fundamental. Since $\tilde{J}$ is a 0-degree derivation of $\mathfrak{m}$ we have

$$\begin{aligned} J_{-3}([X,T]) &= [\tilde{J},[X,T]] = [J_{-1}X,T] + [X, J_{-2}T] \\ &= [JX,T] \quad \forall X \in \mathfrak{g}_{-1},\ \forall T \in \mathfrak{g}_{-2}. \end{aligned}$$

Then we obtain

$$\begin{aligned} J^2_{-3}([X,T]) &= J_{-3}([JX,T]) = [J^2X,T] \\ &= -[X,T] \quad \forall X \in \mathfrak{g}_{-1},\ \forall T \in \mathfrak{g}_{-2}, \end{aligned}$$

from which we have

$$J^2_{-3}Y = -Y \quad \forall Y \in \mathfrak{g}_{-3}$$

because this relation holds true on a set of generators of $\mathfrak{g}_{-3}$.

In general, the argument above shows that, if $p < 0$ and $J_p = 0$, then J_{p-1} is a complex structure in $\mathfrak{g}_{p-1}$.

Let us turn now to the case $p = 1$. Let $X \in \mathfrak{g}_1$. Then we have

$$0 = J_0([X,Y]) = [J_1X,Y] + [X,JY] \quad \forall Y \in \mathfrak{g}_{-1}.$$

This yields

$$[J_1X,Y] = -[X,JY] \quad \forall X \in \mathfrak{g}_1,\ \forall Y \in \mathfrak{g}_{-1}.$$

Then we have

$$\begin{aligned} [J_1^2X,Y] &= -[J_1X,JY] = [X,J^2Y] \\ &= -[X,Y] \quad \forall X \in \mathfrak{g}_1,\ \forall Y \in \mathfrak{g}_{-1}. \end{aligned}$$

Since $\mathfrak{g}$ is transitive, this shows that J_1 is a complex structure in $\mathfrak{g}_1$.

More in general, this argument shows that, if $J_p = 0$ for some $p \geqslant 0$, then J_{p+1} is a complex structure in $\mathfrak{g}_{p+1}$.

Let us turn now to the case where $\mathfrak{m}$ is of kind 2. Then, assuming that $J_p = 0$ for some $p \geqslant 0$, we have

$$[J_{p+2}X_{p+2}, Y_{-2}] = J_p[X_{p+2}, Y_{-2}] - [X_{p+2}, J_{-2}Y_{-2}] = 0 \quad \forall X_{p+2} \in \mathfrak{g}_{p+2}$$

for every $Y_{-2} \in \mathfrak{g}_{-2}$. This implies that $J_{p+2}X_{p+2} \in \mathfrak{h}_{p+2}$, where $\mathfrak{h} = \{Z \in \mathfrak{g} \,|\, [Z, \mathfrak{g}_{-2}] = 0\}$. But we proved (see Theorem 3.1) that $\mathfrak{h}_p = 0$ for $p > 0$ when $\mathfrak{m}$ is nondegenerate. Then we obtain by recurrence that $\mathfrak{g}_p = 0$ for every p even.

By the previous remarks, this gives the proof of the lemma.

Remark 3.12. Assume that J_q is a complex structure for some $q \geqslant 1$. Let $X \in \mathfrak{g}_{q+1}$ and $Y \in \mathfrak{g}_{-1}$. Then we obtain

$$-[X, Y] = J_q^2[X, Y] = [J_{q+1}^2 X, Y] + 2[J_{q+1}X, JY] - [X, Y].$$

From this we derive

$$[J_{q+1}^2 X, Y] = -2[J_{q+1}X, JY] \quad \forall X \in \mathfrak{g}_{q+1}, \forall Y \in \mathfrak{g}_{-1}.$$

Applying this equality we obtain

$$\begin{aligned} [J_{q+1}^3 X, Y] &= -2[J_{q+1}^2 X, JY] \\ &= -4[J_{q+1}X, Y] \quad \forall X \in \mathfrak{g}_{q+1}, \forall Y \in \mathfrak{g}_{-1}. \end{aligned}$$

Because $\mathfrak{g}$ is transitive, we have

$$J_{q+1}^3 + 4J_{q+1} = 0.$$

We obtain the analogous equation for J_{p-1} if we assume that $p \leqslant -3$ and J_p is a complex structure in $\mathfrak{g}_p$.

Remark 3.13. Since $[\tilde{J}, A] = 0$ for every $A \in \mathfrak{g}_0$, the linear endomorphism $\rho_p(A)$ is $\mathbb{C}$-linear in $\mathfrak{g}_p$ whenever J_p defines a complex structure in $\mathfrak{g}_p$.

Remark 3.14. From the lemma above we obtain that $\dim_{\mathbb{R}} \mathfrak{g}_{-3}$ must be even when $\mathfrak{g}$ has the (J) property. In particular we cannot expect (J) to hold for the Levi–Tanaka algebras of a CR manifold M of type (1,2) in which the vector fields in $\Gamma(M, HM)$ generate the Lie algebra of tangent vector fields to M.

3.3. SEMISIMPLE PROLONGATIONS

We first recall a lemma on the structure of semisimple graded Lie algebras. See for instance [15].

LEMMA 3.15. *Let* $\mathfrak{s} = \oplus_{-\mu\leqslant p\leqslant\nu}\mathfrak{s}_p$ *be a finite dimensional semisimple graded Lie algebra over* $\mathbb{R}$ *and let* $\kappa_{\mathfrak{s}}$ *be its Killing form. Then:*

(i) $\mathfrak{s}$ *contains a unique element* $E \in \mathfrak{s}_0$ *such that*

$$[E, X_p] = pX_p \quad \forall \Leftrightarrow \mu \leqslant p \leqslant \nu, \quad \forall X_p \in \mathfrak{s}_p;$$

(ii) $\kappa_{\mathfrak{s}}(\mathfrak{s}_p, \mathfrak{s}_q) = 0$ *for* $p + q \neq 0$;

(iii) $\nu = \mu$ *and the Killing form defines a duality pairing between* $\mathfrak{s}_p$ *and* $\mathfrak{s}_{-p}$*: in particular*

$$\dim_{\mathbb{R}} \mathfrak{s}_p = \dim_{\mathbb{R}} \mathfrak{s}_{-p} \quad \textit{for } 0 \leqslant p \leqslant \mu;$$

(iv) $\mathfrak{g}_0$ *is a reductive Lie algebra, i.e. decomposes into the direct sum of a commutative and a semisimple ideal;*

(v) *if* $\mu > 0$, *then* $\mathfrak{s}$ *is of the noncompact type.*

Proof. (i) The linear operator $T: \mathfrak{s} \to \mathfrak{s}$ defined by $T(X_p) = pX_p$ for $p \in \mathbb{Z}$ and $X_p \in \mathfrak{s}_p$ is a derivation of order zero of $\mathfrak{s}$ and hence, because $\mathfrak{s}$ is semisimple, defines an element E of $\mathfrak{s}_0$.

(ii) If $X_p \in \mathfrak{s}_p$ and $Y_q \in \mathfrak{s}_q$ with $p + q \neq 0$, then the linear operator $\mathrm{ad}_{\mathfrak{g}}(X_p) \circ \mathrm{ad}_{\mathfrak{g}}(Y_q): \mathfrak{s} \to \mathfrak{s}$ is nilpotent because $\mathrm{ad}_{\mathfrak{g}}(X_p) \circ \mathrm{ad}_{\mathfrak{g}}(Y_q)(\mathfrak{s}_h) \subset \mathfrak{s}_{h+p+q}$.

(iii) is a consequence of (2), because $\kappa_{\mathfrak{s}}$ is nondegenerate on $\mathfrak{s}$.

Statement (iv) follows because the restriction to $\mathfrak{s}_0$ of Killing form $\kappa_{\mathfrak{s}}$, which is nondegenerate by (iii), is the invariant bilinear form in $\mathfrak{s}_0$ induced by the adjoint representation. Then we apply [3] Ch. I Section 6 Proposition 5(d). The last statement is a trivial remark, as by (iii) the Witt index of the Killing form is larger or equal to $\dim_{\mathbb{R}} \oplus_{-\mu\leqslant p<0} \mathfrak{s}_p$.

LEMMA 3.16. *Let* $\mathfrak{s} = \oplus_{-\mu\leqslant p\leqslant\mu}\mathfrak{s}_p$ *be a semisimple graded Lie algebra over* $\mathbb{R}$. *Then a necessary and sufficient condition in order that* $\mathfrak{s}$ *be transitive is that*

$$[X, \mathfrak{s}_{-1}] \neq 0 \qquad \forall X \in \mathfrak{s}_0, X \neq 0. \tag{7}$$

Proof. The condition (7) is trivially necessary. Let us prove sufficiency. First we show that, for $X \in \mathfrak{s}$,

$$[X, \mathfrak{s}_{-1}] = 0 \Rightarrow [X, \mathfrak{s}_p] = 0 \quad \forall p < 0.$$

This follows by recurrence: indeed $[\mathfrak{s}_p, \mathfrak{s}_{-1}] = \mathfrak{s}_{p-1}$ for $p < 0$ because $\mathfrak{m}$ is fundamental; then

$$[X, \mathfrak{s}_{p-1}] = [[X, \mathfrak{s}_p], \mathfrak{s}_{-1}] + [\mathfrak{s}_p, [X, \mathfrak{s}_{p-1}]]$$

shows that $[X, \mathfrak{s}_{p-1}] = 0$ when $[X, \mathfrak{s}_p] = [X, \mathfrak{s}_{-1}] = 0$.

Let now X be a nonzero element of $\mathfrak{s}_q$ for some $q > 0$. Since $\mathfrak{m}$ is fundamental, it suffices to show that there is $Y \in \mathfrak{s}_p$ for some $p < 0$ such that $[X, Y] \neq 0$. By

Lemma 3.15 (iii) there is $U \in \mathfrak{s}_{-q}$ such that $\mathrm{ad}_{\mathfrak{s}}(U) \circ \mathrm{ad}_{\mathfrak{s}}(X) \neq 0$. Then we can find $p \in \mathbb{Z}$ and a homogeneous $Z \in \mathfrak{s}_p$, such that

$$\mathrm{ad}_{\mathfrak{s}}(U) \circ \mathrm{ad}_{\mathfrak{s}}(X)(Z) = [U, [X, Z]] \neq 0.$$

Since we obtain in particular $[X, Z] \neq 0$, if $p < 0$ we have finished. Assume now that $p = 0$. Then

$$[U, [X, Z]] = [[U, X], Z] + [X, [U, Z]] \neq 0$$

and the Lie product of X by either U or $[U, Z]$, both belonging to $\mathfrak{s}_{-q}$, is different from zero. If finally $p > 0$, we use again Lemma 3.15 (iii): since $[X, Z] \in \mathfrak{s}_{p+q}$ is different from zero, we can find $V \in \mathfrak{s}_{-(p+q)}$ such that

$$0 \neq \kappa_{\mathfrak{s}}([X, Z], V) = \Leftrightarrow\kappa_{\mathfrak{s}}(Z, [X, V])$$

and $[X, V] \neq 0$ with $V \in \mathfrak{s}_{-(p+q)}$ and $\Leftrightarrow(p+q) < 0$. The proof is complete.

If $\mathfrak{g}$ is a Lie algebra, we define by recurrence $[X] = X$ for every element $X \in \mathfrak{g}$ and $[X_1, X_2, \dots, X_k] = [X_1, [X_2, \dots, X_k]]$ for every $X_1, \dots, X_k \in \mathfrak{g}$ when $k > 1$. For $\mathfrak{a} \subset \mathfrak{g}$ denote by $\mathfrak{a}^{(k)}$ the linear span of $[X_1, \dots, X_k]$ for $X_1, \dots, X_k \in \mathfrak{a}$.

LEMMA 3.17. *Let* $\mathfrak{s} = \oplus_{-\mu \leqslant p \leqslant \mu} \mathfrak{s}_p$ *be a simple graded Lie algebra over* $\mathbb{R}$. *Then* $\mathfrak{m} = \oplus_{-\mu \leqslant p < 0} \mathfrak{s}_p$ *is fundamental if and only if* $\mathfrak{s}_{-1}^{(\mu)} \neq 0$ (*i.e. there exist* $X_1, \dots, X_\mu \in \mathfrak{s}_{-1}$ *such that* $[X_1, \dots, X_\mu] \neq 0$). *In this case* $\mathfrak{s}$ *is nondegenerate and transitive if and only if* $\mu \geqslant 2$.

Proof. In the proof we shall use the following:

CLAIM 3.18. *For every element* X *of a graded Lie algebra* $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ *the elements of the ideal* $\mathfrak{i}(X)$ *generated by* X *are linear combinations of* X *and elements of the form* $[Z_k, \dots, Z_1, X]$ *with* Z_i *homogeneous and* $\deg Z_k \geqslant \cdots \geqslant \deg Z_1$.

(This claim can be easily obtained using induction and Jacobi's identity.) Suppose there exist $X_1, \dots, X_\mu \in \mathfrak{s}_{-1}$ such that $[X_\mu, \dots, X_1] \neq 0$. Then we have that $Y_j = [X_j, \dots, X_1] \neq 0$ for $1 < j \leqslant \mu$ and the ideals $\mathfrak{i}(Y_j)$ generated by the Y_j's are not zero. Because $\mathfrak{s}$ is simple, they coincide with $\mathfrak{s}$ and $\mathfrak{i}(Y_j)_{-p} = \mathfrak{i}(Y_j) \cap \mathfrak{s}_{-p} = \mathfrak{s}_{-p}$. We will prove, by recurrence, that $\mathfrak{s}_{-q} = \mathfrak{s}_{-1}^{(q)}$ for $1 < q \leqslant \mu$. If $q = \mu$, then it follows from the claim that $\mathfrak{s}_{-\mu} = \mathfrak{i}(Y_\mu)_{-\mu}$ is generated by elements of the form $[Z_k, \dots, Z_1, Y_\mu]$ with $Z_i \in \mathfrak{s}_0$ for every i. Because $\mathfrak{s}_{-1}^{(\mu)}$ is invariant under the adjoint action of $\mathfrak{g}_0$, we conclude that $\mathfrak{s}_{-1} = \mathfrak{i}(Y_\mu)_{-\mu} = \mathfrak{s}_{-1}^{(\mu)}$. Assume now that $q < \mu$ and $\mathfrak{s}_{-p} = \mathfrak{s}_{-1}^{(p)}$ for $q < p \leqslant \mu$. We want to prove that $\mathfrak{s}_{-q} = \mathfrak{s}_{-1}^{(q)}$. By the claim $\mathfrak{s}_{-q} = \mathfrak{i}(Y_q)_{-q}$ is generated by linear combinations of Y_q and elements of the form $[Z_k, \dots, Z_1, Y_q]$ with Z_i homogeneous, $\deg Z_k \geqslant \cdots \geqslant \deg Z_1$ and

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

$\Sigma_1^k \deg Z_i = 0$. It suffices to prove that they all belong to $\mathfrak{s}_{-1}^{(q)}$. If $\deg Z_k = 0$, then $Z_1, \ldots, Z_k \in \mathfrak{s}_0$ and therefore $[Z_k, \ldots, Z_1, Y_q] \in \mathfrak{s}_{-1}^{(q)}$. If $\deg Z_k > 0$, then $[Z_k, \ldots, Z_1, Y_q]$ is a linear combination of elements of the form $[Z_k, U_r, \ldots, U_1]$ with $U_j \in \mathfrak{s}_{-1}$ and $r = q + \deg Z_k$. By repeated application of the formula $[V, V_s, \ldots, V_1] = \Sigma_{i=1}^s [V_s, \ldots, V_{i+1}, [V, V_i], V_{i-1}, \ldots, V_1]$, we can show that the commutator $[Z_k, U_r, \ldots, U_1]$ belongs to $\mathfrak{s}_{-1}^{(q)}$.

The converse is obvious.

Suppose now that $\mathfrak{m}$ is fundamental. Assume $\mu \geqslant 2$. First we show that $\mathfrak{s}$ is nondegenerate. If $\mathfrak{s}$ was degenerate then we could find $X \in \mathfrak{s}_{-1}$ such that $[X, \mathfrak{m}] = 0$ and the ideal $\mathfrak{i}(X)$ generated by such an X would be different from zero, hence equal to $\mathfrak{s}$. On the other hand, using the claim above we obtain that $\mathfrak{s}_{-\mu} = \mathfrak{i}(X)_{-\mu} = 0$ and this gives a contradiction.

Next we show that $\mathfrak{s}$ is transitive. Let $\mathfrak{a}$ be equal to $\{A \in \mathfrak{s}_0 | [A, \mathfrak{m}] = 0\}$. By Lemma 3.16, it suffices to prove that $\mathfrak{a}$ is zero. Assume that $A \in \mathfrak{a}$. If $X \in \mathfrak{s}_0$, then $[[A, X], Z] = [[A, Z], X] + [A, [X, Z]] = 0$ for every $Z \in \mathfrak{m}$, so that $[\mathfrak{a}, \mathfrak{s}_0] \subset \mathfrak{a}$. If $X \in \mathfrak{s}$ is homogeneous of positive degree, then we have

$$0 = \kappa_{\mathfrak{s}}([A, Z], X) = \Leftrightarrow \kappa_{\mathfrak{s}}(Z, [A, X]) \quad \forall Z \in \mathfrak{m}$$

and, by Lemma 3.15, we obtain $[A, X] = 0$. It follows that $\mathfrak{a}$ is an ideal of $\mathfrak{s}$. Since it is contained in $\mathfrak{s}_0$ and $\mathfrak{s}$ is simple with $\mu \geqslant 2$, we have $\mathfrak{a} = 0$.

The converse is obvious, as μ is always greater than or equal to 2 for a nondegenerate fundamental graded Lie algebra.

Remark 3.19. If $\mathfrak{s}$ is the Levi–Tanaka algebra at a point x of a CR manifold, the condition in the previous lemma means that the highest order Levi form is not identically zero at x (cf. [11]).

LEMMA 3.20. *Let* $\mathfrak{g} = \oplus \mathfrak{g}_p$ *be a semisimple transitive prolongation of a fundamental graded Lie algebra* $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$. *Then* $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$.

Proof. Setting $\mathfrak{h}_p = \mathfrak{g}_p$ for $p \neq 0$ and $\mathfrak{h}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$, we obtain an ideal $\mathfrak{h} = \oplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ for an ideal $\mathfrak{a} \subset \mathfrak{g}$. As in the proof of Lemma 3.6 it can be proved that each ideal of $\mathfrak{g}$, in particular $\mathfrak{a}$, is graded. Therefore $\mathfrak{a} \subset \mathfrak{g}_0$. Since $[\mathfrak{a}, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{-1} = 0$, we have $\mathfrak{a} = 0$ because $\mathfrak{g}$ is transitive.

We have

THEOREM 3.21. *Let* $\mathfrak{m}$ *be a fundamental graded Lie algebra and let* $\mathfrak{s}$ *be a semisimple transitive prolongation of* $\mathfrak{m}$. *If* $\mathfrak{g}$ *is a finite dimensional transitive prolongation of* $\mathfrak{m}$ *containing* $\mathfrak{s}$, *then* $\mathfrak{g}$ *coincides with* $\mathfrak{s}$.

In particular, if $\mathfrak{m}$ *is also pseudocomplex and nondegenerate and if* $\mathfrak{s}$ *is a semisimple transitive pseudocomplex prolongation, then* $\mathfrak{s}$ *is isomorphic to the canonical pseudocomplex prolongation of* $\mathfrak{m}$.

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

Proof. Assume that $\mathfrak{s}$ is a transitive semisimple prolongation of $\mathfrak{m}$. In this case we can consider $\mathfrak{s}$ as a subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ and $\mathfrak{s}$ coincide. Indeed, by (iii) in Lemma 3.15, $\mathfrak{g}_p$ is equal to $\mathfrak{s}_p$ for $p \neq 0$ (because they have the same dimension as vector spaces) and, by the lemma above, $\mathfrak{g}$ coincides with $\mathfrak{s}$.

Let us prove now that $\mathfrak{g}$ is semisimple. We already know that $\mathfrak{g}$ is finite dimensional. Then it suffices to show that its radical $\mathfrak{r}$ is 0. By Corollary 3.6, $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}$. We have $\mathfrak{r} \cap \mathfrak{s} = 0$ because $\mathfrak{s}$ is semisimple and hence $\oplus_{-\mu \leqslant p<0}\mathfrak{r}_p = 0$ because $\oplus_{-\mu \leqslant p<0}\mathfrak{g}_p \subset \mathfrak{s}$.

Let us show by recurrence that $\mathfrak{r}_p = 0$ also when $p \geqslant 0$. For $p = 0$ we have $[\mathfrak{r}_0, \mathfrak{g}_{-1}] \subset \mathfrak{r}_{-1} = 0$ and hence $\mathfrak{r}_0 = 0$ because $\mathfrak{g}$ is transitive. Assuming $\mathfrak{r}_p = 0$ for some $p \geqslant 0$, we deduce that also $\mathfrak{r}_{p+1} = 0$ from the transitivity of $\mathfrak{g}$ and the fact that $[\mathfrak{r}_{p+1}, \mathfrak{g}_{-1}] \subset \mathfrak{r}_p = 0$.

The following is a criterion for the simplicity of the prolongation, which is close to one which was stated in [12].

THEOREM 3.22. *Let $\mathfrak{g}$ be the canonical pseudocomplex prolongation of a nondegenerate pseudocomplex fundamental graded Lie algebra $\mathfrak{m}$ and assume that ρ_{-1} is irreducible and $\mathfrak{g}_1 \neq 0$. Then $\mathfrak{g}$ is simple.*

Proof. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. We want to show that $\mathfrak{r} = 0$. We consider two cases.

(a) *Assume* $\mathfrak{r}_{-1} = 0$.

In this case, we claim that $\mathfrak{r}_p = 0$ for $p \geqslant -1$. Indeed, we argue by recurrence on $p \geqslant -1$. We have $\mathfrak{r}_{-1} = 0$ by assumption. If $\mathfrak{r}_p = 0$ for some $p \geqslant -1$, we have $[\mathfrak{r}_{p+1}, \mathfrak{g}_{-1}] \subset \mathfrak{r}_p = 0$ and hence $\mathfrak{r}_{p+1} = 0$ because $\mathfrak{g}$ is transitive. This shows that $\mathfrak{r} \subset \mathfrak{n} = \oplus_{-\mu \leqslant p<-1}\mathfrak{g}_p$. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$: $\mathfrak{s}$ is semisimple and $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$. We have $\mathfrak{s} \simeq \mathfrak{g}/\mathfrak{r}$ and, since $\mathfrak{r}_{-1} = 0$, for every $X \in \mathfrak{g}_{-1}$ the subalgebra $\mathfrak{s}$ contains an element of the form $X + Z$ with $Z \in \mathfrak{n}$. Since

$$\begin{aligned}&[X_1 + Z_1, [\ldots, [X_{\mu-1} + Z_{\mu-1}, X_\mu + Z_\mu]\ldots]]\\ &\quad = [X_1, [\ldots, [X_{\mu-1}, X_\mu]\ldots]]\end{aligned}$$

if $X_1, X_2, \ldots, X_\mu \in \mathfrak{g}_{-1}$ and $Z_1, Z_2, \ldots, Z_\mu \in \mathfrak{n}$, we obtain $\mathfrak{g}_{-\mu} \subset \mathfrak{s}$ because $\mathfrak{m}$ is fundamental.

Repeating a similar argument we deduce that also $\mathfrak{g}_{1-\mu}, \ldots, \mathfrak{g}_{-2}$ are contained in $\mathfrak{s}$ and then $\mathfrak{g} = \mathfrak{s}$ and $\mathfrak{r} = 0$.

(b) *Assume* $\mathfrak{r}_{-1} \neq 0$.

Since $\mathfrak{r}_{-1}$ is a $\rho_{-1}(\mathfrak{g}_0)$-invariant subspace of $\mathfrak{g}_{-1}$ and by assumption ρ_{-1} is irreducible, we have in this case $\mathfrak{r}_{-1} = \mathfrak{g}_{-1}$. Let $\mathfrak{r}^{(0)} = \mathfrak{r}$ and define recursively the ideals $\mathfrak{r}^{(\ell)} = [\mathfrak{r}^{(\ell-1)}, \mathfrak{r}^{(\ell-1)}]$ for $\ell > 0$. We have $\mathfrak{r}^{(\ell)} = 0$ for $\ell > 0$ and sufficiently large because $\mathfrak{r}$ is a solvable ideal of $\mathfrak{g}$. Then there is a smallest positive integer h such that $\mathfrak{r}^{(h)}_{-1} = 0$, while $\mathfrak{r}^{(h-1)}_{-1} \neq 0$. We note that $\mathfrak{r}^{(h-1)}$ is an ideal of $\mathfrak{g}$, in particular $\mathfrak{r}^{(h-1)}_{-1}$ is a $\rho_{-1}(\mathfrak{g}_0)$-invariant subspace of $\mathfrak{g}_{-1}$. Therefore $\mathfrak{r}^{(h-1)}_{-1} = \mathfrak{g}_{-1}$.

On the other hand, arguing as in (a), we prove that $\mathfrak{r}^{(h)} \subset \mathfrak{n}$. Therefore we have

$$[\mathfrak{r}_p^{(h-1)}, \mathfrak{g}_{-1}] = [\mathfrak{r}_p^{(h-1)}, \mathfrak{r}_{-1}^{(h-1)}] \subset \mathfrak{r}_{p-1}^{(h)} = 0 \quad \text{for} \quad p \geqslant 0$$

and this implies that $\mathfrak{r}_p^{(h-1)} = 0$ by the transitivity of $\mathfrak{g}$. This gives a contradiction, because

$$\mathfrak{r}_0^{(h-1)} \supset [\mathfrak{r}_{-1}^{(h-1)}, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq 0.$$

This shows that $\mathfrak{r}_{-1} = 0$ and then $\mathfrak{r} = 0$ by (a).

Therefore $\mathfrak{g}$ is semisimple. It is simple, because if it was the direct sum of two semisimple ideals $\mathfrak{s}'$ and $\mathfrak{s}''$, then each of the subspaces $\mathfrak{s}'_{-1}$ and $\mathfrak{s}''_{-1}$ would be $\rho_{-1}(\mathfrak{g}_0)$-invariant. One of these, say $\mathfrak{s}'_{-1}$ is then equal to $\mathfrak{g}_{-1}$ and the other is 0 by the irreducibility of ρ_{-1}. But, since $\mathfrak{m}$ is fundamental, $\mathfrak{s}'$ is then a semisimple pseudocomplex prolongation of $\mathfrak{m}$ and therefore coincides with $\mathfrak{g}$. This gives $\mathfrak{s}'' = 0$ and completes the proof of the theorem.

Remark 3.23. Vice versa, when $\mathfrak{g}$ is semisimple, then the representation ρ_{-1} is completely reducible. Indeed, $\mathfrak{g}_0$ is reductive. Then its radical $\mathfrak{r}(\mathfrak{g}_0)$ is equal to its center $\mathfrak{z}(\mathfrak{g}_0)$ and therefore is contained in every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is contained in $\mathfrak{g}_0$. Hence its elements are semisimple together with their ρ_{-1} representation. Then ρ_{-1} is completely reducible (cf. [3] Ch. I Section 6 Theorem 4).

3.4. Solvable Prolongations

We consider in this subsection criteria for the solvability of the canonical pseudocomplex prolongation.

Let $\mathfrak{m} = \oplus_{-\mu \leqslant p < 0} \mathfrak{g}_p$ be a pseudocomplex fundamental Lie algebra. We denote by $\mathfrak{f}$ the Hermitian symmetric form $\mathfrak{f}\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C} \otimes \mathfrak{g}_{-2}$ such that

$$[X, Y] = \Im \mathfrak{f}(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}_{-1}.$$

Let $\mathfrak{g}^*_{-2}$ be the dual space of $\mathfrak{g}_{-2}$ and, for every $\xi \in \mathfrak{g}^*_{-2}$ denote by $\mathfrak{f}_\xi$ the Hermitian symmetric form

$$\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \ni (X, Y) \to \mathfrak{f}_\xi(X, Y) = \langle \mathfrak{f}(X, Y), \xi \rangle \in \mathbb{C}.$$

Then we have the following:

THEOREM 3.24. *Let* $\mathfrak{m}$ *be a pseudocomplex fundamental Lie algebra of kind* 2 *and let* $\mathfrak{g} = \oplus_{p \geqslant -2} \mathfrak{g}_p$ *be its canonical pseudocomplex prolongation. Assume that*

(i) $\dim_{\mathbb{R}} \mathfrak{g}_{-2} \geqslant 2$;

(ii) *there is* $\xi \in \mathfrak{g}^*_{-2}$ *such that the Hermitian symmetric form* $\mathfrak{f}_\xi$ *is nondegenerate;*
(iii) $\rho_{-2}(\mathfrak{g}_0) = \{\lambda \operatorname{Id}_{\mathfrak{g}_{-2}} \mid \lambda \in \mathbb{R}\}$.
Then $\mathfrak{g}_p = 0$ *for all* $p \geqslant 1$.
Moreover $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{g}_0$ *is solvable.*

Proof. Let us prove that under the assumptions (i), (ii), (iii) we have $\mathfrak{g}_1 = 0$. By the transitivity of $\mathfrak{g}$ this implies that $\mathfrak{g}_p = 0$ for $p > 0$.

Let $V \in \mathfrak{g}_1$ and denote by $A\colon \mathfrak{g}_{-1} \to \mathfrak{g}_0$ and $B\colon \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ the corresponding $\mathbb{R}$-linear homomorphisms. Then the following equations are satisfied

$$\begin{cases} \rho_{-1}(A(X))Y \Leftrightarrow \rho_{-1}(A(Y))X = B([X,Y]) \quad \forall X,Y \in \mathfrak{g}_{-1}, \\ [B(T),X] = \rho_{-2}(A(X))T \quad \forall T \in \mathfrak{g}_{-2}, \quad \forall X \in \mathfrak{g}_{-1}. \end{cases} \tag{8}$$

By assumption (ii) we can find a basis $\xi^1, \ldots, \xi^k$ of $\mathfrak{g}^*_{-2}$ such that $\mathfrak{f}_{\xi^j}$ is nondegenerate for $j = 1, \ldots, k$. We take the dual basis $T_1, \ldots, T_k$ of $\mathfrak{g}_{-2}$ defined by the condition that

$$\langle T_j, \xi^h \rangle = \delta^h_j \quad \text{for} \quad 1 \leqslant j,h \leqslant k.$$

Then the second equation in (8) yields, by assumption (iii):

$$\mathfrak{f}_{\xi^h}(B(T_j),X) = 0 \quad \forall X \in \mathfrak{g}_{-1} \quad \text{for } h \neq j$$

and hence $B(T_1) = \cdots = B(T_k) = 0$. This shows that, with $\mathfrak{h} = \{X \in \mathfrak{g} \mid [X,\mathfrak{g}_{-2}] = 0\}$, we have $V \in \mathfrak{h} \cap \mathfrak{g}_1 = \mathfrak{h}_1$. But $\mathfrak{h}_1 = 0$ by Theorem 3.1. Therefore $V = 0$ and this shows that $\mathfrak{g}_1 = 0$.

In this case we have $[\mathfrak{a},\mathfrak{a}]_0 = [\mathfrak{a}_0,\mathfrak{a}_0]$ for every ideal $\mathfrak{a}$ of $\mathfrak{g}$ and then it is clear that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_0$ is solvable. The proof is complete.

In the following we will assume that $\mathfrak{g}$ is a finite dimensional Levi–Tanaka algebra and denote by $\mathcal{S}$ the set of all semisimple elements of $\mathfrak{g}$ and by $\mathfrak{b}$ the set of all nilpotent elements of $\mathfrak{g}$.

LEMMA 3.25. *Assume that* $\mathfrak{g}$ *is solvable. Then the set* $\mathfrak{b}$ *of its nilpotent elements is the maximal nilpotent ideal of* $\mathfrak{g}$.

Let $\mathcal{T}$ *be the set of commutative Lie subalgebras of* $\mathfrak{g}$ *contained in* $\mathcal{S}$ *and* $\mathcal{T}_1$ *the subset of maximal elements of* $\mathcal{T}$. *Then for every* $\mathfrak{t} \in \mathcal{T}_1$ *we have a decomposition of* $\mathfrak{g}$ *into a semidirect sum*

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{b}.$$

We can find a regular element $X_0 \in \mathcal{S} \cap \mathfrak{g}_0$ *such that the centralizer*

$$C_{\mathfrak{g}}(X_0) = \{Y \in \mathfrak{g} \mid [X_0,Y] = 0\}$$

is a Cartan subalgebra of $\mathfrak{g}$ *containing* E *and contained in* $\mathfrak{g}_0$.

Proof. The fact that $\mathfrak{b}$ is an ideal of $\mathfrak{g}$ follows from [3] Ch. I Section 5 Corollary 7 to Theorem 1. The above decomposition is in [3] Ch. VII Section 5 Corollary 2 to Proposition 6.

The last statement then follows from [3] Ch. VII Section 2 Theorem 1(iv). Indeed (cf. [3] Ch. I Section 5 Corollary 7 to Theorem 1) any Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contains a regular element A of $\mathfrak{g}$. Then, taking a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}_0$, we find a regular element $A \in \mathfrak{g}_0$. Its semisimple component S belongs to $\mathfrak{g}_0$ by Lemma 3.8. Since $\mathrm{ad}_{\mathfrak{g}} S$ has the same characteristic polynomial as $\mathrm{ad}_{\mathfrak{g}} A$, it follows that S is a regular element of $\mathfrak{g}$. Moreover E and S commute and thus the centralizer of S is a Cartan subalgebra of $\mathfrak{g}$ containing E and hence contained in $\mathfrak{g}_0$.

COROLLARY 3.26. *Assume that $\mathfrak{g}$ is solvable and that all elements of $\mathfrak{g}_0$ are semisimple. Then $\mathfrak{g}_p = 0$ for every $p \geqslant 1$.*

Proof. We have $\oplus_{p\neq 0}\mathfrak{g}_p \subset \mathfrak{b}$. Therefore, if $X_1 \in \mathfrak{g}_1$, then

$$[X_1, Y_{-1}] \in [\mathfrak{g}_1, \mathfrak{g}_{-1}] = [\mathfrak{b}_1, \mathfrak{b}_{-1}] \subset \mathfrak{b}_0 = 0 \quad \forall Y_{-1} \in \mathfrak{g}_{-1}.$$

This shows that $\mathfrak{g}_1 = 0$ and hence $\mathfrak{g}_p = 0$ for all $p \geqslant 1$ because $\mathfrak{g}$ is transitive.

3.5. Graded Levi–Malčev decomposition for Levi–Tanaka algebras

We turn in this subsection to the general case. First we prove

THEOREM 3.27. *Let $\mathfrak{g} = \oplus_{-\mu\leqslant p\leqslant \nu}\mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra. Then we can find a pseudocomplex semisimple graded Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s},$$

where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}$.

Proof. As usual we denote by $\mathcal{S}$ the set of all semisimple elements of $\mathfrak{g}$ and by $\mathfrak{r}$ the radical of $\mathfrak{g}$. Being an ideal of $\mathfrak{g}$, the radical $\mathfrak{r}$ is graded. By Lemma 3.9 we can find a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is contained in $\mathfrak{g}_0$. Then, since $\mathfrak{g}$ is splittable,

$$\mathfrak{t} = \mathfrak{a} \cap \mathcal{S}$$

is a maximal commutative Lie subalgebra of $\mathfrak{g}_0$ and

$$\mathfrak{a} = \{X \in \mathfrak{g} \,|\, [X, \mathfrak{t}] = 0\}.$$

Next we note that $\mathfrak{a} \cap \mathfrak{r}$ is a Cartan subalgebra of $\mathfrak{r}$ and that $\mathfrak{r}$ is also splittable. Then

$$\mathfrak{t}' = \mathfrak{a} \cap \mathfrak{r} \cap \mathcal{S}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

is a maximal commutative Lie subalgebra of $\mathfrak{r}_0$ and we have

$$\begin{aligned}\mathfrak{a} \cap \mathfrak{r} &= \{X \in \mathfrak{r} \mid [X, \mathfrak{t}] = 0\} \\ &= \{X \in \mathfrak{r} \mid [X, \mathfrak{t}'] = 0\}.\end{aligned}$$

Let $\mathfrak{b}$ be the ideal of $\mathfrak{r}$ of nilpotent elements of $\mathfrak{r}$. Then $\mathfrak{r} = \mathfrak{t}' \oplus \mathfrak{b}$.

We define the subalgebra $\mathfrak{z}$ of $\mathfrak{g}$ by setting

$$\mathfrak{z} = \{X \in \mathfrak{g} \mid [X, \mathfrak{t}'] = 0\}.$$

We note that $\mathfrak{z} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{z}_p$ is a pseudocomplex graded Lie subalgebra of $\mathfrak{g}$. We denote by $\mathrm{rad}(\mathfrak{z})$ its radical. Then we have:

(i) $\mathrm{rad}(\mathfrak{z}) = \mathfrak{a} \cap \mathfrak{r} = \mathfrak{z} \cap \mathfrak{r} \subset \mathfrak{z}_0$;
(ii) $\mathrm{rad}(\mathfrak{z})$ is a nilpotent ideal in $\mathfrak{z}$;
(iii) $[\mathrm{rad}(\mathfrak{z}), \mathfrak{z}_p] = 0 \quad \text{for} \quad p \neq 0$;
(iv) every Levi subalgebra of $\mathfrak{z}$ is a Levi subalgebra of $\mathfrak{g}$.

To prove (i) and (iv) we use [3] Ch. VII Section 5 Proposition 7: for every Levi subalgebra $\mathfrak{L}$ of $\mathfrak{z}$ we have a direct sum decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{t}' \oplus \mathfrak{b}$, with $\mathfrak{r} = \mathfrak{t}' \oplus \mathfrak{b}$. This implies (iv). Moreover, $\mathfrak{g} = \mathfrak{z} + \mathfrak{r}$. Hence $\mathfrak{z}/\mathfrak{z} \cap \mathfrak{r} \simeq \mathfrak{g}/\mathfrak{r}$, from which (i) follows. Now (ii) is a consequence of the fact that $\mathrm{rad}(\mathfrak{z})$ is contained in the nilpotent Lie algebra $\mathfrak{a}$ and (iii) of the fact that the ideal $\mathrm{rad}(\mathfrak{z})$ is contained in $\mathfrak{z}_0$.

We claim that $\mathfrak{z}$ contains a graded pseudocomplex Levi subalgebra. This result, giving the proof of the theorem, follows from the lemma below.

LEMMA 3.28. *Let $\mathfrak{g} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{g}_p$ be a finite dimensional graded Lie algebra, whose radical $\mathfrak{r}$ is contained in $\mathfrak{g}_0$. Then $\mathfrak{g}$ contains a graded Levi subalgebra $\mathfrak{s} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{s}_p$. If $\mathfrak{g}$ is pseudocomplex, then also $\mathfrak{s}$ is pseudocomplex.*

Proof. We argue by induction on the order of solvability of $\mathfrak{r}$, i.e. the smallest nonnegative integer h such that $\mathfrak{r}^{(h)} = 0$ (by $\mathfrak{r}^{(h)}$ we indicate the hth term of the derived series of $\mathfrak{r}$). If $h = 0$, this means that $\mathfrak{r} = 0$ and then $\mathfrak{g}$ is semisimple and there is nothing to prove.

Assume now that $h > 0$ and that the statement of the theorem is true for graded Lie algebras with the radical composed of homogeneous terms of degree 0 and order of solvability lesser than h. Let $\mathfrak{L}$ be a Levi subalgebra of $\mathfrak{g}$ and set $\mathfrak{L}_0 = \mathfrak{L} \cap \mathfrak{g}_0$. Set $\mathfrak{q}_p = \mathfrak{g}_p$ for $p \neq 0$ and $\mathfrak{q}_0 = \mathfrak{L}_0 \oplus \mathfrak{r}^{(1)}$. We claim that $\mathfrak{q} = \oplus \mathfrak{q}_p$ is a Lie subalgebra of $\mathfrak{g}$ with radical $\mathfrak{r}^{(1)}$.

To prove the first asset, it suffices to show that $\mathfrak{q}$ contains the Lie product $[X, Y]$ of every pair of homogeneous elements $X \in \mathfrak{q}_p$ and $Y \in \mathfrak{q}_q$. This is obviously true when $p + q \neq 0$ because in this case $[\mathfrak{q}_p, \mathfrak{q}_q] \subset [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} = \mathfrak{q}_{p+q}$. It is also obvious when $p = q = 0$ because $\mathfrak{L}_0$ is a Lie subalgebra of $\mathfrak{g}_0$ and therefore also $\mathfrak{L} \oplus \mathfrak{r}^{(1)}$, because $\mathfrak{r}^{(1)}$ is an ideal in $\mathfrak{g}$.

Then we only need to consider the case where $q = \Leftrightarrow p \neq 0$. We can find $\tilde{X}, \tilde{Y} \in \mathfrak{L}$ such that $\tilde{X} \Leftrightarrow X, \tilde{Y} \Leftrightarrow Y \in \mathfrak{r}$. Then we obtain

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

$$\begin{aligned}[\tilde{X}, \tilde{Y}] &= [X, Y] + [\tilde{X} \Leftrightarrow X, Y] + [X, \tilde{Y} \Leftrightarrow Y] + [\tilde{X} \Leftrightarrow X, \tilde{Y} \Leftrightarrow Y] \\ &= [X, Y] + [\tilde{X} \Leftrightarrow X, \tilde{Y} \Leftrightarrow Y] \in \mathfrak{L}_0\end{aligned}$$

because $[\mathfrak{r}, \mathfrak{g}_\ell] = 0$ if $\ell \neq 0$. Therefore

$$[X, Y] = [\tilde{X}, \tilde{Y}] \Leftrightarrow [\tilde{X} \Leftrightarrow X, \tilde{Y} \Leftrightarrow Y] \in \mathfrak{L}_0 \oplus \mathfrak{r}^{(1)}.$$

To show that $\mathfrak{r}^{(1)}$ is the radical of $\mathfrak{q}$, we observe that $\mathfrak{q}/\mathfrak{r}^{(1)}$ is isomorphic to $\mathfrak{g}/\mathfrak{r}$. Indeed the map $\mathfrak{q} \to \mathfrak{g}/\mathfrak{r}$ induced by the projection is clearly surjective and its kernel is given by $\mathfrak{q} \cap \mathfrak{r} = \mathfrak{r}^{(1)}$. From this isomorphism it also follows that every Levi subalgebra of $\mathfrak{q}$ is also a Levi subalgebra of $\mathfrak{g}$. Since ${\mathfrak{r}^{(1)}}^{(h-1)} = \mathfrak{r}^{(h)} = 0$, by the inductive assumption $\mathfrak{q}$ contains a graded Levi subalgebra $\mathfrak{s} = \oplus \mathfrak{s}_p$, which is also a graded Levi subalgebra of $\mathfrak{g}$. We note that $\mathfrak{s}_p = \mathfrak{g}_p$ for $p \neq 0$ and therefore $\mathfrak{s}$ is pseudocomplex when $\mathfrak{g}$ is pseudocomplex.

We give now a refinement of the theorem above:

THEOREM 3.29. *Let $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra and let $\mathfrak{L}$ be a pseudocomplex graded Levi subalgebra of $\mathfrak{g}$. Then*

$$\mathfrak{a} = \{X \in \mathfrak{L}_0 \mid [X, \mathfrak{L}_{-1}] = 0\} \subset \mathfrak{b} = \{X \in \mathfrak{L} \mid [(X)_{\mathfrak{L}} \cap \mathfrak{L}_{-2}] = 0\}$$

(*where $(X)_{\mathfrak{L}}$ denotes the ideal of $\mathfrak{L}$ generated by X*) *are ideals of $\mathfrak{L}$ and there is a Levi–Tanaka semisimple graded subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that*

$$\mathfrak{L} = \mathfrak{b} \oplus \mathfrak{s}.$$

Proof. First we note that $\mathfrak{a}$ is an ideal in $\mathfrak{L}$. Indeed the subalgebra $\mathfrak{m}' = \oplus_{p<0} \mathfrak{L}_p$ of $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is fundamental and therefore we obtain $[X, \mathfrak{m}'] = 0$ for all $X \in \mathfrak{a}$. Next we show that $[\mathfrak{a}, \mathfrak{L}_p] = 0$ for $p > 0$. Indeed, if $[X, Y] \neq 0$ for some $X \in \mathfrak{a}$ and some $Y \in \mathfrak{L}_p$, there is $Z \in \mathfrak{L}_{-p}$ such that $\kappa_{\mathfrak{L}}([X, Y], Z) \neq 0$, $\kappa_{\mathfrak{L}}$ being the Killing form of the semisimple Lie algebra $\mathfrak{L}$. But then

$$\kappa_{\mathfrak{L}}([X, Y], Z) = \Leftrightarrow \kappa_{\mathfrak{L}}(Y, [X, Z]) = 0$$

gives a contradiction. Finally the fact that $[\mathfrak{a}, \mathfrak{L}_0] \subset \mathfrak{a}$ follows because

$$\begin{aligned}&[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]] = 0 \\ &\quad \forall X \in \mathfrak{a},\ Y \in \mathfrak{L}_0,\ Z \in \mathfrak{L}_{-1}.\end{aligned}$$

We note that the graded semisimple Lie algebra $\mathfrak{L}$ contains an element $E_{\mathfrak{L}} \in \mathfrak{L}_0$ such that $[E_{\mathfrak{L}}, X] = pX$ for each $p \in \mathbb{Z}$ and $X \in \mathfrak{L}_p$. Thus every ideal of $\mathfrak{L}$ is graded. We write $\mathfrak{b}$ as the direct sum of $\mathfrak{a}$ and a graded semisimple ideal $\mathfrak{b}'$ of $\mathfrak{L}$. We note that $\mathfrak{b}$ and hence $\mathfrak{b}'$ are pseudocomplex by Claim 3.18. If $\mathfrak{s} = \{X \in \mathfrak{L} \mid [X, \mathfrak{b}] = 0\}$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

is the complement ideal of $\mathfrak{b}$ in $\mathfrak{L}$, one verifies that it is pseudocomplex because, for $X \in \mathfrak{s}_{-1}$ it is clear that $[JX, \mathfrak{b}_0] = [JX, \mathfrak{b}_{-1}] = 0$, while $[JX, \mathfrak{b}_1] = 0$ because $[JX, \mathfrak{b}_1] \subset \mathfrak{a} \cap \mathfrak{b}'_0 = 0$. The conclusion follows from Lemma 3.17 and Theorem 3.21.

A finite dimensional Levi–Tanaka algebra $\mathfrak{g} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{g}_p$ with a graded Levi subalgebra $\mathfrak{L}$ contained in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ will be called *weakly solvable* and *almost solvable* if $\mathfrak{L} \subset \mathfrak{g}_0$.

COROLLARY 3.30. *Let $\mathfrak{g} = \oplus_p \mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra with radical $\mathfrak{r}$. The following statements are equivalent:*

(i) *$\mathfrak{g}$ is semisimple;*

(ii) *$\mathfrak{r}_{-1} = 0$;*

(iii) $\oplus_{p<0} \mathfrak{r}_p = 0$.

Proof. Clearly (i) ⇒ (iii) ⇒ (ii). Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{L}$ be a graded Levi–Malčev decomposition. Let $\mathfrak{L} = \mathfrak{b} \oplus \mathfrak{s}$ be the decomposition given in Theorem 3.29. If $\mathfrak{r}_{-1} = 0$, then we have $\mathfrak{s}_{-1} = \mathfrak{g}_{-1}$ and therefore $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p = \oplus_{p<0} \mathfrak{s}_p$ because $\mathfrak{m}$ is nondegenerate. Since $\mathfrak{s}$ is transitive, we have $\mathfrak{g} = \mathfrak{s}$ by Theorem 3.21. This shows that (ii) ⇒ (i). The proof is complete.

COROLLARY 3.31. *Let $\mathfrak{g} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra. If the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is irreducible, then $\mathfrak{g}$ is either simple or almost solvable.*

More precisely, it is simple when $\mathfrak{g}_1 \neq 0$ and almost solvable when $\mathfrak{g}_1 = 0$.

Proof. By Theorem 3.22, if $\mathfrak{g}$ is not simple, then $\mathfrak{g}_1 = 0$ and so $\mathfrak{g}$ is almost solvable.

COROLLARY 3.32. *Let $\mathfrak{g} = \oplus_{-\mu \leqslant p \leqslant \nu} \mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra. If the representation ρ_{-2} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-2}$ is irreducible, then $\mathfrak{g}$ is either simple or weakly solvable.*

Proof. Using Theorem 3.27 and Theorem 3.29 we obtain the decomposition $\mathfrak{g} = \mathfrak{s} \oplus (\mathfrak{r} \oplus \mathfrak{b})$ where $\mathfrak{s}$ is a semisimple Levi–Tanaka algebra, $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{b}$ a semisimple Lie algebra contained in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. By the assumption, we have either $\mathfrak{g}_{-2} = \mathfrak{r}_{-2}$ or $\mathfrak{g}_{-2} = \mathfrak{s}_{-2}$. In the first case, we get $\mathfrak{s} = 0$ because $\mathfrak{s}$ is a Levi–Tanaka algebra by Theorem 3.29, and so $\mathfrak{g}$ is weakly solvable. In the second case, we obtain $\mathfrak{r}_{-1} = 0$ because $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is nondegenerate, and hence $\mathfrak{g}$ semisimple by Corollary 3.30. It is a sum of simple graded ideals by Corollary 3.6, which are not included in $\mathfrak{g}_0$ by Lemma 3.16. Since $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is fundamental, these ideals have a nonzero component in $\mathfrak{g}_{-1}$ and, since $\mathfrak{g}$ is nondegenerate, they have also a nonzero component in $\mathfrak{g}_{-2}$, which is an invariant subspace of $\mathfrak{g}_{-2}$ for ρ_{-2}. Since it is irreducible, we have that $\mathfrak{g}$ has to be simple.

COROLLARY 3.33. *Let $\mathfrak{g}$ be a finite dimensional Levi–Tanaka algebra with radical*

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

$\mathfrak{r} = \oplus_{p\in\mathbb{Z}}\mathfrak{r}_p$. *If the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is completely reducible, then* $\mathfrak{r}_1 = 0$.

Proof. Indeed in this case $\mathfrak{r}_0$ is an Abelian algebra whose elements are semisimple (see [3] Ch. VII Section 5 Proposition 7 (i)). But $[X, Y]$ is a nilpotent element of $\mathfrak{r}_0$ for every $X \in \mathfrak{r}_1$ and every $Y \in \mathfrak{g}_{-1}$, because $\mathfrak{r}_1$ is contained in the maximal nilpotent ideal of the adjoint representation of $\mathfrak{g}$. Therefore $[X, Y] = 0$ for every $X \in \mathfrak{r}_1$ and every $Y \in \mathfrak{g}_{-1}$ and hence $\mathfrak{r}_1 = 0$.

3.6. PROPERTIES OF SEMISIMPLE LEVI–TANAKA ALGEBRAS

In this subsection we investigate some structural properties of semisimple Levi–Tanaka algebras.

LEMMA 3.34. *Let $\mathfrak{g} = \oplus_p\mathfrak{g}_p$ be a semisimple Levi–Tanaka algebra. Then there is a unique complex structure J_1 in $\mathfrak{g}_1$ such that:*

(i) *$\rho_1(\mathfrak{g}_0)$ is a real subalgebra of the algebra $\mathfrak{gl}_{\mathbb{C}}(\mathfrak{g}_1)$ of endomorphisms of $\mathfrak{g}_1$ which are $\mathbb{C}$-linear for the complex structure defined by J_1;*

(ii) $[J_1X, Y] = \Leftrightarrow[X, JY] \quad \forall X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1};$

(iii) $\kappa_{\mathfrak{g}}(J_1X, Y) = \Leftrightarrow\kappa_{\mathfrak{g}}(X, JY) \quad \forall X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1},$

where $\kappa_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$.

Proof. Since the Killing form is nondegenerate, we can use (iii) to define $J_1: \mathfrak{g}_1 \to \mathfrak{g}_1$. The proof of (i) and (ii) is then straightforward.

In the following we will write for simplicity JX instead of J_1X for $X \in \mathfrak{g}_1$.

By an easy computation we obtain:

LEMMA 3.35. *Let $\mathfrak{g} = \oplus_{\mu\leqslant p\leqslant\mu}\mathfrak{g}_p$ be a semisimple Levi–Tanaka algebra and let J be the complex structure on $\mathfrak{g}_1$ defined in the previous lemma. Then we obtain*:

(1) $[JX, JY] = [X, Y] \quad \forall X, Y \in \mathfrak{g}_1;$

(2) $J[X, Y] = [X, JY] \quad \forall X \in \mathfrak{g}_{-2},\ Y \in \mathfrak{g}_1;$

(3) $J[X, Y] = [X, JY] \quad \forall X \in \mathfrak{g}_2,\ Y \in \mathfrak{g}_{-1}.$

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ (cf. Lemma 3.9). Then $\mathfrak{h}^{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{h}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g}$ of $\mathfrak{g}$. Setting $\mathfrak{g}_p^{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g}_p$, $\mathfrak{g}^{\mathbb{C}} = \oplus_p\mathfrak{g}_p^{\mathbb{C}}$ is a graded complex Lie algebra and $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$.

We denote by $\mathcal{R} \subset \hom_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ the set of nonzero roots of $\mathfrak{h}^{\mathbb{C}}$. Assuming that $\mathfrak{g}$ is semisimple, $\mathfrak{g}^{\mathbb{C}}$ is also semisimple and, denoting for every $\alpha \in \mathcal{R}$

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{h}^{\mathbb{C}}\}, \tag{9}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

we have that $\mathfrak{g}^\alpha$ is a 1-dimensional complex subspace of $\mathfrak{g}^{\mathbb{C}}$ and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}^\alpha .$$

LEMMA 3.36. *For every $\alpha \in \mathcal{R}$, we have $\alpha(E) \in \mathbb{Z}$, where E is the element considered in Lemma* 3.5, *and $\mathfrak{g}^\alpha \subset \mathfrak{g}^{\mathbb{C}}_{\alpha(E)}$.*

When $\alpha(E) = \pm 1$, all vectors of $\mathfrak{g}^\alpha$ are either of the form $X + \sqrt{-1}JX$, or of the form $X - \sqrt{-1}JX$ with $X \in \mathfrak{g}^{\mathbb{C}}_{\alpha(E)}$.

Proof. We have $\mathfrak{g}^\alpha$ included in $\mathfrak{g}^{\mathbb{C}}_p$ for a certain $p \in \mathbb{Z}$ because all subspaces $\mathfrak{g}^{\mathbb{C}}_p$ are invariant under $\mathrm{ad}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}_0)$. As $E \in \mathfrak{h} \subset \mathfrak{g}_0$, from (9) with $H = E$ it follows that $p = \alpha(E)$.

The second statement is a consequence of the fact that for every $A \in \mathfrak{g}^{\mathbb{C}}_0$, the representation $\tilde{\rho}_{\pm 1}(A)$ of A in $\mathfrak{g}^{\mathbb{C}}_{\pm 1}$ commutes with the complexification of the operator J. Therefore, if $X + \sqrt{-1}Y \in \mathfrak{g}^\alpha$ for some $\alpha \in \mathcal{R}$ with $\alpha(E) = \pm 1$, also $JX + iJY \in \mathfrak{g}^\alpha$. But $\mathfrak{g}^\alpha$ has dimension 1 and this implies that $Y = \pm JX$.

We introduce the notation

$$\mathfrak{g}^{(0,1)}_{-1} = \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\}, \qquad \mathfrak{g}^{(0,1)}_{1} = \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{1}\},$$

$$\mathfrak{g}^{(1,0)}_{-1} = \{X - \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\}, \qquad \mathfrak{g}^{(1.0)}_{1} = \{X - \sqrt{-1}JX \mid X \in \mathfrak{g}_{1}\}.$$

We note that $\mathfrak{g}^{(0,1)}_{-1} \oplus \mathfrak{g}^{(0,1)}_{1}$ and $\mathfrak{g}^{(1,0)}_{-1} \oplus \mathfrak{g}^{(1,0)}_{1}$ are commutative Lie subalgebras of $\mathfrak{g}^{\mathbb{C}}$. This yields:

Remark 3.37. If $\alpha \in \mathcal{R}$ with $\alpha(E) = -1$, then

$$\mathfrak{g}^\alpha \subset \mathfrak{g}^{(0,1)}_{-1} \Longrightarrow \mathfrak{g}^{-\alpha} \subset \mathfrak{g}^{(1,0)}_{1} \quad \text{and} \quad \mathfrak{g}^\alpha \subset \mathfrak{g}^{(1,0)}_{-1} \Longrightarrow \mathfrak{g}^{-\alpha} \subset \mathfrak{g}^{(0,1)}_{1}.$$

We define an involution on $\hom_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ by associating to any $\mathbb{C}$-linear functional α on $\mathfrak{h}^{\mathbb{C}}$ the unique $\mathbb{C}$-linear functional $\bar{\alpha}$ on $\mathfrak{h}^{\mathbb{C}}$ such that

$$\bar{\alpha}(H) = \overline{\alpha(H)} \quad \forall H \in \mathfrak{h}.$$

LEMMA 3.38. *Let $\mathfrak{g} = \oplus \mathfrak{g}_p$ be a semisimple Levi–Tanaka algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ and $\mathcal{R} \subset \hom_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ the set of nonzero roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Then $\bar{\alpha} \in \mathcal{R}$ for every $\alpha \in \mathcal{R}$.*

Proof. We consider first the case of a root $\alpha \in \mathcal{R}$ with $\alpha(E) = -1$. Assume that $\mathfrak{g}^\alpha \subset \mathfrak{g}^{(0,1)}_{-1}$ and let $\tilde{X}_\alpha = X_\alpha + \sqrt{-1}JX_\alpha$ be a basis of g^α. For every $H \in \mathfrak{h}$ we obtain

$$[H, X_\alpha - \sqrt{-1}JX_\alpha] = \overline{[H, X_\alpha + \sqrt{-1}JX_\alpha]} = \bar{\alpha}(H)(X_\alpha - \sqrt{-1}JX_\alpha),$$

where we have used complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. By $\mathbb{C}$-linearity this equality extends to $H \in \mathfrak{h}^{\mathbb{C}}$, showing that $\bar{\alpha} \in \mathcal{R}$ and that $\mathfrak{g}^{\bar{\alpha}}$ is the conjugated of $\mathfrak{g}^\alpha$ with respect to the real form $\mathfrak{g}$.

The case $\mathfrak{g}^{\alpha} \subset \mathfrak{g}_{-1}^{(1,0)}$ is analogous.

Next we note that $\mathfrak{m}^{\mathbb{C}} = \oplus_{p<0}\mathfrak{g}_p^{\mathbb{C}}$ is a complex fundamental graded Lie algebra and therefore all roots α with $\alpha(E) < -1$ can be decomposed as

$$\alpha = \alpha_1 + \cdots + \alpha_{-\alpha(E)}$$

with $\alpha_i(E) = -1$ and, for generators $\tilde{X}_{\alpha_1}, \ldots, \tilde{X}_{\alpha_{-\alpha(E)}}$ of $\mathfrak{g}^{\alpha_1}, \ldots, \mathfrak{g}^{\alpha_{-\alpha(E)}}$, $\mathfrak{g}^{\alpha}$ is generated by

$$[\tilde{X}_{\alpha_1}, [\ldots, \tilde{X}_{\alpha_{-\alpha(E)}}] \ldots].$$

Using conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$ we obtain that also

$$\bar{\alpha} = \bar{\alpha}_1 + \cdots + \bar{\alpha}_{-\alpha(E)}$$

is a root.

To conclude the proof of the lemma, we need only to consider the case where $\alpha \in \mathcal{R}$ and $\alpha(E) = 0$. Since $\mathfrak{g}^{\mathbb{C}}$ is transitive, there exists $\beta \in \mathcal{R}$ with $\beta(E) = -1$ such that $\alpha + \beta \in \mathcal{R}$. Then $\bar{\alpha} + \bar{\beta} \in \mathcal{R}$ and again we conclude by complex conjugation with respect to $\mathfrak{g}$ that

$$[\mathfrak{g}^{\bar{\alpha}+\bar{\beta}}, \mathfrak{g}^{-\bar{\beta}}] = \mathfrak{g}^{\bar{\alpha}} \neq 0.$$

The proof is complete.

PROPOSITION 3.39. *If the complexification $\mathfrak{g}^{\mathbb{C}}$ of a Levi–Tanaka algebra $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is a simple Lie algebra of type A_ℓ, or D_ℓ, or E_6.*

Proof. Indeed the conjugation map $\alpha \to \bar{\alpha}$ on the roots, described in the previous lemma, permits to define an order two automorphism of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, which is different from the identity. This defines an automorphism of a Weyl chamber. Hence the result follows from the classification of the automorphisms of simple complex Lie algebras (cf. [H], Ch. X).

We turn now to the Cartan decomposition of semisimple Levi–Tanaka algebras.

LEMMA 3.40. *Let $\mathfrak{g} = \oplus \mathfrak{g}_p$ be a semisimple Levi–Tanaka algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ and $\mathcal{R}$ the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Then $\mathcal{R}$ admits a basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_\ell\}$ with $\alpha_i(E) \in \{-1, 0\}$ for every $i = 1, \ldots, \ell$.*

Proof. We note that $\mathcal{R}^- = \{\alpha \in \mathcal{R} \mid \alpha(E) < 0\}$ and $\mathcal{R}^+ = \{\alpha \in \mathcal{R} \mid \alpha(E) > 0\}$ generate two disjoint convex cones in $\hom_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ considered as a real vector space. Then we can find a real linear functional $\gamma\colon \hom_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C}) \to \mathbb{R}$ which is different from 0 on every $\alpha \in \mathcal{R}$ and is positive on $\mathcal{R}^-$ and negative on $\mathcal{R}^+$. A basis $\mathcal{B}$ consisting of all simple roots contained in $\{\alpha \mid \gamma(\alpha) > 0\}$ satisfies the conditions of the statement.

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

PROPOSITION 3.41. *A semisimple Levi–Tanaka algebra* $\mathfrak{g} = \oplus_{-\mu\leqslant p\leqslant\mu}\mathfrak{g}_p$ *admits a Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where:

(i) $\mathfrak{k}$ *is a Lie subalgebra of* $\mathfrak{g}$ *of the compact type on which the Killing form* $\kappa_{\mathfrak{g}}$ *is negative definite;*
(ii) $\mathfrak{k} = \oplus_{0\leqslant p\leqslant\mu}\mathfrak{k}_{|p|}$ *with* $\mathfrak{k}_{|0|} = \mathfrak{k}\cap\mathfrak{g}_0$ *and* $\mathfrak{k}_{|p|} \subset \mathfrak{g}_{-p}\oplus\mathfrak{g}_p$ *for* $p > 0$*;*
(iii) $\mathfrak{p}$ *is the orthogonal complement of* $\mathfrak{k}$ *with respect to the Killing form* $\kappa_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *and* $\kappa_{\mathfrak{g}}$ *is positive definite on* $\mathfrak{p}$*;*
(iv) $\mathfrak{p} = \oplus_{0\leqslant p\leqslant\mu}\mathfrak{p}_{|p|}$ *with* $\mathfrak{p}_{|0|} = \mathfrak{p}\cap\mathfrak{g}_0$ *and* $\mathfrak{p}_{|p|} \subset \mathfrak{g}_{-p}\oplus\mathfrak{g}_p$ *for* $p > 1$*;*
(v) *the natural projections* $\mathfrak{k}_{|p|} \to \mathfrak{g}_{\pm p}$ *and* $\mathfrak{p}_{|p|} \to \mathfrak{g}_{\pm p}$ *are isomorphisms for* $p > 0$*;*
(vi) *the associated Cartan involution* $\theta\colon\mathfrak{g}\to\mathfrak{g}$ *such that* $\mathfrak{k}$ *is the set of fixed point of* $\theta, \theta(X) = -X$ *for* $X\in\mathfrak{p}$*, and for which*

$$\mathfrak{g}\times\mathfrak{g} \ni (X,Y) \to -\kappa_{\mathfrak{g}}(X,\theta(Y)) \in \mathbb{R}$$

is a positive definite real symmetric form, has the properties

$$\theta(\mathfrak{g}_p) = \mathfrak{g}_{-p} \quad \textit{for} \quad -\mu\leqslant p\leqslant\mu,$$

$$\mathfrak{g}_{-1}\ni X\to\theta(X)\in\mathfrak{g}_1 \quad \textit{and} \quad \mathfrak{g}_1\ni X\to\theta(X)\in\mathfrak{g}_{-1}$$

are $\mathbb{C}$*-linear for the complex structures of* $\mathfrak{g}_{-1}$ *and* $\mathfrak{g}_1$ *defined by* J.

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ and let $\mathfrak{h}^{\mathbb{C}}$ be the corresponding Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Let $\mathcal{R}$ be the set of nonzero roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and H_α, for $\alpha\in\mathcal{R}$ the element of $\mathfrak{h}^{\mathbb{C}}$ such that

$$\kappa_{\mathfrak{g}^{\mathbb{C}}}(H,H_\alpha) = \alpha(H) \quad \forall H\in\mathfrak{h}^{\mathbb{C}}.$$

The form $\kappa_{\mathfrak{g}^{\mathbb{C}}}$ is positive definite on the real subspace $\mathfrak{h}^{\mathbb{R}}$ of $\mathfrak{h}^{\mathbb{C}}$ generated by the H_α's.

For each $\alpha\in\mathcal{R}$ we can choose a basis $\tilde{X}_\alpha$ of $\mathfrak{g}^\alpha$ in such a way that

$$[\tilde{X}_\alpha,\tilde{X}_{-\alpha}] = H_\alpha, \quad \kappa_{\mathfrak{g}^{\mathbb{C}}}(\tilde{X}_\alpha,\tilde{X}_{-\alpha}) = 1.$$

According to Lemma 3.36 and Remark 3.37, we can split the set of roots α with $\alpha(E) = \pm 1$ into two disjoint subsets, the first $\mathcal{R}_{0,1}$ consisting of roots α for which $\tilde{X}_\alpha = X_\alpha + \sqrt{-1}JX_\alpha$, the second $\mathcal{R}_{1,0}$ consisting of roots α for which $\tilde{X}_\alpha = X_\alpha - \sqrt{-1}JX_\alpha$ with $X_\alpha\in\mathfrak{g}_{\pm 1}$.

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

Then we obtain a compact form $\mathfrak{u}$ by

$$\mathfrak{u} = \oplus_{p=0}^{\mu} \mathfrak{u}_{|p|},$$

where

$$\begin{aligned}
\mathfrak{u}_{|0|} &= \sqrt{\Leftrightarrow 1}\, \mathfrak{h}^{\mathbb{R}} \bigoplus \sum_{\alpha(E)=0} (\mathbb{R}(\tilde{X}_\alpha \Leftrightarrow \tilde{X}_{-\alpha}) \oplus \sqrt{\Leftrightarrow 1}\mathbb{R}(\tilde{X}_\alpha + \tilde{X}_{-\alpha})),\\
\mathfrak{u}_{|1|} &= \bigoplus_{\alpha \in \mathcal{R}_{0,1}} (\mathbb{R}(X_\alpha \Leftrightarrow X_{-\alpha} + \sqrt{\Leftrightarrow 1} J(X_\alpha + X_{-\alpha}))\\
&\qquad \oplus \sqrt{\Leftrightarrow 1}\mathbb{R}(X_\alpha + X_{-\alpha} + \sqrt{\Leftrightarrow 1} J(X_\alpha \Leftrightarrow X_{-\alpha}))),\\
\mathfrak{u}_{|p|} &= \bigoplus_{\alpha(E)=-p} (\mathbb{R}(\tilde{X}_\alpha \Leftrightarrow \tilde{X}_{-\alpha}) \oplus \sqrt{\Leftrightarrow 1}\mathbb{R}(\tilde{X}_\alpha + \tilde{X}_{-\alpha})) \quad \text{for} \quad p > 0.
\end{aligned}$$

Let us denote by $\tau\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{u}$ and by $\sigma\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. We set

$$\begin{cases} \mathfrak{g}^{\mathbb{C}}_{|0|} = \mathfrak{g}^{\mathbb{C}}_0, \\ \mathfrak{g}^{\mathbb{C}}_{|p|} = \mathfrak{g}^{\mathbb{C}}_{-p} \oplus \mathfrak{g}^{\mathbb{C}}_p \quad \text{for } p > 0. \end{cases}$$

Then we have

$$\tau(\mathfrak{g}^{\mathbb{C}}_{|p|}) = \sigma(\mathfrak{g}^{\mathbb{C}}_{|p|}) = \mathfrak{g}^{\mathbb{C}}_{|p|} \quad \text{for } p = 0, \ldots, \mu.$$

Moreover we note that J defines an antiinvolution on $\mathfrak{u}_{|1|}$ and therefore, by $\mathbb{C}$-linearity, also on $\sqrt{\Leftrightarrow 1}\mathfrak{u}_{|1|}$. From this we derive that

$$\Leftrightarrow J \circ \tau \circ J = \tau, \quad \text{i.e.} \quad J \circ \tau = \tau \circ J \quad \text{on } \mathfrak{g}^{\mathbb{C}}_{|1|}.$$

Obviously the conjugation σ commutes with J on $\mathfrak{g}^{\mathbb{C}}_{|1|}$. This property is therefore shared by the composed $\mathbb{C}$-linear automorphism $a = \sigma \circ \tau$ of $\mathfrak{g}^{\mathbb{C}}$. This is a selfadjoint map for the Hermitian scalar product

$$B_\tau\colon \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \ni (X, Y) \to \Leftrightarrow \kappa_{\mathfrak{g}^{\mathbb{C}}}(X, \tau Y) \in \mathbb{C}$$

and therefore a^2 is selfadjoint and positive definite for B_τ. We denote by ϕ the positive selfadjoint fourth root of a^2. This is still an automorphism of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ such that $\phi(\mathfrak{u})$ is a compact form of $\mathfrak{g}^{\mathbb{C}}$ that is invariant under σ. Moreover, by the construction,

$$\phi(\mathfrak{g}^{\mathbb{C}}_{|p|}) = \mathfrak{g}^{\mathbb{C}}_{|p|} \quad \text{and} \quad \phi \circ J = J \circ \phi \quad \text{on } \mathfrak{g}^{\mathbb{C}}_{|1|}.$$

A Cartan decomposition of $\mathfrak{g}$ is obtained by setting $\mathfrak{k} = \phi(\mathfrak{u}) \cap \mathfrak{g}$ and $\mathfrak{p} = \sqrt{-1}\phi(\mathfrak{u}) \cap \mathfrak{g}$. Then the Cartan involution θ on $\mathfrak{g}$ is defined by $\phi \circ \tau \circ \phi^{-1}$ and therefore commutes with J on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

We note that the positive definite symmetric real form

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \to g(X, Y) = -\kappa_{\mathfrak{g}}(X, \theta(Y))$$

satisfies

$$g(JX, Y) = -g(X, JY) \quad \text{for } X, Y \in \mathfrak{g}_{\pm 1}$$

and therefore is on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ the real part of a Hermitian scalar product for the respective complex structures.

4. Homogeneous CR manifolds

4.1. Standard homogeneous CR manifolds

Let $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra. In this section we construct homogeneous CR manifolds $\mathbf{M} = (M, HM, J)$ having at each point $x \in M$ a Levi–Tanaka algebra $\mathfrak{g}(x)$ isomorphic to $\mathfrak{g}$ and such that the group of CR automorphisms of $\mathbf{M}$ is a Lie group with Lie algebra isomorphic to $\mathfrak{g}$.

Let us set

$$\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p \quad \text{and} \quad \mathfrak{g}_+ = \oplus_{p \geqslant 0} \mathfrak{g}_p.$$

We denote by $\mathbf{G}$ a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. We note that $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{g}$ and therefore generates a connected Lie subgroup $\mathbf{G}_+$ of $\mathbf{G}$.

LEMMA 4.1. $\mathbf{G}_+$ *is a closed subgroup of* $\mathbf{G}$.

Proof. Let

$$\mathrm{Ad}\colon \mathbf{G} \to \mathbf{GL}(\mathfrak{g})$$

denote the adjoint representation of $\mathbf{G}$. Then

$$\mathbf{H} = \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(\mathfrak{g}_+) = \mathfrak{g}_+\}$$

is a closed subgroup of $\mathbf{G}$ and hence a Lie subgroup of $\mathbf{G}$. Clearly the Lie algebra of $\mathbf{H}$ is $\mathfrak{g}_+$ and then $\mathbf{G}_+$, being the connected component of the identity in $\mathbf{H}$, is closed in $\mathbf{G}$.

We identify $\mathfrak{g}$ to the Lie algebra of left invariant vector fields on $\mathbf{G}$. For $-\mu \leqslant p \leqslant 0$ we set $\mathfrak{g}_{(p)} = \oplus_{q \geqslant p} \mathfrak{g}_q$ and denote by $\tilde{\mathfrak{g}}_{(p)}$ the vector distribution generated by $\mathfrak{g}_{(p)}$. For $g \in \mathbf{G}$, we denote by L_g and R_g respectively the left and right translations

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

with respect to g.

LEMMA 4.2. *For every* $\Leftrightarrow\mu \leqslant p \leqslant 0$ *the vector distribution* $\tilde{\mathfrak{g}}_{(p)}$ *is invariant with respect to left translations by elements of* $\mathbf{G}$ *and right translations by elements of* $\mathbf{G}_+$.

Proof. The invariance under $(L_g)_*$ for $g \in \mathbf{G}$ is obvious. For $X \in \mathfrak{g}$ and $g \in \mathbf{G}$, we have

$$(R_{g^{-1}})_*(X) = \mathrm{Ad}(g)(X).$$

Since

$$\mathrm{ad}_{\mathfrak{g}}(Y)(X) = [Y, X] \in \mathfrak{g}_{(p)} \quad \forall X \in \mathfrak{g}_{(p)},\ Y \in \mathfrak{g}_+,$$

the Lie algebra of the Lie subgroup $\mathbf{A}$ of the elements $g \in \mathbf{G}$ such that $(R_g)_*(\mathfrak{g}_{(p)}) \subset \mathfrak{g}_{(p)}$ contains $\mathfrak{g}_+$. Hence $\mathbf{G}_+ \subset \mathbf{A}$ because $\mathbf{G}_+$ is connected.

Using these lemmas we obtain:

THEOREM 4.3. *The homogeneous space* $M = \mathbf{G}/\mathbf{G}_+$ *is a simply connected real analytic manifold. We can endow* M *by a natural* CR *structure, in such a way that* $\mathbf{G}$ *acts on* M *as a group of* CR *automorphisms and the Levi–Tanaka algebra* $\mathfrak{g}(x)$ *of* M *at every point* x *of* M *is isomorphic to* $\mathfrak{g}$.

Proof. Since $\mathbf{G}_+$ is a closed subgroup of $\mathbf{G}$, the homogeneous space $M = \mathbf{G}/\mathbf{G}_+$ is a real analytic manifold, on which the elements of $\mathbf{G}$ define real analytic diffeomorphisms. Moreover, M is simply connected because $\mathbf{G}$ is simply connected and $\mathbf{G}_+$ is connected.

Let us describe the CR structure of M. We denote by $\pi\colon \mathbf{G} \to M$ the natural projection, and by $\mathbf{G} \times M \ni (g, x) \to g \cdot x \in M$ the left action of $\mathbf{G}$ on M. Let $\tilde{\mathfrak{g}}_{-1}$, $\tilde{\mathfrak{g}}_+ = \tilde{\mathfrak{g}}_{(0)}$ and $\tilde{\mathfrak{g}}_{(-1)}$ denote the vector distribution generated respectively by $\mathfrak{g}_{-1}$, $\mathfrak{g}_+$ and $\mathfrak{g}_{(-1)} = \oplus_{p \geqslant -1} \mathfrak{g}_p$. They are all invariant by left translations and $\tilde{\mathfrak{g}}_+$ is the vertical distribution of the $\mathbf{G}_+$-principal bundle $\mathbf{G} \xrightarrow{\pi} M$.

Let $o = \pi(e)$ be the image of the identity of $\mathbf{G}$ in M and $H_o M = \pi_*((\tilde{\mathfrak{g}}_{-1})_e)$. If $h \in \mathbf{G}_+$, we have

$$\pi_*((\tilde{\mathfrak{g}}_{-1})_h) = H_o M.$$

Indeed, since $\pi \circ R_{h^{-1}} = \pi$ for $h \in \mathbf{G}_+$, we obtain

$$\begin{aligned} \pi_*((\tilde{\mathfrak{g}}_{-1})_h) &= \pi_*((\tilde{\mathfrak{g}}_{(-1)})_h) = \pi_* \circ (R_{h^{-1}})_*((\tilde{\mathfrak{g}}_{(-1)})_h) \\ &= \pi_*((\tilde{\mathfrak{g}}_{(-1)})_e) = \pi_*((\tilde{\mathfrak{g}}_{-1})_e) = H_o M. \end{aligned}$$

This implies that

$$H_{\pi(g)} M = g_* H_o M$$

is well defined at all points of M and is invariant by the action of $\mathbf{G}$ on M.

If X_x^* is in H_xM and $g \in \mathbf{G}$ is such that $x = g \cdot o$, then we can find a unique $X \in \mathfrak{g}_{-1}$ such that $X_x^* = g_* \pi_*(X_e)$. We want to define the partial complex structure J_M of M in such a way that

$$J_M X_x^* = g_* \pi_*(JX_e).$$

This would imply also that $M \ni x \to g \cdot x \in M$ is a CR diffeomorphism for every $g \in \mathbf{G}$.

To this aim, we only need to show that the definition is consistent, i.e. that, if γ is another element of $\mathbf{G}$ such that $\gamma \cdot o = x$ and $Y \in \mathfrak{g}_{-1}$ is such that $\gamma_* \pi_*(Y_e) = X_x^*$, then

$$\gamma_* \pi_*(JY_e) = g_* \pi_*(JX_e).$$

We note that $\gamma^{-1} g \in \mathbf{G}_+$ and thus we are reduced to show that

$$\pi_*(\mathrm{Ad}(h)(JX_e)) = \pi_*(JY_e) \tag{10}$$

if $h \in \mathbf{G}_+, X, Y \in \tilde{\mathfrak{g}}_{-1}$, and $Y \Leftrightarrow \mathrm{Ad}(h)X \in \tilde{\mathfrak{g}}_+$. Let $H = \Sigma_{p \geqslant 0} H_p \in \mathfrak{g}_+$, expressed as a sum of its homogeneous components. Then we have $\mathrm{Ad}(\exp(tH))X \Leftrightarrow \mathrm{Ad}(\exp(tH_0))X \in \mathfrak{g}_+$ for $X \in \mathfrak{g}_{-1}$ and $t \in \mathbb{R}$. This shows that (10) holds for the elements of $\mathbf{G}_+$ which are of the form $\exp(H)$ for $H \in \mathfrak{g}_+$ and therefore for all $h \in \mathbf{G}_+$ because $\mathbf{G}_+$ is connected.

To show that the Levi–Tanaka algebra $\mathfrak{g}(x)$ of M at every point $x \in M$ is isomorphic to $\mathfrak{g}$, it suffices to note that by construction $\mathfrak{m}(o)$ is isomorphic to $\mathfrak{m}$ and hence $\mathfrak{g}(o) \simeq \mathfrak{g}$: the general statement follows because $\mathbf{G}$ operates on M as a group of CR diffeomorphisms.

The $\mathbf{G}$-homogeneous CR manifold obtained in Theorem 4.3 will be denoted by $M_{\mathfrak{g}}$ and called the *standard (homogeneous) CR manifold associated to the Levi–Tanaka algebra* $\mathfrak{g}$. We have

THEOREM 4.4. *Let $\mathbf{\Gamma}$ be the kernel of the representation of $\mathbf{G}$ as a group of CR automorphisms of the standard CR manifold $M_{\mathfrak{g}}$. Then $\mathbf{\Gamma}$ is the discrete subgroup $\mathbf{Z}(\mathbf{G}) \cap \mathbf{G}_+$, where $\mathbf{Z}(\mathbf{G})$ denotes the center of $\mathbf{G}$, and $\mathbf{G}/\mathbf{\Gamma}$ is the connected component of the identity in the group of CR automorphisms of $M_{\mathfrak{g}}$.*

If N is another connected $\mathbf{G}$-homogeneous CR manifold with the same Levi–Tanaka algebra $\mathfrak{g}$, then there is a CR covering map $M \to N$ commuting to the action of $\mathbf{G}$.

Proof. We note that $\mathbf{\Gamma} = \bigcap_{g \in \mathbf{G}} (g\mathbf{G}_+ g^{-1})$ is a closed normal subgroup of $\mathbf{G}$ contained in $\mathbf{G}_+$. Its Lie algebra is an ideal contained in $\mathfrak{g}_+$ and then is null because $\mathfrak{g}$ is transitive. This shows that $\mathbf{\Gamma}$ is a normal discrete subgroup of the connected Lie group $\mathbf{G}$ and hence is contained in its center. So we have $\mathbf{\Gamma} = \mathbf{Z}(\mathbf{G}) \cap \mathbf{G}_+$. Vive versa every element of $\mathbf{Z}(\mathbf{G}) \cap \mathbf{G}_+$ is obviously in the kernel $\mathbf{\Gamma}$.

To show that $\mathbf{G}$ is the component of the identity in the group of CR automorphisms of M we essentially follow [13]; the proof in the case of homogeneous manifolds is actually simpler.

(a) Let us denote by $\mathbf{A}$ the connected subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{m}$. If θ is the Maurer–Cartan form of $\mathbf{G}$, then the Maurer–Cartan form ξ of $\mathbf{A}$ is the pullback of θ to $\mathbf{A}$. The natural projection $\pi\colon \mathbf{G} \to \mathbf{G}/\mathbf{G}_+ = M$ induces a diffeomorphism of an open neighborhood $\mathbf{U}_e$ of e in $\mathbf{A}$ onto an open neighborhood $\mathbf{U}_o$ of $o = \pi(e)$ in M. Let $\tilde{\xi} = (\pi|_{U_e})_*\xi$ and set $\tilde{\xi}^p = (\pi|_{U_e})_*\xi^p$, where $\xi = \Sigma_{p<0}\xi^p$ is the decomposition of ξ according to the graduation of the fundamental algebra $\mathfrak{m}$. We note that we obtain the equations

$$\mathrm{d}\tilde{\xi}^p = \Leftrightarrow \frac{1}{2}\sum_{r+s=p}[\tilde{\xi}^r, \tilde{\xi}^s] \quad \text{for } p < 0.$$

(b) Let X be a vector field defined on an open neighborhood of o in M. We can as well assume that X is defined on U_o. We want to take X as the infinitesimal generator of a 1-parameter family of local CR diffeomorphisms on M. If $\phi_X(t)$ is the local 1-parameter group defined by X, this condition means that $\mathrm{d}\phi_X(t)\colon T_xM \to T_{\phi_X(t)(x)}M$ induces, by passing to the quotient, an isomorphism of pseudocomplex fundamental graded Lie algebras

$$\mathrm{d}\widehat{\phi_X(t)}\colon \mathfrak{m}(x) \to \mathfrak{m}(\phi_X(t)x)$$

for x in a small neighborhood of o and t in a small neighborhood of 0. In particular, using the identification of $\mathfrak{m}(x)$ to $\mathfrak{m}$ for all $x \in U_o$, the differential at o of the map $\mathfrak{m} \to \mathfrak{m}$ induced by the diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[r] \arrow[d] & \mathfrak{m} \arrow[d] \\ \mathfrak{m}(x) \arrow[r, "\mathrm{d}\widehat{\phi_X(t)}"] & \mathfrak{m}(\phi_X(t)x) \end{tikzcd}$$

gives a map $f^0\colon U_o \to \mathfrak{g}_0$.

Let us set, for $p < 0$, $f^p(x) = \tilde{\xi}^p(X_x) \in \mathfrak{g}_p$. Then the definition of f^0 can be rewritten by

$$\mathrm{d}f^p(x) = \sum_{r=p}^{-1}[f^{p-r}(x), \tilde{\xi}^r] \bmod \tilde{\xi}^{p-1}, \ldots, \tilde{\xi}^{-\mu} \quad \text{for } p < 0.$$

Indeed, we have for every $Y \in \mathfrak{X}(M)$

$$\begin{aligned}(L_X\tilde{\xi}^p)(Y) &= X(\tilde{\xi}^p(Y)) \Leftrightarrow \tilde{\xi}^p[X, Y] = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_X(t)^*\tilde{\xi}^p)(Y)|_{t=0} \\ &= [f^0(x), \tilde{\xi}^p](Y) \bmod \tilde{\xi}^{p-1}, \ldots, \tilde{\xi}^{-\mu} \quad \text{for } p < 0.\end{aligned}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

Hence we deduce that

$$\begin{aligned} \mathrm{d}f^p(x) &= \mathrm{d}(\tilde{\xi}^p(X)) = \mathrm{d}(X\rfloor\,\tilde{\xi}^p) = L_X\tilde{\xi}^p \Leftrightarrow X\rfloor\,\mathrm{d}\tilde{\xi}^p \\ &= L_X\tilde{\xi}^p + X\rfloor\left(\frac{1}{2}\sum_{r+s=p}[\tilde{\xi}^r,\tilde{\xi}^s]\right) \\ &= [f^0(x),\tilde{\xi}^p] + \sum_{r+s=p}[f^r(x),\tilde{\xi}^s] \bmod \tilde{\xi}^{p-1},\ldots,\tilde{\xi}^{-\mu}. \end{aligned}$$

Then we can define f^p also for $p > 0$ in such a way that

$$\mathrm{d}f^p = \sum_{r<0}[f^{p-r}(x),\tilde{\xi}^r] \quad \forall p \in \mathbb{Z}.$$

We have already constructed f^p for $p \leqslant 0$. Now we note that these equations yield

$$\begin{aligned} \mathrm{d}f^0(x) &= \sum_{r<0}[f^{-r}(x),\tilde{\xi}^r], \\ \mathrm{d}f^1(x) &= \sum_{r<0}[f^{1-r}(x),\tilde{\xi}^r], \\ &\cdots \end{aligned}$$

which is a completely integrable system (see [13]).

(c) Let us denote by $\tilde{\mathfrak{x}}_o$ the Lie algebra of germs at o of infinitesimal generators of 1-parameter groups of local CR diffeomorphisms. By Lemma 6.4 in [13], we have

$$f^p_{[X,Y]} = \Leftrightarrow \sum_{r+s=p}[f^r_X, f^s_Y] \ \ \forall p \in \mathbb{Z} \ \ \forall X, Y \in \tilde{\mathfrak{x}}_o,$$

where for $Z \in \tilde{\mathfrak{x}}_o$ we used f^p_Z for the set of functions associated to Z as in (b).

The map $\tilde{\mathfrak{x}}_o \ni X \to \Sigma f^p_X(0) \in \mathfrak{g}$ is therefore an anti-homomorphism of Lie algebras and is injective by Lemma 6.3 in [13]. But this map is trivially surjective and therefore is an anti-isomorphism. This proves the first statement.

To prove the last statement of the theorem, it suffices to note that $N \cong \mathbf{G}/\mathbf{Q}$ for a closed subgroup $\mathbf{Q}$ of $\mathbf{G}$ whose Lie algebra is isomorphic to $\mathfrak{g}_+$. Indeed from (a), (b), (c) above we deduce that M and N are locally CR diffeomorphic and therefore the Lie algebras of the stabilizer of a point in the group of local CR automorphisms of M and N (respectively) are isomorphic.

The following theorem is a slight extension of a result in [13]:

THEOREM 4.5 *If the Levi–Tanaka algebra* $\mathfrak{g}$ *is semisimple, then the standard*

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

homogeneous CR manifold $M_{\mathfrak{g}}$ is compact.

The proof of this theorem relies on the following

LEMMA 4.6. *Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional semisimple Levi–Tanaka algebra and let*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is a maximal Lie subalgebra of $\mathfrak{g}$ on which the Killing form $\kappa_{\mathfrak{g}}$ is negative defined. Then, for $\mathfrak{g}_+ = \oplus_{p\geqslant 0}\mathfrak{g}_p$, we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_+.$$

Proof. Let $\mathrm{d} = \dim_{\mathbb{R}}\mathfrak{g}_0$ and $m = \dim_{\mathbb{R}}\mathfrak{m}$ where $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$. The Killing form $\kappa_{\mathfrak{g}}$ is nondegenerate on $\mathfrak{g}_0$ and therefore its restriction to $\mathfrak{g}_0$ has a signature (σ^+, σ^-) with $\sigma^+ + \sigma^- = \mathrm{d}$. Since $\mathfrak{m}$ is totally isotropic, the Killing form $\kappa_{\mathfrak{g}}$ has signature $(\sigma^+ + m, \sigma^- + m)$. Given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we claim that $\mathfrak{k} \cap \mathfrak{g}_+$ is a Lie subalgebra of dimension σ^- of $\mathfrak{g}$. Indeed, if $X = \Sigma_{p\geqslant 0}X_p$ is a nonzero vector in $\mathfrak{k} \cap \mathfrak{g}_+$ decomposed into its homogeneous components, then

$$0 > \kappa_{\mathfrak{g}}(X, X) = \kappa_{\mathfrak{g}}(X_0, X_0)$$

shows that the natural projection $\mathfrak{k}\cap\mathfrak{g}_+ \to \mathfrak{g}_0$ is injective and its image is a subspace of $\mathfrak{g}_0$ on which $\kappa_{\mathfrak{g}}$ is negative definite. This shows that $\dim_{\mathbb{R}} \mathfrak{k} \cap \mathfrak{g}_+ \leqslant \sigma^-$. On the other hand, the projection $\mathfrak{k} \to \mathfrak{g}/\mathfrak{g}_+$, having kernel $\mathfrak{k}\cap\mathfrak{g}_+$, is necessarily surjective and therefore has rank m and σ^--dimensional kernel. In particular we obtain that $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_+$.

Proof (*of Theorem* 4.5). Let $\mathbf{G}$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathbf{G}_+$ and $\mathbf{K}$ be the connected Lie subgroups of $\mathbf{G}$ having Lie algebras $\mathfrak{g}_+$ and $\mathfrak{k}$ respectively, with $\mathfrak{k}$ the direct summand in a Cartan decomposition of $\mathfrak{g}$. Then $\mathbf{K}$ is a compact subgroup of $\mathbf{G}$. We consider the map $\mathbf{K} \to M_{\mathfrak{g}} = \mathbf{G}/\mathbf{G}_+$ induced by the restriction of the natural projection. Its image is compact and hence closed. On the other hand, the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_+$ shows that this map is a submersion and then open. Therefore, since $M_{\mathfrak{g}}$ is connected, this map is onto and $M_{\mathfrak{g}}$ is compact.

Denote by $\mathbf{K}_0$ the connected Lie subgroup of $\mathbf{G}$ having Lie algebra $\mathfrak{k} \cap \mathfrak{g}_+$. (Note that $\mathfrak{k} \cap \mathfrak{g}_+ \subset \mathfrak{g}_0$ if we use a Cartan decomposition with the properties of Proposition 3.41.) Then the natural map $\mathbf{K}/\mathbf{K}_0 \to M_{\mathfrak{g}}$ is a diffeomorphism because is a connected covering of a simply connected manifold.

4.2. Canonical immersions of standard CR manifolds

Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra and $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ be its complexification. We denote by $\mathbf{G}^{\mathbb{C}}$ a connected and simply connected Lie group

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

having Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and by $\mathbf{G}^{\mathbb{R}}$ the connected Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ having Lie algebra $\mathfrak{g}$. This is a closed Lie subgroup of $\mathbf{G}^{\mathbb{C}}$, as $\mathbf{G}^{\mathbb{R}}$ is the connected component of the identity of the closed subgroup of $\mathbf{G}^{\mathbb{C}}$

$$\{g \in \mathbf{G}^{\mathbb{C}} \mid \mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(g)(\mathfrak{g}) = \mathfrak{g}\},$$

where $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}\colon \mathbf{G}^{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$ is the adjoint representation. We also use the notation $\mathfrak{g}_+^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_+$ for the complexification of the Lie subalgebra $\mathfrak{g}_+ = \oplus_{p \geqslant 0}\mathfrak{g}_p$ and $\mathbf{G}_+^{\mathbb{R}}$ for the connected Lie subgroup of $\mathbf{G}^{\mathbb{R}}$ having Lie algebra $\mathfrak{g}_+$.

LEMMA 4.7. *Let* $\mathfrak{g}_{-1}^{(0,1)} = \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\}$. *Then* $\mathfrak{q} = \mathfrak{g}_{-1}^{(0,1)} \oplus \mathfrak{g}_+^{\mathbb{C}}$ *is a complex Lie subalgebra of* $\mathfrak{g}^{\mathbb{C}}$.

Proof. First we remark that $\mathfrak{g}_{-1}^{(0,1)}$ is a complex subspace of $\mathfrak{g}^{\mathbb{C}}$. Indeed, for $X \in \mathfrak{g}_{-1}$ we have

$$\sqrt{-1}(X + \sqrt{-1}JX) = (-JX) + \sqrt{-1}J(-JX) \quad \text{and} \quad JX \in \mathfrak{g}_{-1}.$$

Moreover

$$[X + \sqrt{-1}JX, Y + \sqrt{-1}JY] = 0 \quad \forall X, Y \in \mathfrak{g}_{-1}$$

and $[\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0, \mathfrak{g}_{-1}^{(0,1)}] \subset \mathfrak{g}_{-1}^{(0,1)}$ because $\mathfrak{g}_{-1}^{(0,1)}$ is a complex subspace of $\mathfrak{g}^{\mathbb{C}}$ and the elements of $\rho_{-1}(\mathfrak{g}_0)$ commute with J on $\mathfrak{g}_{-1}$. Finally, it is obvious that $[\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_p, \mathfrak{g}_{-1}^{(0,1)}] \subset \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{p-1} \subset \mathfrak{q}$ for $p > 0$.

Let $\mathbf{Q}$ be the connected complex Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ corresponding to the Lie subalgebra $\mathfrak{q}$.

LEMMA 4.8. $\mathbf{Q}$ *is a closed Lie subgroup of* $\mathbf{G}^{\mathbb{C}}$.

Proof. We consider the adjoint representation $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}\colon \mathbf{G}^{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$. Then

$$\mathbf{H} = \{g \in \mathbf{G}^{\mathbb{C}} \mid \mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(g)(\mathfrak{q}) = \mathfrak{q}\}$$

is a closed subgroup of $\mathbf{G}^{\mathbb{C}}$ and $\mathbf{Q}$ is the connected component of the identity of $\mathbf{H}$.

THEOREM 4.9. *The* $\mathbf{G}^{\mathbb{C}}$*-homogeneous space* $\hat{M}_{\mathfrak{g}}^{\mathbb{C}} = \mathbf{G}^{\mathbb{C}}/\mathbf{Q}$ *is a complex manifold.*

The $\mathbf{G}^{\mathbb{R}}$*-homogeneous space* $M_{\mathfrak{g}}^{\mathbb{R}} = \mathbf{G}^{\mathbb{R}}/\mathbf{G}_+^{\mathbb{R}}$ *is a differentiable manifold with a unique* CR *structure which makes the covering map*

$$M_{\mathfrak{g}} \to M_{\mathfrak{g}}^{\mathbb{R}}$$

defined by the commutative diagram

$$\begin{tikzcd}
\mathbf{G} \arrow[r] \arrow[d] & \mathbf{G}^{\mathbb{R}} \arrow[d] \\
M_{\mathfrak{g}} \arrow[r] & M_{\mathfrak{g}}^{\mathbb{R}}
\end{tikzcd}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

a local CR diffeomorphism.

The composition $\mathbf{G} \to \mathbf{G}^{\mathbb{R}} \to \mathbf{G}^{\mathbb{C}}$ *induces a CR immersion*

$$M_{\mathfrak{g}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{C}},$$

whose image $M_{\mathfrak{g}}^{\mathbb{C}}$ is a locally closed CR submanifold of $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$.

Proof. $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ is a connected smooth complex manifold because $\mathbf{Q}$ is a closed subgroup of $\mathbf{G}^{\mathbb{C}}$. Analogously $M_{\mathfrak{g}}^{\mathbb{R}}$ is a connected real analytic CR manifold because $\mathbf{G}_{+}^{\mathbb{R}}$ is a closed subgroup of $\mathbf{G}^{\mathbb{R}}$.

The group $\mathbf{G}$ is a covering of $\mathbf{G}^{\mathbb{R}}$ and $M_{\mathfrak{g}}^{\mathbb{R}}$ is $\mathbf{G}$-homogeneous by the action

$$\mathbf{G} \times M_{\mathfrak{g}}^{\mathbb{R}} \ni (g, x) \to p(g) \cdot x \in M_{\mathfrak{g}}^{\mathbb{R}},$$

where $p\colon \mathbf{G} \to \mathbf{G}^{\mathbb{R}}$ is the covering map.

We consider the orbit $M_{\mathfrak{g}}^{\mathbb{C}}$ in $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ of the image o of the identity of $\mathbf{G}^{\mathbb{C}}$ in $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ with respect to the closed subgroup $\mathbf{G}^{\mathbb{R}}$. Since $\mathfrak{g}_{+}$ is the Lie algebra of the stabilizer in $\mathbf{G}^{\mathbb{R}}$ of o, we obtain an immersion $M_{\mathfrak{g}}^{\mathbb{R}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ which is a surjective local diffeomorphism onto the orbit $M_{\mathfrak{g}}^{\mathbb{C}}$. Let $\alpha\colon \mathbf{G}^{\mathbb{R}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ denote the map

$$g \to g \cdot o.$$

We note that for the elements X of $\mathfrak{g}_{-1}$ we obtain, by the definition of $\mathfrak{q}$, $\alpha_*(JX) = \sqrt{\Leftrightarrow 1}\alpha_*(X)$ and therefore the map $M_{\mathfrak{g}}^{\mathbb{R}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ is a CR immersion.

Let $\mathbf{A}$ and $\tilde{\mathbf{A}}$ be the connected Lie subgroups of $\mathbf{G}^{\mathbb{C}}$ having Lie algebras $\mathfrak{m}$ and $\mathfrak{l} = \mathfrak{g}_{-1} \oplus (\oplus_{p<-1}\mathfrak{g}_p^{\mathbb{C}})$ respectively. We fix convex open neighborhoods U_0 of 0 in $\mathfrak{g}^{\mathbb{C}}$ and V_0 of 0 in $\mathfrak{l}$ such that the exponential maps

$$\exp\colon U_0 \to U_e \subset \mathbf{G}^{\mathbb{C}}, \qquad \exp\colon V_0 \to V_e \subset \tilde{\mathbf{A}},$$
$$\exp\colon U_0 \cap \mathfrak{g} \to U_e \cap \mathbf{G}^{\mathbb{R}}$$

be diffeomorphisms. We can assume that $V_0 = U_0 \cap \mathfrak{l}$, so that $V_e = U_e \cap \tilde{\mathbf{A}}$. If $a \in \mathbf{G}^{\mathbb{R}} \cap \tilde{\mathbf{A}} \cap U_e$, we have $a = \exp(Z) = \exp(X + \sqrt{\Leftrightarrow 1}Y)$ with $Z \in \mathfrak{g} \cap U_0$, $X \in \mathfrak{m}$, $Y \in \oplus_{p<-1}\mathfrak{g}_p$ and $Z, X + \sqrt{\Leftrightarrow 1}Y \in U_0$. By the injectivity of the exponential on U_0, we obtain $Z = X + \sqrt{\Leftrightarrow 1}Y$, hence $Y = 0$ and $Z \in \mathfrak{m}$. This shows that $\mathbf{G}^{\mathbb{R}} \cap \tilde{\mathbf{A}} \cap U_e = \mathbf{A} \cap U_e$. Moreover, $\mathbf{A} \cap U_e$ is closed and connected in $\tilde{\mathbf{A}} \cap U_e$. We note now that the projection $\pi\colon \mathbf{G}^{\mathbb{C}} \to M_{\mathfrak{g}}^{\mathbb{C}}$ induces a diffeomorphism of a neighborhood $W \subset V_e$ of e in $\tilde{\mathbf{A}}$ onto a neighborhood W_o of $o = \pi(e)$ in $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ and, since $\mathbf{Q} \cap \tilde{\mathbf{A}} \cap V_e = \{e\}$, we have $(\pi_{|W_o})^{-1}(M_{\mathfrak{g}}^{\mathbb{R}}) = V_e \cap \mathbf{G}^{\mathbb{R}}$. This shows that $W_o \cap M_{\mathfrak{g}}^{\mathbb{R}}$ is closed in W_o. Since $M_{\mathfrak{g}}^{\mathbb{R}}$ is homogeneous, it is locally closed in $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$.

We call $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ the *standard* (*homogeneous*) *complex manifold* associated to $\mathfrak{g}$ and the map $M_{\mathfrak{g}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ the *canonical immersion* of $M_{\mathfrak{g}}$.

THEOREM 4.10. *Il $\mathfrak{g}$ is semisimple, then the standard homogeneous complex manifold $\hat{M}^{\mathbb{C}}_{\mathfrak{g}}$ associated to $\mathfrak{g}$ is compact.*

Proof. Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution found in Proposition 3.41 and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Then $\mathfrak{u} = \mathfrak{k} \oplus \sqrt{\Leftrightarrow 1}\mathfrak{p}$ is a compact form of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. We set

$$\begin{aligned}
\mathfrak{u}^{(1,0)} &= \{X \Leftrightarrow \sqrt{\Leftrightarrow 1}JX + \theta(X) + \sqrt{\Leftrightarrow 1}J\theta(X) \mid X \in \mathfrak{g}_{-1}\},\\
\mathfrak{u}^{(0,1)} &= \{X + \sqrt{\Leftrightarrow 1}JX + \theta(X) \Leftrightarrow \sqrt{\Leftrightarrow 1}J\theta(X) \mid X \in \mathfrak{g}_{-1}\},\\
\mathfrak{u}_p &= \mathfrak{u} \cap (\mathfrak{g}^{\mathbb{C}}{}_{-p} + \mathfrak{g}^{\mathbb{C}}{}_{p}) \quad \text{for} \quad p \geqslant 0.
\end{aligned}$$

Next we define the real Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{h} = \mathfrak{u} \cap \mathfrak{q} = \mathfrak{u}_0 \oplus \mathfrak{u}^{(0,1)}.$$

Let $\mathbf{U}$ denote the connected Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ having Lie algebra $\mathfrak{u}$ and $\mathbf{H}$ the connected Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ having Lie algebra $\mathfrak{h}$. The group $\mathbf{U}$ is compact and hence closed in $\mathbf{G}^{\mathbb{C}}$, and also $\mathbf{H}$ is compact, being the connected component of the identity in the intersection $\mathbf{U} \cap \mathbf{Q}$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{U} \arrow[r] \arrow[d] & \mathbf{G}^{\mathbb{C}} \arrow[d] \\
\mathbf{U}/_{\mathbf{H}} \arrow[r] & \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}.
\end{tikzcd}$$

Since $\mathfrak{u} + \mathfrak{q} = \mathfrak{g}^{\mathbb{C}}$, the map $\mathbf{U} \to \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is a submersion and therefore is open. It is also closed, being a continuous map from a compact space into a Hausdorff space. Since $\hat{M}^{\mathbb{C}}_{\mathfrak{g}} = \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is connected, this map is surjective and therefore $\mathbf{U}/_{\mathbf{H}} \to \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is a covering map. Since $\mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is simply connected, this map is a diffeomorphism. This proves the theorem.

PROPOSITION 4.11. *If the component $\mathfrak{g}_1$ of $\mathfrak{g}$ is null, then the manifold $M^{\mathbb{C}}_{\mathfrak{g}}$ is embedded in $\hat{M}^{\mathbb{C}}_{\mathfrak{g}}$ as a closed submanifold and euclidean.*

Proof. By Lemma 3.18.4 of [16], $M^{\mathbb{C}}_{\mathfrak{g}}$ is closed in $\hat{M}^{\mathbb{C}}_{\mathfrak{g}}$ and simply connected and, by Lemma 3.18.11 of [16], it is also euclidean. Indeed, $\mathfrak{m}$ is an ideal in $\mathfrak{g}$ and therefore the map $\mathfrak{m} \oplus \mathfrak{g}_0 \ni (X, Y) \to \exp(X)\exp(Y) \in \mathbf{G}$ is a diffeomorphism.

4.3. Canonical projective immersions of standard CR manifolds

The problem of finding an immersion of the standard homogeneous CR manifold $M_{\mathfrak{g}}$ into a complex projective space is equivalent, by Theorem 4.4, to the one of finding, given a Levi–Tanaka algebra $\mathfrak{g}$, $\mathbf{G}$-homogeneous CR submanifolds of complex projective spaces having at each point a Levi–Tanaka algebra isomorphic to $\mathfrak{g}$.

Our construction is akin to the one used in [1]. We use the complexification of the adjoint representation $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}: \mathbf{G}^{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$ and denote by $\mathbf{G}^{\mathbb{P}}_{\mathbb{C}}$ and $\mathbf{G}^{\mathbb{P}}$ respectively the image $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(\mathbf{G}^{\mathbb{C}})$ and $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(\mathbf{G}^{\mathbb{R}})$. They are Lie subgroups of $\mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$. We also set $\mathbf{Q}^{\mathbb{P}}$ and $\mathbf{G}^{\mathbb{P}}_{+}$ for the connected Lie subgroups of $\mathbf{G}^{\mathbb{P}}_{\mathbb{C}}$ having Lie algebra equal respectively to the Lie subalgebra $\mathfrak{q}$ defined in Lemma 4.7 and to $\mathfrak{g}_{+}$.

We consider the Grassmannian $\mathbf{Gr}_{\ell}(\mathfrak{g}^{\mathbb{C}})$ of complex subspaces of $\mathfrak{g}^{\mathbb{C}}$ having dimension ℓ equal to the complex dimension of $\mathfrak{q}$. The orbits $\hat{M}^{\mathbb{P}}_{\mathfrak{g}}$ and $M^{\mathbb{P}}_{\mathfrak{g}}$ of $\mathfrak{q}$ by the action of $\mathbf{G}^{\mathbb{P}}_{\mathbb{C}}$ and $\mathbf{G}^{\mathbb{P}}$ are respectively a $\mathbf{G}^{\mathbb{P}}_{\mathbb{C}}$-homogeneous complex manifold and a $\mathbf{G}^{\mathbb{R}}$-homogeneous CR submanifold (and therefore $\mathbf{G}^{\mathbb{C}}$ and $\mathbf{G}$-homogeneous). In this way we obtain a CR submanifold of a projective manifold having the prescribed Levi–Tanaka algebra $\mathfrak{g}$ at each point.

We take up now the question of the existence of a closed embedding into a projective space in the case where the Levi–Tanaka algebra is semisimple.

We recall that a *Borel subalgebra* $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is said to be *parabolic* if it contains a Borel subalgebra. Accordingly, a connected Lie subgroup **B** (resp. **Q**) of a Lie group **G** is a Borel (resp. parabolic) subgroup if its Lie algebra $\mathfrak{b}$ (resp. $\mathfrak{q}$) is Borel (resp. parabolic). In particular a Borel subgroup of **G** is a maximal connected solvable subgroup of **G**.

LEMMA 4.12. *Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional Levi–Tanaka algebra. Then the following facts are equivalent:*

(i) *$\mathfrak{g}$ is semisimple;*

(ii) *$\mathfrak{g}_{+} = \oplus_{p\geqslant 0}\mathfrak{g}_p$ is parabolic;*

(iii) *$\mathfrak{q} = \mathfrak{g}^{(0,1)}_{-1} \oplus \mathfrak{g}^{\mathbb{C}}_{+}$ is a parabolic Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. (i) $\Leftrightarrow$ (ii). Let E be the element of $\mathfrak{g}_0$ described in Lemma 3.5. Then $\oplus_{p>0}\mathfrak{g}_p \oplus \mathbb{R}\cdot E$ is a solvable Lie subalgebra of $\mathfrak{g}$ and hencefore is contained in a Borel subalgebra $\mathfrak{b}$. If $\mathfrak{r}$ is the radical of $\mathfrak{g}$, then $\mathfrak{r} \subset \mathfrak{b}$. By Corollary 3.30, $\mathfrak{r}$ is contained in $\mathfrak{g}_{+}$ if and only if $\mathfrak{g}$ is semisimple and $\mathfrak{r} = 0$. The condition is therefore necessary.

To prove sufficiency, we first note that the representation $\rho: \mathfrak{b} \to \mathfrak{gl}(\mathfrak{g})$ obtained by restriction from the adjoint representation is faithful. Then, by the criterion of Cartan, $\rho(\mathfrak{b})$, and thus $\mathfrak{b}$, is solvable if and only if $[\mathfrak{b}, \mathfrak{b}]$ is orthogonal to $\mathfrak{b}$ with respect to the Killing form $\kappa_{\mathfrak{g}}$ of $\mathfrak{g}$. Assume by contradiction that $\mathfrak{b}$ contains an element $X = \Sigma_p X_p$ with homogeneous component $X_q \neq 0$ for some $q < 0$. Since $\mathfrak{g}$ was assumed to be semisimple, we can find $Y_{-q} \in \mathfrak{g}_{-q} \subset \mathfrak{b}$ such that $\kappa_{\mathfrak{g}}(X_q, Y_{-q}) \neq 0$. Then we obtain

$$\kappa_{\mathfrak{g}}([E, X], Y_{-q}) = q\,\kappa_{\mathfrak{g}}(X_q, Y_{-q}) \neq 0,$$

which contradicts the Cartan criterion.

(i) ⇔ (iii). If $\mathfrak{r}$ is the radical of $\mathfrak{g}$, then $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{r}$ is the radical of $\mathfrak{g}^{\mathbb{C}}$. Clearly, if $X \in \mathfrak{r}_{-1}$, then $X \Leftrightarrow \sqrt{\Leftrightarrow 1}JX$ belongs to the radical of $\mathfrak{g}^{\mathbb{C}}$ and therefore, if $\mathfrak{q}$ is parabolic, the radical of $\mathfrak{g}$ is contained in $\mathfrak{g}_+$. The proof is complete.

THEOREM 4.13. *A necessary and sufficient condition in order that $\hat{M}_{\mathfrak{g}}^{\mathbb{P}}$ be compact is that $\mathfrak{g}$ is semisimple.*

If $\mathfrak{g}$ is semisimple, then $M_{\mathfrak{g}}^{\mathbb{P}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{P}}$ is a closed embedding of $M_{\mathfrak{g}}^{\mathbb{P}}$ into a compact projective complex manifold.

Proof. The first part of the statement is a consequence of Lemma 4.12 and of [2] (Theorem 11.1 and Corollary 11.2) because $\mathbf{G}_{\mathbb{C}}^{\mathbb{P}}$ is an algebraic group.

The second part follows because $M_{\mathfrak{g}}^{\mathbb{P}}$ is compact when $\mathfrak{g}$ is semisimple because it is the quotient of $M_{\mathfrak{g}}$ with respect to the action of a discrete subgroup of $\mathbf{G}$.

We call $\hat{M}_{\mathfrak{g}}^{\mathbb{P}}$ the *standard* (*homogeneous*) *projective manifold* associated to the Levi–Tanaka algebra $\mathfrak{g}$ and the map $M_{\mathfrak{g}} \to M_{\mathfrak{g}}^{\mathbb{P}} \to \hat{M}_{\mathfrak{g}}^{\mathbb{P}}$ the *canonical projective immersion* of $M_{\mathfrak{g}}$.

THEOREM 4.14. *Let $\mathfrak{g}$ be a finite dimensional Levi–Tanaka algebra. Then a necessary and sufficient condition in order that $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ be compact is that $\mathfrak{g}$ be semisimple.*

Proof. We already proved that $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ is compact when $\mathfrak{g}$ is semisimple. When $\mathfrak{g}$ is not semisimple, then $\hat{M}_{\mathfrak{g}}^{\mathbb{P}}$ is not compact and hence also $\hat{M}_{\mathfrak{g}}^{\mathbb{C}}$ is not compact, because it is a covering space of $\hat{M}_{\mathfrak{g}}^{\mathbb{P}}$.

Remark 4.15. It follows from [6] that, when $\mathfrak{g}$ is semisimple, the standard homogeneous projective manifold $M_{\mathfrak{g}}^{\mathbb{P}}$ associated to a semisimple Levi–Tanaka algebra $\mathfrak{g} = \oplus_{-\mu \leqslant p \leqslant \mu} \mathfrak{g}_p$ with $\dim_{\mathbb{C}} \mathfrak{g}_{-1} = n$ and $\dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{R}} \oplus_{p<-1} \mathfrak{g}_p = k$, has a CR embedding in the space $\mathbb{CP}^{[2n+(3/2)k]}$.

Remark 4.16. Every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is splittable. Indeed the splittable envelope of a solvable subalgebra of $\mathfrak{g}$ is still solvable and therefore the splittable envelope of $\mathfrak{b}$ is equal to $\mathfrak{b}$ by maximality.

5. Examples

Let $\mathfrak{m} = \oplus_{-\mu \leqslant p < 0} \mathfrak{m}_p$ be a pseudocomplex fundamental graded Lie algebra. By Proposition 1.1 the alternating map

$$\mathfrak{m}_{-1} \times \mathfrak{m}_{-1} \ni (X, Y) \to [X, Y] \in \mathfrak{m}_{-2},$$

uniquely defines a Hermitian symmetric form

$$\mathfrak{f}\colon \mathfrak{m}_{-1} \times \mathfrak{m}_{-1} \ni (X, Y) \to \mathfrak{f}(X, Y) \in \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{m}_{-2}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

such that

$$[X,Y] = \Im \mathfrak{f}(X,Y) \quad \forall X,Y \in \mathfrak{m}_{-1}.$$

We consider the natural map

$$\lambda\colon \mathfrak{m}_{-2}^{*} \ni \xi \to \mathfrak{f}_{\xi} \in \mathfrak{H}_{\mathfrak{s}}(\mathfrak{m}_{-1})$$

from the dual space $\mathfrak{m}_{-2}^{*}$ of $\mathfrak{m}_{-2}$ to the real linear space $\mathfrak{H}_{\mathfrak{s}}(\mathfrak{m}_{-1})$ of Hermitian symmetric forms on $\mathfrak{m}_{-1}$, which is given by

$$\mathfrak{f}_{\xi}(X,Y) = \langle \mathfrak{f}(X,Y), \xi \rangle \quad \forall \xi \in \mathfrak{m}_{-2}^{*}, \forall X,Y \in \mathfrak{m}_{-1}.$$

Viceversa, given a finite dimensional $\mathbb{C}$-linear space V and a linear subspace L of the space $\mathfrak{H}_{\mathfrak{s}}(V)$ of Hermitian symmetric forms on V, there is a pseudocomplex fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ of kind 2 such that

$$\mathfrak{m}_{-1} = V \quad \text{and} \quad \lambda(\mathfrak{m}_{-2}^{*}) = L.$$

This algebra is unique up to isomorphisms and can be described by setting

$$\mathfrak{m}_{-2} = L^{*}, \quad [\mathfrak{m}_{-2}, \mathfrak{m}_{-2}] = [\mathfrak{m}_{-2}, \mathfrak{m}_{-1}] = 0$$

and defining the Lie product $[X,Y]$ of two elements $X,Y \in \mathfrak{m}_{-1} = V$ as the $\mathbb{R}$-linear functional on L

$$[X,Y]\colon L \ni h \to \Im h(X,Y) \in \mathbb{R}.$$

We say that $\mathfrak{m}$ is *of type* (n,k) where $n = \dim_{\mathbb{C}} V$ and $k = \dim_{\mathbb{R}} L$.

The group $\mathbf{GL}_{\mathbb{C}}(V)$ of $\mathbb{C}$-linear automorphisms of V acts on the space $\mathfrak{H}(V)$ of the Hermitian forms on V by

$$\mathbf{GL}_{\mathbb{C}}(V) \times \mathfrak{H}(V) \ni (a,h) \to a \cdot h \in \mathfrak{H}(V),$$

where

$$a \cdot h(X,Y) = h(a^{-1}(X), a^{-1}(Y))$$
$$\forall a \in \mathbf{GL}_{\mathbb{C}}(V), \forall h \in \mathfrak{H}(V), \forall X,Y \in V.$$

Clearly $\mathfrak{H}_{\mathfrak{s}}(V)$ is stable under this action of $\mathbf{GL}_{\mathbb{C}}(V)$. Moreover, $\mathbf{GL}_{\mathbb{C}}(V)$ transforms k-dimensional subspaces of $\mathfrak{H}_{\mathfrak{s}}(V)$ into k-dimensional subspaces of $\mathfrak{H}_{\mathfrak{s}}(V)$. We denote by $\mathfrak{H}_{\mathfrak{s}k}(V)$ the Grassmannian of k-dimensional subspaces of $\mathfrak{H}_{\mathfrak{s}}(V)$ and by $\mathfrak{O}_k(\mathfrak{H}_{\mathfrak{s}}(V))$ the space of orbits of $\mathfrak{H}_{\mathfrak{s}k}(V)$ for the action of the linear group $\mathbf{GL}_{\mathbb{C}}(V)$.

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

PROPOSITION 5.1. *Let n, k be positive integers, with $1 \leqslant k \leqslant n^2$ and let V be a complex vector space of dimension n. There is a* 1*-to-*1 *correspondence between pseudocomplex fundamental graded Lie algebras of kind* 2 *and type (n, k) modulo isomorphisms and the orbits in $\mathfrak{O}_k(\mathfrak{H}_\mathfrak{s}(V))$.*

Proof. Let $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ be a pseudocomplex fundamental graded Lie algebra. Let $a \in \mathbf{GL}_\mathbb{C}(\mathfrak{m}_{-1})$ and $b \in \mathbf{GL}_\mathbb{R}(\mathfrak{m}_{-2})$. Then we obtain another isomorphic fundamental graded Lie algebra $\tilde{\mathfrak{m}} = \tilde{\mathfrak{m}}_{-2} \oplus \tilde{\mathfrak{m}}_{-1}$ by setting $\tilde{\mathfrak{m}}_{-1} = \mathfrak{m}_{-1}$ as $\mathbb{C}$-linear spaces and $\tilde{\mathfrak{m}}_{-2} = \mathfrak{m}_{-2}$ as $\mathbb{R}$-linear spaces and defining the Lie product by

$$[\tilde{\mathfrak{m}}_{-1}, \tilde{\mathfrak{m}}_{-2}]' = [\tilde{\mathfrak{m}}_{-2}, \tilde{\mathfrak{m}}_{-2}]' = 0$$

and

$$[X, Y]' = b([a(X), a(Y)]) \qquad \forall X, Y \in \mathfrak{m}_{-1} = \tilde{\mathfrak{m}}_{-1}.$$

The isomorphism $\phi: \tilde{\mathfrak{m}} \to \mathfrak{m}$ is given by

$$\tilde{\mathfrak{m}}_{-1} \ni X \to a(X) \in \mathfrak{m}_{-1} \quad \text{and} \quad \tilde{\mathfrak{m}}_{-2} \ni T \to b^{-1}(T) \in \mathfrak{m}_{-2}.$$

Indeed the equation $\phi([X, Y]') = [\phi(X), \phi(Y)]$ reduces then to the definition of the Lie product in $\tilde{\mathfrak{m}}$.

By this remark, the statement of the proposition becomes clear.

Using this proposition, we can parametrize pseudocomplex fundamental graded Lie algebras of kind 2 and type (n, k) modulo isomorphisms by fixing a complex n-dimensional vector space V and a point L in one of the orbits of $\mathfrak{O}_k(\mathfrak{H}_\mathfrak{s}(V))$. We will denote by $\mathfrak{m}(L)$ the corresponding pseudocomplex fundamental graded Lie algebra and by $\mathfrak{g}(L)$ its canonical pseudocomplex prolongation.

Let $\mathbb{P}\mathfrak{H}_\mathfrak{s}(V)$ denote the projective $(n^2 \Leftrightarrow 1)$-dimensional space corresponding to the linear space $\mathfrak{H}_\mathfrak{s}$. The action of $\mathbf{GL}_\mathbb{C}(V)$ defines, by passing to the quotient, an action on $\mathbb{P}\mathfrak{H}_\mathfrak{s}(V)$. Let us denote by $\mathcal{C}$ the image in $\mathbb{P}\mathfrak{H}_\mathfrak{s}(V)$ of the cone of positive definite Hermitian symmetric forms on V. This is a convex body in $\mathbb{P}\mathfrak{H}_\mathfrak{s}(V)$. The corresponding Hilbert distance in $\mathcal{C}$ is given by

$$\mathrm{d}([h_1], [h_2]) = \sup_{1 \leqslant i,j \leqslant n} \log \frac{\lambda_i}{\lambda_j},$$

where $[h_1]$ and $[h_2]$ are the points of $\mathcal{C}$ corresponding to two positive definite Hermitian symmetric forms h_1, h_2 on V and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of h_2 with respect to h_1 (i.e., denoting still by h_1 and h_2 the anti-$\mathbb{C}$-linear maps $V \to V^*$ corresponding to the forms h_1 and h_2, the eigenvalues of the $\mathbb{C}$ linear endomorphism $h_1^{-1} \circ h_2$ of V). The group $\mathbf{GL}_\mathbb{C}(V)$ operates on $\mathcal{C}$. Its image in its representation in the group of permutations of $\mathcal{C}$ is the connected component of the identity in the Lie group of isometries of the Hilbert metric.

We devote the rest of this section mostly to the study of the canonical pseudocomplex prolongations in the kind 2 case. In the following we will denote by V an n dimensional complex space and use L for a k-dimensional real linear subspace of $\mathfrak{H}_{\mathfrak{s}}(V)$ and $\mathbb{P}L$ for its projective image in $\mathbb{P}\mathfrak{H}_{\mathfrak{s}}(V)$.

5.1. Levi–Tanaka algebras of kind 2 isomorphic to $\mathfrak{su}(p+m, q+m)$

Let m, p, q be nonnegative integers with $m > 0$ and $\ell = p + q > 0$. With I_t denoting the $t \times t$ identity matrix, we set

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

We consider the Hermitian symmetric matrix

$$Q = \begin{pmatrix} 0 & 0 & I_m \\ 0 & I_{p,q} & 0 \\ I_m & 0 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ of matrices A in $\mathfrak{sl}(\ell + 2m, \mathbb{C})$ satisfying

$$A^*Q + QA = 0$$

is isomorphic to $\mathfrak{su}(p + m, q + m)$, so it is simple. Its elements are null-trace matrices of the form

$$\begin{pmatrix} a_{11} & -a_{23}^* I_{p,q} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & -a_{21}^* I_{p,q} & -a_{11}^* \end{pmatrix}$$

with blocks a_{13}, $a_{31} \in \mathfrak{u}(m)$ and $a_{22} \in \mathfrak{u}(p, q)$. We obtain a structure of Levi–Tanaka algebra of type $(\ell m, m^2)$ by defining the elements E and $\tilde{J}$ by

$$E = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_m \end{pmatrix} \quad \text{and} \quad \tilde{J} = \frac{\sqrt{-1}}{\ell + 2m} \begin{pmatrix} -\ell I_m & 0 & 0 \\ 0 & 2m I_\ell & 0 \\ 0 & 0 & -\ell I_m \end{pmatrix}.$$

This case generalizes the case of CR hypersurfaces, i.e. of type $(n, 1)$, with nondegenerate Levi form, that was fully discussed in [14] and [4] and corresponds to the choice $m = 1$. We note that the space of orbits of $\mathfrak{O}_1(\mathfrak{H}_{\mathfrak{s}}(V))$ contains only finitely many elements. In order that the canonical pseudocomplex prolongation be finite dimensional, it is necessary and sufficient to start from $\mathbb{P}L = \{[h]\}$ with h nondegenerate, i.e. of signature (p, q) with $p + q = n$. In this case $\mathfrak{g}(L)$ is isomorphic to the simple Lie algebra $\mathfrak{su}(p + 1, q + 1)$.

5.2. LEVI-TANAKA ALGEBRAS OF KIND 2 ISOMORPHIC TO $\mathfrak{sl}(n, \mathbb{C})$

Let $n \geqslant 3$ and let us fix two positive integers m, ℓ with $2m + \ell = n$. We write a matrix $A \in \mathfrak{sl}(n, \mathbb{C})$ in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{with} \quad \begin{cases} a_{11},\, a_{13},\, a_{31},\, a_{33} \in \mathfrak{gl}(m, \mathbb{C}) \\ a_{12},\, a_{32} \quad m \times \ell \text{ complex matrices} \\ a_{21},\, a_{23} \quad \ell \times m \text{ complex matrices} \\ a_{22} \in \mathfrak{gl}(\ell, \mathbb{C}) \\ \operatorname{tr}(a_{11}) + \operatorname{tr}(a_{22}) + \operatorname{tr}(a_{33}) = 0. \end{cases}$$

We graduate the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ by setting

$$\mathfrak{g}_p = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid a_{ij} = 0 \quad \text{for} \quad j \Leftrightarrow i \neq p \right\}.$$

The elements E and $\tilde{J}$ are like in the previous example. We denote this pseudo-complex graded Lie algebra by $\mathfrak{sl}(2m + \ell, \mathbb{C})$.

We consider the $2\ell m$-dimensional complex vector space V of pairs of $\ell \times m$ complex matrices and the map

$$\mathfrak{g}_{-1} \ni \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} \to (a_{21}, a_{32}^*) \in V,$$

where a_{32}^* denotes the conjugated transpose of a_{32}. This map is $\mathbb{C}$-linear for the complex structure of $\mathfrak{g}_{-1}$ and the canonical complex structure of V. Identifying the space of $m \times m$ complex matrices to a $2m^2$-dimensional real space, we obtain the Levi–Tanaka form on V

$$\Im \mathfrak{f}((v_1, v_2), (w_1, w_2)) = v_2^* w_1 \Leftrightarrow w_2^* v_1.$$

It is convenient to represent $\mathfrak{g}_{-2}$ as the direct sum of two copies of the Hermitian symmetric $m \times m$ matrices. We obtain a (vector-valued) Levi form that can be written as

$$V \ni \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} (v_1^*, v_2^*) \begin{pmatrix} 0 & \sqrt{\Leftrightarrow 1} I_\ell \\ \Leftrightarrow \sqrt{\Leftrightarrow 1} I_\ell & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ (v_1^*, v_2^*) \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} \in \mathfrak{H}_{\mathfrak{s}}(\mathbb{C}^m)^2$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

and therefore $\mathfrak{sl}(2m+\ell,\mathbb{C})$ is the Levi–Tanaka algebra of a CR manifold **M** of type $(2\ell m, 2m^2)$ which is ℓ-pseudoconcave. It is also $m\ell$-pseudoconvex.

Note that the algebra considered in this example can be obtained by considering the complexification of the algebra in the previous one, where $\ell = p+q$.

Remark 5.2. The simple algebra $\mathfrak{sl}(n,\mathbb{C})$ admits at least one structure of Levi–Tanaka algebra of kind μ for $1<\mu<n$. Moreover, there exist several nonequivalent structures for the same μ if $1<\mu<n-1$.

Indeed, given a partition $(n_0,\dots,n_\mu)$ of n, i.e. positive integers n_j with $0\leqslant j\leqslant\mu$ such that $\Sigma_{j=0}^{\mu} n_j = n$, we consider

$$E = \tfrac{1}{2}\,\mathrm{diag}(\mu I_{n_0},\dots,(\mu-2j)I_{n_j},\dots,-\mu I_{n_\mu}) + c_E I_n,$$

$$\tilde{J} = \frac{\sqrt{-1}}{2}\,\mathrm{diag}(I_{n_0},\dots,(-1)^j I_{n_j},\dots,(-1)^\mu I_{n_\mu}) + \sqrt{-1}\,c_{\tilde{J}} I_n,$$

where $c_E, c_{\tilde{J}}\in\mathbb{R}$ are such that $E,\tilde{J}\in\mathfrak{sl}(n,\mathbb{C})$ and $\mathrm{diag}(a_0,\dots,a_\mu)$ denotes the block-diagonal matrix of entries $a_0,\dots,a_\mu$. If we denote by $\mathfrak{g}_p$ the eigenspace of the adjoint representation of $\mathfrak{sl}(n,\mathbb{C})$ of the element E associated to the eigenvalue $-\mu\leqslant p\leqslant\mu$, then $\mathfrak{g} = \oplus_{-\mu\leqslant p\leqslant\mu}\mathfrak{g}_p$, with the pseudocomplex structure on $\mathfrak{g}_{-1}$ given by the adjoint representation of the element $\tilde{J}$, is a Levi–Tanaka algebra of kind μ.

We note that if in addition $n_j = n_{\mu-j}$ for every $0\leqslant j\leqslant\mu$, denoting by

$$Q = \begin{pmatrix} & & I_{n_\mu} \\ & \ddots & \\ I_{n_0} & & \end{pmatrix},$$

we have that the algebra

$$\{A\in\mathfrak{sl}(n,\mathbb{C}) \mid A^*Q + QA = 0\},$$

with the graduation and the pseudocomplex structure similarly defined, is a Levi–Tanaka algebra of kind μ. If in addition μ is even, we may take in the definition of Q the matrix $I_{p,q}$ instead of $I_{\mu/2}$. These algebras are all isomorphic to $\mathfrak{su}(p,q)$ for suitable p and q.

5.3. Levi–Tanaka algebra of kind 2 isomorphic to $\mathfrak{so}(n+2,n)$

Let V be a complex linear space of dimension $n\geqslant 2$ and let W be a totally real subspace of V of real dimension n. We consider the $n(n-1)/2$ dimensional subspace L of $\mathfrak{H}_{\mathfrak{s}}(V)$ of Hermitian symmetric forms h such that $h(X,X)=0$ for all $X\in W$.

In a basis $e_1, \ldots, e_n$ of V contained in W, the matrices associated to the elements of L are of the form $\sqrt{-1}\,A$ for a matrix $A \in \mathfrak{so}(n)$. We call such a subspace L of $\mathfrak{H}_{\mathfrak{s}}(V)$ a *skew subspace* of $\mathfrak{H}_{\mathfrak{s}}(V)$. Clearly, all skew subspaces of $\mathfrak{H}_{\mathfrak{s}}(V)$ belong to the same orbit under the action of $\mathbf{GL}_{\mathbb{C}}(V)$ and therefore define isomorphic pseudocomplex fundamental graded Lie algebras $\mathfrak{m}(L)$.

PROPOSITION 5.3. *The canonical pseudocomplex prolongation of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m}(L)$ associated to a skew subspace L of $\mathfrak{H}_{\mathfrak{s}}(V)$ is a simple graded Lie algebra, isomorphic to the Lie algebra $\mathfrak{so}(n+2, n)$.*

Proof. We consider on the real vector space $\mathbb{R}^{2n+2}$ the symmetric bilinear form of signature $(n+2, n)$ defined by the Hermitian symmetric matrix

$$Q = \begin{pmatrix} 0 & 0 & I_n \\ 0 & I_2 & 0 \\ I_n & 0 & 0 \end{pmatrix},$$

where I_ℓ is the identity $\ell \times \ell$ matrix. Then $\mathfrak{so}(n+2, n)$ is identified to the space of matrices of the form

$$\begin{pmatrix} \alpha & \delta & \gamma \\ \beta & \varepsilon & -{}^t\delta \\ \theta & -{}^t\beta & -{}^t\alpha \end{pmatrix} \quad \text{where} \quad \begin{cases} \alpha \in \mathfrak{gl}(n, \mathbb{R}) \\ \beta \quad \text{is a } 2 \times n \text{ real matrix} \\ \delta \quad \text{is a } n \times 2 \text{ real matrix} \\ \gamma,\ \theta \in \mathfrak{so}(n) \\ \varepsilon \in \mathfrak{so}(2, \mathbb{R}). \end{cases}$$

We denote by $\mathfrak{g}$ the Lie algebra of $(2n+2) \times (2n+2)$ matrices defined above. We consider the element $E \in \mathfrak{g}$

$$E = \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & -I_n \end{pmatrix}.$$

Then $\mathrm{ad}_{\mathfrak{g}}(E)$ is semisimple with eigenvalues $-2,\ -1,\ 0,\ 1,\ 2$ and we denote by $\mathfrak{g}_p$ the eigenspace corresponding to its integral eigenvalues $-2 \leqslant p \leqslant 2$. In this way $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ has the structure of a simple graded Lie algebra. We note that

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -{}^t\alpha \end{pmatrix} \mid \alpha \in \mathfrak{gl}(n, \mathbb{R}),\ \varepsilon \in \mathfrak{so}(2) \right\}$$

and

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & -{}^t\beta & 0 \end{pmatrix} \mid \beta \text{ is a } 2 \times n \text{ matrix} \right\}.$$

Let

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and consider

$$\tilde{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_0.$$

We have $\rho_{-1}(\tilde{J})^2 = -\mathrm{Id}|_{\mathfrak{g}_{-1}}$ and $[\rho_{-1}(\tilde{J})X, \rho_{-1}(\tilde{J})Y] = [X, Y]$ for every $X, Y \in \mathfrak{g}_{-1}$, therefore $\rho_{-1}(\tilde{J})$ defines a complex structure in $\mathfrak{g}_{-1}$. If we associate to the matrix β parametrizing $\mathfrak{g}_{-1}$ the element $Z \in \mathbb{C}^n$ obtained by adding to its first row $\sqrt{-1}$ times its second row, the way the element

$$X_0 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -{}^t\alpha \end{pmatrix} \in \mathfrak{g}_0$$

acts on $\mathfrak{g}_{-1}$ can be described by

$$\rho_{-1}(X_0)(Z) = -\alpha\, Z + \sqrt{-1}\tau\, Z$$

if

$$\varepsilon = \begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix}.$$

It is clear then that $[X_0, \mathfrak{g}_{-1}] \neq 0$ if $X_0 \in \mathfrak{g}_0$ is different from zero. Moreover, the matrices of the form $\beta{}^t\gamma - \gamma{}^t\beta$, for β, γ varying in the space of $n \times 2$ real matrices, are a basis of $\mathfrak{so}(n)$ as a real vector space and the bilinear form $(\beta, \gamma) \to \beta{}^t\gamma - \gamma{}^t\beta$ is nondegenerate. Then $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a nondegenerate fundamental graded Lie algebra. By Lemma 3.16, it follows that $\mathfrak{g}$ is transitive. From $j{}^tj = I_2$ we obtain also that $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is pseudocomplex. By Theorem 3.21, it is sufficient then to establish an isomorphism between the pseudocomplex fundamental graded Lie algebras $\mathfrak{m}(L)$ and $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

To this aim we choose a basis $e_1, \ldots, e_n$ of V contained in W and associate to every vector $v \in V$ the $n \times 2$ real matrix β whose first column is the real and the second the immaginary part of the components of v in this basis. The identification of $\mathfrak{g}_{-2}$ and L^* is the standard identification of the dual of real alternating forms on W with the real alternating forms on W^*. The proof is complete.

5.4. LEVI–TANAKA ALGEBRAS OF TYPE $(n, 2)$ WITH $n > 1$

Let $\mathfrak{m} = \oplus_{p\geqslant -2}\mathfrak{g}_p$ be a fundamental graded Lie algebra of type $(n, 2)$. Assume that $\mathfrak{m}$ is nondegenerate so that its canonical pseudocomplex prolongation is finite dimensional. The structure of $\mathfrak{m}$ can be given by a real 2-dimensional subspace L of Hermitian symmetric forms on a complex vector space V with $\dim_{\mathbb{C}} V = n$. Assume that there exists a nondegenerate form belonging to L and let L_1 and L_2 be a basis of L with L_1 nondegenerate. By Theorem 4.5.19 of [7] we can choose a basis of V such that L_1 and L_2 are represented by two matrices in the diagonal form with $\ell_i \times \ell_i$ blocks A_i, respectively B_i, where

$$A_i = \varepsilon_i \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}, \quad B_i = \varepsilon_i \begin{pmatrix} 0 & & & \alpha_i \\ & & \ddots & 1 \\ & \ddots & \ddots & \\ \alpha_i & 1 & & 0 \end{pmatrix},$$

with $\alpha_i \in \mathbb{R}$ and $\varepsilon_i = \pm 1$, for $1 \leqslant i \leqslant r$, and $2\ell_i \times 2\ell_i$ blocks

$$A_i = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} & & & & & & & \alpha_i \\ & & & & & & \ddots & 1 \\ & & & & & \ddots & \ddots & \\ & & & & \alpha_i & 1 & & \\ & & & \bar{\alpha}_i & & & & \\ & & \ddots & 1 & & & & \\ & \ddots & \ddots & & & & & \\ \bar{\alpha}_i & 1 & & & & & & \end{pmatrix}$$

with $\alpha_i \in \mathbb{C} \setminus \mathbb{R}$, for $r + 1 \leqslant i \leqslant r + s$.

We assume that $\ell_i = 1$ for every $1 \leqslant i \leqslant r + s$. The case $s = 0$ and $\alpha_1 = \cdots = \alpha_n$ is not possible because L has dimension 2. In the case $s = 0$ with $\alpha_1 = \cdots = \alpha_p \neq \alpha_{p+1} = \cdots = \alpha_{p+q}$, the algebra $\mathfrak{m}$ is the direct sum of two ideals and $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(p+1, 1) \oplus \mathfrak{su}(q+1, 1)$ (see Proposition 3.3). When $r = 0, n$ is even and if $\alpha_1 = \cdots = \alpha_{n/2}$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2 + n/2, \mathbb{C})$ as in example in 5.2 (using Theorem 3.21). In all other cases with $n \geqslant 3$ it can be proved that $\rho_{-2}(\mathfrak{g}_0) = \mathbb{R}\,\mathrm{Id}_{\mathfrak{g}_{-2}}$ and so, by Theorem 3.24, we have that $\mathfrak{g}_p = 0$ for every $p > 0$.

When $n \geqslant 3$ and all ℓ_i's are equal to 1 and the α_i's are distinct, the corresponding standard homogeneous CR manifolds are euclidean and are parametrized, modulo CR diffeomorphisms, by a moduli space of real dimension $n \Leftrightarrow 3$. This space is indeed the quotient of the set of n-tuple of distinct points of $\mathbb{CP}^1$, symmetrical for the involution defined by $\mathbb{RP}^1 \subset \mathbb{CP}^1$, under the action of the group of automorphisms of $\mathbb{CP}^1$ which leave invariant the Poincaré half-plane. This has been shown in [9] for the case $n \geqslant 7$ and in general by one of the authors in his laurea dissertation (1991).

For $n = 2$, assuming again that the ℓ_i's are equal to 1, we obtain Levi–Tanaka algebras isomorphic either to $\mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1)$ (the pseudoconvex case) or to $\mathfrak{sl}(3,\mathbb{C})$ (the 1-pseudoconcave case).

We consider a case where $\ell_i \neq 1$ in the example below, that completes the description of all Levi–Tanaka algebras of type (2,2) and kind $\mu = 2$.

5.5. The weakly pseudoconcave Levi–Tanaka algebra of type $(2,2)$

Let L be the linear subspace of $\mathfrak{H}_{\mathfrak{s}}(\mathbb{C}^2)$ generated by the Hermitian forms associated to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To compute the Levi–Tanaka algebra $\mathfrak{g}(L)$ we first introduce some notation. We denote by $T_2\mathbb{C}$ the unitary associative $\mathbb{C}$-algebra of lower triangular 2×2 matrices with complex coefficients. We consider on $T_2\mathbb{C}$ the two antilinear maps $T_2\mathbb{C} \ni \alpha \to \bar{\alpha} \in T_2\mathbb{C}$ and $T_2\mathbb{C} \ni \alpha \to \tilde{\alpha} \in T_2\mathbb{C}$ associating to the matrix

$$\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

the matrices

$$\bar{\alpha} = \begin{pmatrix} \bar{\alpha}_{11} & 0 \\ \bar{\alpha}_{21} & \bar{\alpha}_{22} \end{pmatrix}, \quad \tilde{\alpha} = \begin{pmatrix} \bar{\alpha}_{22} & 0 \\ \bar{\alpha}_{21} & \bar{\alpha}_{11} \end{pmatrix}.$$

Then we define the two subrings of $T_2\mathbb{C}$

$$N_2\mathbb{C} = \left\{ \begin{pmatrix} z_1 & 0 \\ z_2 & z_1 \end{pmatrix} \,\middle|\, z_1,\, z_2 \in \mathbb{C} \right\} = \{\alpha \in T_2\mathbb{C} \mid \bar{\alpha} = \tilde{\alpha}\},$$

$$N_2\mathbb{R} = \left\{ \begin{pmatrix} t_1 & 0 \\ t_2 & t_1 \end{pmatrix} \,\middle|\, t_1, t_2 \in \mathbb{R} \right\} = \{\alpha \in T_2\mathbb{C} \mid \alpha = \bar{\alpha} = \tilde{\alpha}\}.$$

Remark 5.4. We have:

(1) $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta} \quad \forall \alpha,\, \beta \in T_2\mathbb{C}$;

(2) $\widetilde{\alpha\beta} = \tilde{\beta}\tilde{\alpha} \quad \forall \alpha,\, \beta \in T_2\mathbb{C}$;

(3) $\alpha\beta = \beta\alpha \quad \forall \alpha,\, \beta \in N_2\mathbb{C}$;

(4) if $\alpha,\, \beta \in T_2\mathbb{C}$ and $\alpha\zeta = \zeta\beta \quad \forall \zeta \in N_2\mathbb{R}$, then $\alpha = \beta \in N_2\mathbb{C}$.

PROPOSITION 5.5. *The Levi–Tanaka algebra $\mathfrak{g}(L)$ is isomorphic to the subalgebra of $\mathfrak{gl}(6, \mathbb{C})$ of matrices of the form*

$$\begin{pmatrix} \alpha & \eta & \sigma \\ \zeta & \beta & \sqrt{\Leftrightarrow 1}\bar{\eta} \\ \tau & \Leftrightarrow\sqrt{\Leftrightarrow 1}\bar{\zeta} & \Leftrightarrow\tilde{\alpha} \end{pmatrix}, \tag{11}$$

where $\tau,\, \sigma \in N_2\mathbb{R}$, $\zeta,\, \eta \in N_2\mathbb{C}$, and

$$\alpha = \begin{pmatrix} a + \sqrt{\Leftrightarrow 1}b & 0 \\ c & d + \sqrt{\Leftrightarrow 1}b \end{pmatrix},$$

$$\beta = \begin{pmatrix} \frac{a-d}{2} \Leftrightarrow 2\sqrt{\Leftrightarrow 1}b & 0 \\ \Leftrightarrow 2\sqrt{\Leftrightarrow 1}\Im c & \frac{d-a}{2} \Leftrightarrow 2\sqrt{\Leftrightarrow 1}b \end{pmatrix}, \tag{12}$$

with $a, b, d \in \mathbb{R}$ and $c \in \mathbb{C}$.

We note that $\tilde{\beta} = \Leftrightarrow\beta$ and that $\alpha + \beta \Leftrightarrow \tilde{\alpha}$ is a diagonal 2×2 matrix with 0 trace.

The operators $E, \tilde{J} \in \mathfrak{g}_0(L)$ are described by the matrices

$$E = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & \Leftrightarrow I_2 \end{pmatrix} \quad \textit{and} \quad \tilde{J} = \begin{pmatrix} \Leftrightarrow\frac{\sqrt{-1}}{3}I_2 & 0 & 0 \\ 0 & \frac{2\sqrt{-1}}{3}I_2 & 0 \\ 0 & 0 & \Leftrightarrow\frac{\sqrt{-1}}{3}I_2 \end{pmatrix}.$$

We have

$$\mathfrak{g}_{-2}(L) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{pmatrix} \mid \tau \in N_2\mathbb{R} \right\} \simeq \mathbb{R}^2;$$

$$\mathfrak{g}_{-1}(L) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \zeta & 0 & 0 \\ 0 & \Leftrightarrow\sqrt{\Leftrightarrow 1}\bar{\zeta} & 0 \end{pmatrix} \mid \zeta \in N_2\mathbb{C} \right\} \simeq \mathbb{C}^2;$$

$$\mathfrak{g}_0(L) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\tilde{\alpha} \end{pmatrix} \mid \alpha, \beta \textit{ as in } (12) \right\};$$

$$\mathfrak{g}_1(L) = \left\{ \begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & \sqrt{-1}\eta \\ 0 & 0 & 0 \end{pmatrix} \mid \eta \in N_2\mathbb{C} \right\} \simeq \mathbb{C}^2;$$

$$\mathfrak{g}_2(L) = \left\{ \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \sigma \in N_2\mathbb{R} \right\} \simeq \mathbb{R}^2.$$

The Levi–Tanaka algebra $\mathfrak{g}(L)$ *admits a graded Levi–Malčev decomposition* $\mathfrak{g}(L) = \mathfrak{s} \oplus \mathfrak{r}$ *with* $\mathfrak{s} \simeq \mathfrak{su}(1,2)$ *and* $\mathfrak{r} \neq 0$.

Proof. Using the previous remark, one easily checks by direct computation that the matrix algebra defined above is a pseudocomplex prolongation of the fundamental pseudocomplex Lie algebra $\mathfrak{m}(L)$. We define $\mathfrak{r}$ as the set of matrices as in (11) with

$$\tau = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \ t \in \mathbb{R}; \qquad \sigma = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}, \ s \in \mathbb{R};$$

$$\zeta = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \ z \in \mathbb{C}; \qquad \eta = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}, \ w \in \mathbb{C};$$

$$\alpha = \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \text{ and } \beta = \begin{pmatrix} a & 0 \\ -2\sqrt{-1}\Im c & -a \end{pmatrix} \quad \text{for } a \in \mathbb{R}, c \in \mathbb{C}.$$

It is easy to verify that $\mathfrak{r}$ is an ideal of $\mathfrak{g}(L)$.

Next we denote by $\mathfrak{s}$ the set of matrices of the form (11) with

$$\tau = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{R}; \quad \sigma = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \ s \in \mathbb{R};$$

$$\zeta = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \ z \in \mathbb{C}; \quad \eta = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \ w \in \mathbb{C};$$

$$\alpha = \begin{pmatrix} a + \sqrt{-1}b & 0 \\ 0 & a + \sqrt{-1}b \end{pmatrix} \text{ and } \beta = \begin{pmatrix} -2\sqrt{-1}b & 0 \\ 0 & -2\sqrt{-1}b \end{pmatrix}$$

https://doi.org/10.1023/A:1000166119593 Published online by Cambridge University Press

for $a, b \in \mathbb{R}$. We observe that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}(L)$ which is semisimple being isomorphic to $\mathfrak{su}(1, 2)$. To prove that the algebra $\mathfrak{g}(L)$ defined above is the Levi–Tanaka algebra of the second kind associated to L, we have to show that it is a maximal prolongation. First we remark that the canonical pseudocomplex prolongation of $\mathfrak{m}(L)$ is not semisimple because $\mathfrak{g}_0(L)$ is not reductive (and $\mathfrak{g}_0(L)$ is the degree 0 component of the canonical pseudocomplex prolongation). By the graded Levi–Malčev decomposition and the fact that the canonical prolongation is semisimple when the radical has no degree ⇔1-component, we deduce that $\mathfrak{s}$ is the semisimple part of the canonical pseudocomplex prolongation. Knowing that a prolongation of $\mathfrak{g}(L)$ would be a prolongation of its radical, we conclude by an explicit computation that the $\mathfrak{g}(L)$ we constructed is indeed the canonical pseudocomplex prolongation of $\mathfrak{m}(L)$.

5.6. Finite dimensional Levi–Tanaka algebras $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ with $\dim_{\mathbb{R}}\mathfrak{g}_{-1} = 2$

Let $\mathfrak{m} = \oplus_{-\mu\leqslant p<0}\mathfrak{g}_p$ be a pseudocomplex fundamental graded Lie algebra with $n = \frac{1}{2}\dim_{\mathbb{R}}\mathfrak{g}_{-1} = 1$ and let $\mathfrak{g} = \oplus_{p\geqslant-\mu}\mathfrak{g}_p$ be its canonical pseudocomplex prolongation. Suppose that $\mathfrak{m}$ is nondegenerate. This is equivalent to $\mu \geqslant 2$ and, by Theorem 3.1, to $\mathfrak{g}$ finite dimensional. Note that $\dim_{\mathbb{R}}\mathfrak{g}_0 \leqslant 2$. We will prove that $\mathfrak{g}$ is either solvable with $\mathfrak{g}_p = 0$ for every $p > 0$, or simple and isomorphic to $\mathfrak{su}(2, 1)$ with the graduation given in example in 5.1. Indeed, since ρ_{-2} is irreducible, if $\mathfrak{g}$ is not simple, then, by Corollary 3.32, $\mathfrak{g}$ is almost solvable, i.e. $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a semisimple subalgebra contained in $\mathfrak{g}_0$. If $\dim_{\mathbb{R}}\mathfrak{g}_0 = 1$, using that $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = [\mathfrak{r}_1, \mathfrak{r}_{-1}] \subset \mathfrak{g}_0$, we obtain that $\mathfrak{g}$ has to be solvable. By the Corollary 3.26, $\mathfrak{g}_1 = 0$. If $\dim_{\mathbb{R}}\mathfrak{g}_0 = 2$, then ρ_{-1} is irreducible, hence, by Corollary 3.31, $\mathfrak{g}_1 = 0$ and, since $\mathfrak{g}_0$ is Abelian, $\mathfrak{g}$ is solvable.

Let us assume now that $\mathfrak{g}$ is simple. Then the complexification $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ of $\mathfrak{g}$ is a semisimple complex Lie algebra and a Levi–Tanaka algebra. By Lemma 3.9, $\mathfrak{g}^{\mathbb{C}}$ has a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ contained in $\mathfrak{g}_0^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$ and then its rank $\ell = \dim_{\mathbb{C}}\mathfrak{h}^{\mathbb{C}}$ is less than or equal to $\dim_{\mathbb{C}}\mathfrak{g}_0^{\mathbb{C}} \leqslant 2$. By the classification of simple complex Lie algebras we have that $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to one of the following: $\mathfrak{so}(5, \mathbb{C})$, $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{sl}(3, \mathbb{C})$ or the exceptional Lie algebra G_2. Since $\dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}}\mathfrak{g} \geqslant 7$, and using Proposition 3.39, we obtain that $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(3, \mathbb{C})$. This implies $\mu = 2$ and $\dim_{\mathbb{R}}\mathfrak{g}_{-2} = 1$. These two conditions characterize a fundamental graded Lie algebra $\mathfrak{m}$ whose prolongation is isomorphic to $\mathfrak{su}(2, 1)$ (cf. example in 5.1). In conclusion, we proved that a Levi–Tanaka algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ with $n = \frac{1}{2}\dim_{\mathbb{R}}\mathfrak{g}_{-1} = 1$ is either solvable with $\mathfrak{g}_p = 0$ for every $p > 0$, or isomorphic to $\mathfrak{su}(2,\ 1)$.

References

1. Azad, H., Huckleberry, A., and Richthofer, W.: Homogeneous CR-manifolds. *J. Reine Angew. Math.* 358 (1985) 125–154.

2. Borel, A.: *Linear algebraic groups*. Springer, New York, 1991.
3. Bourbaki, N.: *Groupes et algèbres de Lie*. Éleménts de mathématique. Hermann, Paris, 1960.
4. Chern, S. S., and Moser, J. K.: Real hypersurfaces in complex manifolds. *Acta Math.* 133 (1975) 219–270.
5. Guillemin, V. W., and Sternberg, S.: An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.* 70 (1964) 16–47.
6. Hill, C. D., and Nacinovich, M.: The topology of Stein CR manifolds and the Lefschetz theorem. *Ann. Inst. Fourier (Grenoble) 43*, 2 (1993) 459–468.
7. Horn, R. A., and Johnson, C. R.: *Matrix analysis*. Cambridge University Press, Cambridge, 1985.
8. Medori, C., and Nacinovich, M.: Pluriharmonic functions on abstract CR manifolds. *Ann. Mat. Pura Appl. (IV)* 170 (1996) 377–394.
9. Mizner, R. I.: CR structures of codimension 2. *J. Diff. Geom.* 30 (1989) 167–190.
10. Nacinovich, M.: Poincaré lemma for tangential Cauchy–Riemann complexes. *Math. Ann.* 268 (1984) 449–471.
11. Taiani, G.: *Cauchy–Riemann (CR) manifolds*. Math. Dept., Pace Univ., New York, 1989.
12. Tanaka, N.: Graded Lie algebras and geometric structures I. *J. Math. Soc. Japan* 17 (1967) 215–264.
13. Tanaka, N.: On differential systems, graded Lie algebras and pseudogroups. *J. Math. Kyoto Univ.* 10 (1970) 1–82.
14. Tanaka, N.: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections. *Japan J. Math.* 2 (1976) 131–190.
15. Tanaka, N.: On the equivalence problems associated with simple graded Lie algebras. *Hokkaido Math. J.* 8 (1979) 23–84.
16. Varadarajan, V. S.: *Lie groups, Lie algebras, and their representations*. Springer-Verlag, New York, 1984.