ON THE EXISTENCE OF UNI-INSTANTANEOUS
\(Q\)-PROCESSES WITH A GIVEN finite
\(\mu\)-INVARIANT MEASURE

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Abstract

Let \(S\) be a countable set and let \(Q = (q_{ij}, i, j \in S)\) be a conservative \(q\)-matrix over \(S\) with a single instantaneous state \(b\). Suppose that we are given a real number \(\mu \geq 0\) and a strictly positive probability measure \(m = (m_j, j \in S)\) such that 
\[
\sum_{i \in S} m_i q_{ij} = -\mu m_j, \quad j \neq b.
\]
We prove that there exists a \(Q\)-process \(P(t) = (p_{ij}(t), i, j \in S)\) for which \(m\) is a \(\mu\)-invariant measure, that is
\[
\sum_{i \in S} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in S.
\]
We illustrate our results with reference to the Kolmogorov ‘K1’ chain and a birth–death process with catastrophes and instantaneous resurrection.

Keywords: Markov chain; \(q\)-matrix; birth–death process; construction theory

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1. Introduction

We begin with a conservative \(q\)-matrix over a countable set \(S\); that is, a collection \(Q = (q_{ij}, i, j \in S)\) of real numbers that satisfy \(0 \leq q_{ij} < \infty, i, j \in S, j \neq i; q_i := -q_{ii} \leq \infty, i \in S;\) and \(\sum_{j \neq i} q_{ij} = q_i, i \in S.\)

We shall assume that \(Q\) has a single instantaneous state; that is, a state \(b \in S\) such that \(q_b = \infty\) and \(q_i < \infty\) for \(i \neq b\). A set of real-valued functions \(P(t) = (p_{ij}(t), i, j \in S)\) defined on \((0, \infty)\) is called a standard transition function or process if
\[
\begin{align*}
p_{ij}(t) & \geq 0, & i, j \in S, t > 0, \\
\sum_{j \in S} p_{ij}(t) & \leq 1, & i \in S, t > 0, \\
p_{ij}(s + t) & = \sum_{k \in S} p_{ik}(s)p_{kj}(t), & i, j \in S, s, t > 0, \\
\lim_{t \downarrow 0} p_{ij}(t) & = \delta_{ij}, & i, j \in S,
\end{align*}
\]

where \(\delta_{ij}\) is the Kronecker delta. The process \(P\) is then honest if equality holds in (2) for some (and, thus, all) \(t > 0\), and it is called a \(Q\)-transition function (or \(Q\)-process) if \(p_{ij}'(0^+) = q_{ij}\) for each \(i, j \in S.\)
If \( \mu \) is some fixed nonnegative real number, a collection of strictly positive numbers \( m = (m_j, j \in S) \) is called a \( \mu \)-subinvariant measure (on \( S \)) for \( Q \) if \( \sum_{i \in S} m_i q_{ij} \leq -\mu m_j, \quad j \in S, \) and is called \( \mu \)-invariant if

\[
\sum_{i \in S} m_i q_{ij} = -\mu m_j, \quad j \in S. \tag{5}
\]

Here, we shall suppose that \( m \) is a finite measure (i.e. \( \sum_{i \in S} m_i < \infty \)) which is almost \( \mu \)-invariant for \( Q \), that is

\[
\sum_{i \in S} m_i q_{ij} = -\mu m_j, \quad j \neq b, \tag{6}
\]

and we will show that there always exists a \( Q \)-process \( P \) such that \( m \) is a \( \mu \)-invariant measure (on \( S \)) for \( P \), that is

\[
\sum_{i \in S} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in S, \quad t > 0. \tag{7}
\]

(When \( \mu = 0 \), all of the above notions reduce to the more common ones of invariance and subinvariance.) Note that if we were given a \( \mu \)-invariant measure \( m \) for a particular \( Q \)-process \( P \), then, since (7) may be rewritten as

\[
\sum_{i \neq j} m_i q_{ij} + (1 - e^{-\mu t}) m_j = (1 - p_{jj}(t)) m_j,
\]

Fatou’s lemma would give

\[
\sum_{i \neq j} m_i q_{ij} + \mu m_j \leq q_j m_j
\]

for all \( j \in S \), meaning that \( m \) would be \( \mu \)-subinvariant for \( Q \). However, under what conditions is \( m \) \( \mu \)-invariant for \( Q \)? In Section 2, we provide necessary and sufficient conditions for \( m \) to be almost invariant for \( Q \) and delay addressing the interesting question of whether or not \( \sum_{i \neq b} m_i q_{ib} = \infty \), which would be the remaining requirement for (5) to hold; this question will be considered in Section 6.

Here, we are assuming that \( Q \) is uni-instantaneous. When \( Q \) is totally stable, that is \( q_i < \infty \) for all \( i \in S \), the relationship between (5) and (7) is well understood, and has been divined completely for the minimal \( Q \)-process \( F \). It was shown by Tweedie [14] that if \( m \) is a \( \mu \)-invariant measure for \( F \), then it is also \( \mu \)-invariant for \( Q \). Conversely [8], [9], if \( m \) is \( \mu \)-invariant for \( Q \), then it is \( \mu \)-subinvariant for \( F \) and \( \mu \)-invariant for \( F \) if and only if the equations

\[
\sum_{i \in S} y_i q_{ij} = -vy_j, \quad 0 \leq y_j \leq m_j, \quad j \in S,
\]

have only the trivial solution for some (and, thus, all) \( v < \mu \). This result holds whether or not \( S \) is irreducible and does not require \( m \) to be finite. If, as we are assuming here, \( m \) is finite, then, for \( \mu \) to be strictly positive, it is necessary that \( F \) be dishonest. Furthermore, if \( F \) is the unique \( Q \)-process satisfying the forward equations, then \( m \) is \( \mu \)-invariant for \( F \).
Recently, Zhang, Lin and Hou solved the existence problem for the case \( \mu = 0 \) in the totally stable case \([17]\) and the uni-instantaneous case \([18]\). They proved that if \( m \) is a strictly positive, (almost-)invariant probability measure for \( Q \), then there exists a \( Q \)-process \( P \) for which \( m \) is an invariant measure (and, hence, a stationary distribution). We will extend their results to the case \( \mu > 0 \).

The structure of the paper is as follows. We begin, in Section 2, by examining the relationship between (6) and (7). Next, we recall the resolvent decomposition theorem of \([2]\), which is the major tool for constructing uni-instantaneous \( Q \)-processes. This, and some other preliminary results, are presented in Section 3. Our main result on the existence of a \( Q \)-process with a given finite, almost-\( \mu \)-invariant measure for \( Q \) is proved in Section 4. In Section 5, we discuss two examples illustrative of our results and, finally, in Section 6, we provide some necessary conditions for \( \mu \)-invariance. The terminology and notation used will follow that established by Anderson \([1]\) and Yang \([16]\).

2. Almost \( \mu \)-invariance

Our aim here is to provide necessary and sufficient conditions for a measure \( m \) that satisfies (7) (but is not necessarily finite) to be almost \( \mu \)-invariant for \( Q \). To do so, we recall the notions of an almost-\( B \)-type and an almost-\( F \)-type \( Q \)-process.

**Definition 1.** (Chen and Renshaw \([3]\).) A uni-instantaneous \( Q \)-process \( P \) with instantaneous state \( b \) is called almost \( B \)-type if it satisfies the Kolmogorov backward equations over the noninstantaneous states, that is if

\[
p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t), \quad i \neq b, \ j \in S. \tag{8}
\]

The process \( P \) is called almost \( F \)-type if it satisfies the Kolmogorov forward equations over the noninstantaneous states, that is if

\[
p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \quad i \neq b, \ j \in S.
\]

By adapting the proof of Theorem 1 of \([11]\), we can establish the following result.

**Theorem 1.** If \( m \) is a \( \mu \)-invariant measure for \( P \), then \( m \) is almost \( \mu \)-invariant for \( Q \) if and only if \( P \) is almost \( F \)-type.

**Proof.** Since (7) holds, we may define an honest standard transition function \( P^*(t) = (p^*_{ij}(t), i, j \in S) \) over \( S \) by

\[
p^*_{ij}(t) = e^{\mu t} \frac{m_{ij} p_{ji}(t)}{m_j}, \quad i, j \in S, \ t > 0.
\]

Indeed, \( P^* \) is a \( Q^* \)-transition function, where \( Q^* = (q^*_{ij}, i, j \in S) \) is the \( q \)-matrix with entries

\[
q^*_{ij} = \frac{m_{ij} q_{ji}}{m_i} + \mu \delta_{ij}, \quad i, j \in S.
\]
(\(P^*\) is called the \(\mu\)-reverse of \(P\) with respect to \(m\) and \(Q^*\) the \(\mu\)-reverse of \(Q\) with respect to \(m\); see [9]). It is easy to see that \(Q^*\) is uni-instantaneous with instantaneous state \(b\) and, for \(i \neq b\), that

\[
m_i \sum_{j \in S} q_{ij}^* = \sum_{j \neq i} m_j q_{ji} + \mu m_i - m_i q_i \leq 0.
\]

Moreover, all of the states \(i \neq b\) are conservative states for \(Q^*\) if and only if (6) holds. It is easy to verify that \(P^*\) is almost \(B\)-type if and only if \(P\) is almost \(F\)-type. Thus, if (6) holds then \(Q^*\) is conservative for the states \(i \neq b\). Hence, the backward equations (8) hold for \(P^*\) over the states \(i \neq b\), implying that \(P\) is almost \(F\)-type. Conversely, if \(P\) is almost \(F\)-type then \(P^*\) is almost \(B\)-type; however, \(P^*\) is honest, implying that the states \(i \neq b\) are conservative states for \(Q^*\) and, hence, (6) holds.

3. The resolvent decomposition theorem

Henceforth, we will find it convenient to specify transition functions through their Laplace transforms. If \(P\) is a specified transition function, then the function \(\Psi_1(\alpha) = (\psi_{ij}(\alpha), i, j \in S)\) given by

\[
\psi_{ij}(\alpha) = \int_0^\infty e^{-\alpha t} p_{ij}(t) \, dt, \quad i, j \in S, \quad \alpha > 0,
\]

is called the resolvent of \(P\). Indeed, if \(i, j \in C\), where \(C\) is any irreducible class, then the integral in (9) converges for all \(\alpha > -\lambda_P(C)\), where \(\lambda_P(C)\) is the decay parameter of \(C\) (for \(P\)); see [6]. In analogy to properties (1)–(4) of \(P\), the resolvent satisfies

\[
\psi_{ij}(\alpha) \geq 0, \quad i, j \in S, \quad \alpha > 0,
\]

\[
\sum_{j \in S} \alpha \psi_{ij}(\alpha) \leq 1, \quad i \in S, \quad \alpha > 0,
\]

\[
\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in S} \psi_{ik}(\alpha) \psi_{kj}(\beta) = 0, \quad i, j \in S, \quad \alpha, \beta > 0,
\]

\[
\lim_{\alpha \to \infty} \alpha \psi_{ij}(\alpha) = \delta_{ij}, \quad i, j \in S.
\]

(Note that (12) is called the resolvent equation.) Indeed, any \(\Psi\) that satisfies (10)–(13) is the resolvent of a standard transition function \(P\); see Lemma 1.1 of [12]. Furthermore, (11) is satisfied with equality if and only if \(P\) is honest, in which case the resolvent is said to be honest. Also, the \(q\)-matrix of \(P\) can be recovered from \(\Psi\) using the following identity:

\[
q_{ij} = \lim_{\alpha \to \infty} \alpha(\alpha \psi_{ij}(\alpha) - \delta_{ij}).
\]

Finally, a resolvent \(\Psi\) that satisfies (14) is called a \(Q\)-resolvent.

We can identify \(\mu\)-invariant measures using resolvents. If \(P\) is a \(Q\)-process with resolvent \(\Psi\) and \(m = (m_j, j \in S)\) is a \(\mu\)-invariant measure for \(P\), then \(\mu \leq \lambda_P(S)\), where \(\lambda_P(S) = \inf_C \lambda_P(C)\) (the infimum being taken over all the irreducible classes comprising \(S\)); see Lemma 4.1 of [15]. Furthermore, since the integral in (9) converges for all \(\alpha > -\lambda_P(S)\), we have

\[
\sum_{i \in S} m_i \alpha \psi_{ij}(\alpha - \mu) = m_j.
\]
for all \( j \in S \) and \( \alpha > 0 \). We refer to \( m \) as \( \mu \)-invariant for \( \Psi \) if (15) is satisfied. Finally, a simple extension of Lemma 1 of [10] establishes both that \( m \) is \( \mu \)-invariant for \( \Psi \) if it is \( \mu \)-invariant for \( P \), and that if \( m \leq \lambda P(S) \) then \( m \) is \( \mu \)-invariant for \( P \) if it is \( \mu \)-invariant for \( \Psi \).

We are assuming that \( Q \) is a uni-instantaneous \( q \)-matrix with instantaneous state \( b \), so let us write \( N = S \setminus \{b\} \) and denote by \( Q_N = (q_{ij}, i, j \in N) \) the restriction of \( Q \) to \( N \). If \( m = (m_i, i \in S) \) is a measure on \( S \), then \( m_N = (m_i, i \in N) \) will be the restriction of \( m \) to \( N \).

The following important result combines Theorems 7.7 and 7.8 of [2]. It characterizes \( Q \)-processes with a single instantaneous state. In preparation, define families \( H/\Psi_1 \) and \( K/\Psi_1 \), for a given \( Q_N \)-resolvent \( \Psi_1(\alpha) = (\psi_{ij}(\alpha), i, j \in N) \), as follows: \( H/\Psi_1 \) is the set of all nonnegative row vectors \( \eta(\alpha) = (\eta_i(\alpha), i \in N), \alpha > 0, \) satisfying \( \sum_{j \in N} \eta_j(\alpha) < \infty \) and

\[
\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in N} \eta_k(\alpha) \psi_{kj}(\beta) = 0, \quad j \in N,
\]

and \( K/\Psi_1 \) is the set of all column vectors \( \xi(\alpha) = (\xi_i(\alpha), i \in N), \alpha > 0, \) satisfying \( 0 \leq \xi_i(\alpha) \leq 1, i \in N, \) and

\[
\xi_i(\alpha) - \xi_i(\beta) + (\alpha - \beta) \sum_{k \in N} \psi_{ki}(\alpha) \xi_k(\beta) = 0, \quad i \in N.
\]

**Theorem 2.** (Resolvent decomposition theorem.) For the uni-instantaneous \( q \)-matrix \( Q \), every \( Q \)-resolvent \( R(\alpha) = (r_{ij}(\alpha), i, j \in S) \) can be decomposed uniquely as

\[
R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \psi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) \eta(\alpha) \end{pmatrix},
\]

where \( \psi(\alpha) = (\psi_{ij}(\alpha), i, j \in N) \) is a \( Q_N \)-resolvent and \( \eta(\alpha) = (\eta_i(\alpha), i \in N) \) and \( \xi(\alpha) = (\xi_i(\alpha), i \in N) \) satisfy the following conditions:

(i) \( \eta(\alpha) \in H/\Psi_1 \) and \( \xi(\alpha) \in K/\Psi_1 \),

(ii) \( \xi_i(\alpha) \leq 1 - \sum_{j \in N} \alpha \psi_{ij}(\alpha), i \in N, \)

(iii) \( \lim_{\alpha \to \infty} \alpha \eta_j(\alpha) = q_{bj}, j \in N, \)

(iv) \( \lim_{\alpha \to \infty} \alpha \xi_i(\alpha) = q_{ib}, i \in N, \) and

(v) \( r_{bb}(\alpha) = (C + \alpha + \alpha \sum_{j \in N} \eta_j(\alpha) \xi_j)^{-1}, \) where \( \xi_j := \lim_{\alpha \to 0} \xi_j(\alpha) \) and \( C < \infty \) satisfy

\[
C \geq \lim_{\alpha \to \infty} \alpha \sum_{j \in N} \eta_j(\alpha)(1 - \xi_j),
\]

\[
\lim_{\alpha \to \infty} \alpha \sum_{j \in N} \eta_j(\alpha) \xi_j = \infty \quad \text{(or, equivalently,} \lim_{\alpha \to \infty} \alpha \sum_{j \in N} \eta_j(\alpha) = \infty \text{)}.
\]

Conversely, if there exists a \( Q_N \)-resolvent \( \Psi \), and vectors \( \eta(\alpha) \) and \( \xi(\alpha) \) satisfying the above conditions, then \( R \), defined by (17), is a \( Q \)-resolvent.
Our main result rests on the following three lemmas.

**Lemma 1.** Suppose that the uni-instantaneous $q$-matrix $Q$ admits an almost-$\mu$-invariant measure $m = (m_i, i \in S)$. Then $d_i(\alpha) = (d_i(\alpha), i \in N)$, defined by

$$d_i(\alpha) = m_i - (\alpha + \mu) \sum_{k \in N} m_k \phi_{ki}(\alpha), \quad i \in N, \alpha > 0,$$

(19)

where $\Phi_N(\alpha) = (\phi_{ij}(\alpha), i, j \in N)$ is the minimal $Q_N$-resolvent, satisfies

$$\lim_{\alpha \to \infty} \alpha d_i(\alpha) = m_i q_{bi}, \quad i \in N.$$

**Proof.** Since $m$ is almost $\mu$-invariant for $Q$, it is clear that the restriction $m_N = (m_i, i \in N)$ is a $\mu$-subinvariant measure for $Q_N$. Therefore, because $m_N$ is then $\mu$-subinvariant for $\Phi_N$, we find that $d_i(\alpha) \geq 0, i \in N, \alpha > 0$. Also, since $\Phi_N$ is the minimal $Q_N$-resolvent, it satisfies the resolvent equation

$$\phi_{ij}(\alpha) - \phi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in N} \phi_{ik}(\alpha) \phi_{kj}(\beta) = 0, \quad i, j \in N, \alpha, \beta > 0,$$

and, therefore,

$$\bar{d}_i(\alpha) - \bar{d}_i(\beta) + (\alpha - \beta) \sum_{k \in N} \bar{d}_k(\alpha) \phi_{kj} (\beta) = 0, \quad i \in N, \alpha, \beta > 0,$$

(20)

where

$$\bar{d}_i(\alpha) = m_i - \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha), \quad i \in N, \alpha > 0.$$

Since $d_i(\alpha) \geq 0, i \in N, \alpha > 0$, we have $\bar{d}_i(\alpha) \geq 0, i \in N, \alpha > 0$. Using (20) and, hence, $\alpha \sum_{k \in N} m_k \phi_{ki}(\alpha)$ is nondecreasing in $\alpha$. Therefore, $\lim_{\alpha \to \infty} \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha)$ exists. However, by Fatou’s lemma, $\lim_{\alpha \to \infty} \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha) = m_i$, and, hence, $\lim_{\alpha \to \infty} \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha) = m_i$. Since $\Phi_N$ satisfies the forward equation

$$\alpha \phi_{ij}(\alpha) = \delta_{ij} + \sum_{k \in N} \phi_{ik}(\alpha) q_{kj}, \quad i, j \in N, \alpha > 0,$$

and (19) can be rewritten as

$$d_i(\alpha) = \sum_{k \in N} m_k (\delta_{ki} - (\alpha + \mu) \phi_{ki}(\alpha)), \quad i \in N, \alpha > 0,$$

we deduce that

$$\alpha d_i(\alpha) = -\alpha \sum_{k \in N} m_k \sum_{j \in N} \phi_{kj}(\alpha) q_{ji} - \alpha \mu \sum_{k \in N} m_k \phi_{ki}(\alpha),$$

$$= -\sum_{j \in N} q_{ji} \alpha \sum_{k \in N} m_k \phi_{kj}(\alpha) - \mu \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha),$$

which leads to

$$\lim_{\alpha \to \infty} \alpha d_i(\alpha) = -\sum_{j \in N} m_j q_{ji} - \mu m_i = m_b q_{bi}, \quad i \in N.$$

This completes the proof.
Lemma 2. Let $\Psi(\alpha) = (\psi_{ij}(\alpha), i, j \in N)$ be a $Q_N$-resolvent and let $\xi_i(\alpha) = \lim_{\alpha \to 0} \xi_i(\alpha)$, where $\xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \psi_{ij}(\alpha), i \in N$. If $\eta(\alpha) \in H/\Psi_1$ then $\alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i)$ is finite and does not depend on $\alpha$.

Proof. By the dominated convergence theorem,

$$
\lim_{\beta \to 0} \alpha \beta \sum_{i \in N} \sum_{j \in N} \eta_i(\alpha)\psi_{ij}(\beta) = \alpha \sum_{i \in N} \eta_i(\alpha) \lim_{\beta \to 0} \beta \sum_{j \in N} \psi_{ij}(\beta)
$$

$$
= \alpha \sum_{i \in N} \eta_i(\alpha) \lim_{\beta \to 0} (1 - \xi_i(\beta))
$$

$$
= \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i).
$$

On the other hand, using (16), we obtain

$$
\lim_{\beta \to 0} \alpha \beta \sum_{i \in N} \sum_{j \in N} \eta_i(\alpha)\psi_{ij}(\beta) = \lim_{\beta \to 0} \alpha \beta \sum_{i \in N} \sum_{j \in N} \eta_i(\alpha)\psi_{ij}(\beta)
$$

$$
= \lim_{\beta \to 0} \frac{\alpha \beta}{\beta - \alpha} \sum_{j \in N} (\eta_j(\alpha) - \eta_j(\beta))
$$

$$
= \lim_{\beta \to 0} \frac{\alpha \beta}{\beta - \alpha} \sum_{j \in N} \eta_j(\alpha) + \lim_{\beta \to 0} \frac{\alpha \beta}{\alpha - \beta} \sum_{j \in N} \eta_j(\beta).
$$

The first term vanishes because $\sum_{j \in N} \eta_j(\alpha) < \infty$. The second term equals

$$
\lim_{\beta \to 0} \frac{\beta}{\beta} \sum_{j \in N} \eta_j(\beta),
$$

which exists, because it is easy to deduce, from (16), that $\beta \sum_{j \in N} \eta_j(\beta)$ is nondecreasing in $\beta$. Since this limit does not depend on $\alpha$, the proof is complete.

Lemma 3. Suppose that $m = (m_i, i \in S)$ is a strictly positive probability measure. If $m$ is $\mu$-invariant for the $Q$-resolvent $R$ defined in (17), then

(i) $m_N = (m_i, i \in N)$ is a $\mu$-subinvariant measure for $\Psi$, and

(ii) $\eta_i(\alpha) = d_i(\alpha)/m_b$, where $d_i(\alpha) = m_i - (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha), i \in N, \alpha > 0$.

Conversely, if (i) and (ii) hold, then, on setting $\xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \psi_{ij}(\alpha), i \in N$, and $C = \mu/m_b + \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i)$, where $\xi_i = \lim_{\alpha \to 0} \xi_i(\alpha), (17)$ determines a $Q$-resolvent $R$ for which $m$ is a $\mu$-invariant measure.

Proof. If $m$ is $\mu$-invariant for $R$, that is

$$
(\alpha + \mu) \sum_{i \in S} m_ir_{ij}(\alpha) = m_j, \quad j \in S, \alpha > 0,
$$

(21)
then \((\alpha + \mu) \sum_{i \in N} m_i \psi_{ij}(\alpha) \leq m_j, j \in N\), since, from (17), we have \(\psi_{ij}(\alpha) \leq r_{ij}(\alpha), i, j \in N\). This proves part (i). Next, from (17) and (21), we have

\[
(\alpha + \mu) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{k \in N} m_k \xi_k(\alpha) r_{bb}(\alpha) = m_b
\]  

(22)

and, for all \(i \in N\) and \(\alpha > 0\),

\[
(\alpha + \mu) \eta_i(\alpha) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{j \in N} m_j \psi_{ij}(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \xi_k(\alpha) r_{bb}(\alpha) \eta_i(\alpha) = m_i.
\]

(23)

These equations combine to give

\[
m_b \eta_i(\alpha) + (\alpha + \mu) \sum_{j \in N} m_j \psi_{ij}(\alpha) = m_i, i \in N,
\]

and, hence, part (ii) holds.

To prove the converse, set \(\xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \psi_{ij}(\alpha)\) in (17) and take \(\eta(\alpha)\) to satisfy (16). Then, by Lemma 2, \(\alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i)\) is finite and independent of \(\alpha\), and, so, the given \(C\) satisfies (18). It follows that

\[
r_{bb}(\alpha) = \left(\frac{\mu}{m_b} + \alpha \sum_{i \in N} \eta_i(\alpha)\right)^{-1}.
\]

Since parts (i) and (ii) hold and \(\sum_{i \in S} m_i = 1\), we have

\[
(\alpha + \mu) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{i \in N} m_i \xi_i(\alpha) r_{bb}(\alpha) = r_{bb}(\alpha) \left((\alpha + \mu) m_b + (\alpha + \mu)(1 - m_b) - \alpha \sum_{j \in N} (\alpha + \mu) \sum_{i \in N} m_i \psi_{ij}(\alpha)\right) = r_{bb}(\alpha) \left(\mu + \alpha m_b + \alpha m_b \sum_{j \in N} \eta_j(\alpha)\right) = m_b
\]

and, for \(i \in N\),

\[
(\alpha + \mu) \eta_i(\alpha) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \xi_k(\alpha) r_{bb}(\alpha) \eta_i(\alpha) = (\alpha + \mu) r_{bb}(\alpha) d_i(\alpha) + (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + (\alpha + \mu) r_{bb}(\alpha) \frac{d_i(\alpha)}{m_b} \sum_{k \in N} m_k \xi_k(\alpha) = (\alpha + \mu) \sum_{k \in N} m_k \psi_{ki}(\alpha) + \frac{d_i(\alpha)}{m_b} \left((\alpha + \mu) r_{bb}(\alpha) m_b + (\alpha + \mu) \sum_{i \in N} m_i \xi_i(\alpha) r_{bb}(\alpha)\right) = m_i.
\]

Thus, (22) and (23) hold. These in turn imply that (21) holds, meaning that \(m\) is a \(\mu\)-invariant measure for \(R\).
4. Existence

We are now ready to state our main result.

**Theorem 3.** Let $\mu \geq 0$ and suppose that the uni-instantaneous $q$-matrix $Q$ admits a finite, almost-$\mu$-invariant measure $m = (m_i, i \in S)$. Then there exists a $Q$-process for which $m$ is a $\mu$-invariant measure.

**Proof.** Without loss of generality, we may assume that $\sum_{i \in S} m_i = 1$. Let $\Phi(\alpha) = (\phi_{ij}(\alpha), i, j \in N)$ be the minimal $Q_N$-resolvent. Since $m$ is almost $\mu$-invariant for $Q$, the restriction $m_N = (m_i, i \in N)$ is a $\mu$-subinvariant measure for $Q_N$ and, hence, is $\mu$-subinvariant for $\Phi$. Set

$$
\begin{align*}
d_i(\alpha) &= m_i - (\alpha + \mu) \sum_{k \in N} m_k \phi_{ki}(\alpha), \quad i \in N, \alpha > 0, \\
\eta_i(\alpha) &= \frac{d_i(\alpha)}{m_b}, \quad i \in N, \alpha > 0, \\
\xi_i(\alpha) &= 1 - \alpha \sum_{j \in N} \phi_{ij}(\alpha), \quad i \in N, \alpha > 0,
\end{align*}
$$

and

$$
r_{bb}(\alpha) = \left( \frac{\mu}{m_b} + \alpha \sum_{i \in N} \eta_i(\alpha) \right)^{-1}.
$$

Since $\Phi$ satisfies the resolvent equation, $\eta(\alpha)$ and $\xi(\alpha)$ given in (25) and (26) satisfy

$$
\eta_i(\alpha) - \eta_i(\beta) + (\alpha - \beta) \sum_{k \in N} \eta_k(\alpha) \phi_{ki}(\beta) = 0, \quad i \in N,
$$

and

$$
\xi_i(\alpha) - \xi_i(\beta) + (\alpha - \beta) \sum_{k \in N} \phi_{ik}(\alpha) \xi_k(\beta) = 0, \quad i \in N.
$$

Using Lemma 1, we see that

$$
\lim_{\alpha \to \infty} \alpha \eta_j(\alpha) = \lim_{\alpha \to \infty} \alpha \frac{d_j(\alpha)}{m_b} = q_{bj}, \quad j \in N,
$$

and

$$
\lim_{\alpha \to \infty} \alpha \sum_{j \in N} \eta_j(\alpha) = \lim_{\alpha \to \infty} \frac{1}{m_b} \sum_{j \in N} \alpha d_j(\alpha) = \sum_{j \in N} q_{bj} = \infty.
$$

Also,

$$
\lim_{\alpha \to \infty} \alpha \xi_i(\alpha) = \lim_{\alpha \to \infty} \sum_{k \in N} \alpha (\delta_{ik} - \alpha \phi_{ik}(\alpha)) = - \sum_{k \in N} q_{ik} = q_{ib}, \quad i \in N.
$$
Therefore, using (26), (28), and Lemma 2, we deduce that \( \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i) \) is finite and independent of \( \alpha \). Now set

\[
C = \frac{\mu}{m_b} + \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i),
\]

where \( \xi = \lim_{\alpha \to 0} \xi(\alpha) \), and observe that \( C \) satisfies (18). Hence, in view of Theorem 2, we may use (24)–(27) to construct a \( Q \)-resolvent \( R \) by setting

\[
R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha) \end{pmatrix},
\]

and then use the second part of Lemma 3 to deduce that \( m \) is a \( \mu \)-invariant measure for \( R \). This completes the proof.

**Remark 1.** When \( \mu = 0 \), Theorem 3 reduces to the result of [18].

### 5. Examples

**Example 1.** We will begin with an example, generally known as the ‘K1’ chain, described by Kolmogorov [7] and analysed by Kendall and Reuter [5] and Reuter [13] (see also the discussions in [4] and [1]). The chain has a \( q \)-matrix over the nonnegative integers given by

\[
Q = \begin{pmatrix} -\infty & 1 & 1 & 1 & \cdots \\ q_1 & -q_1 & 0 & 0 & \cdots \\ q_2 & 0 & -q_2 & 0 & \cdots \\ q_3 & 0 & 0 & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

(29)

where \( q_i > 0, \ i \geq 1 \). If a \( \mu \)-subinvariant measure exists for \( Q \) then \( \mu \leq \inf_i q_i \); see Corollary 1 of [6]. We will assume that \( \mu < q_i \) for all \( i \geq 1 \). Then, for any such \( \mu \), \( Q \) admits a \( \mu \)-invariant measure \( m = (m_i, \ i \geq 0) \) given by \( m_i = m_0/(q_i - \mu), \ i \geq 1 \), with \( m_0 \) arbitrary. This is finite if and only if

\[
\sum_{i=1}^{\infty} \frac{1}{q_i} < \infty,
\]

(30)

in which case \( Q \) has the unique \( \mu \)-invariant probability measure

\[
m_0 = \frac{1}{A}, \quad m_i = \frac{m_0}{q_i - \mu}, \ i \geq 1,
\]

(31)

where \( A = 1 + \sum_{i=1}^{\infty} 1/(q_i - \mu) \). Therefore, an immediate consequence of Theorems 2 and 3 and Lemma 3 is the following simple result.

**Proposition 1.** If \( Q \) defined in (29) satisfies (30), then there exists a \( Q \)-process for which \( m \), defined by (31), is a \( \mu \)-invariant probability measure. The resolvent of one such process is given by

\[
R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha) \end{pmatrix},
\]
where
\[ \phi_{ij}(\alpha) = \frac{\delta_{ij}}{\alpha + q_i}, \quad i, j \geq 1, \alpha > 0, \]
\[ \xi_i(\alpha) = \frac{q_i}{\alpha + q_i}, \quad i \geq 1, \alpha > 0, \]
\[ \eta_j(\alpha) = \frac{1}{\alpha + q_j}, \quad j \geq 1, \alpha > 0, \]
and
\[ r_{bb}(\alpha) = \left( \frac{\mu}{m_0} + \alpha + \alpha \sum_{i=1}^{\infty} \eta_i(\alpha) \right)^{-1}. \]

**Example 2.** Next we consider the following \(q\)-matrix, describing a birth–death process incorporating catastrophes to state 0 and instantaneous resurrection from state 0:
\[
Q = \begin{pmatrix}
-\infty & h_1 & h_2 & h_3 & \cdots \\
 d_1 & -(d_1 + b_1) & b_1 & 0 & \cdots \\
 d_2 & a_2 & -(a_2 + b_2 + d_2) & b_2 & \cdots \\
 d_3 & 0 & a_3 & -(a_3 + b_3 + d_3) & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \quad (32)
\]

Here, \(d_i > 0, b_i > 0, i \geq 0, a_i > 0, i \geq 1, h_j \geq 0, j \geq 1, \) and \(\sum_{j=1}^{\infty} h_j = \infty.\) Define \(\pi = (\pi_i, i \geq 1)\) by \(\pi_1 = 1\) and
\[ \pi_i = \prod_{j=2}^{i} \frac{b_{j-1}}{a_j}, \quad i \geq 2. \]

It is easy to show that if \(\mu\) satisfies \(0 \leq \mu \leq \inf_{i \geq 1} d_i\) and if \(h_i = c\pi_i(d_i - \mu), \quad i \geq 1,\) where \(c\) is a positive constant, then \(m = (m_i, i \geq 0)\) given by
\[ m_0 = 1, \quad m_i = c\pi_i, \quad i \geq 1, \quad (33) \]
is a \(\mu\)-invariant measure for \(Q.\)

**Proposition 2.** If \(\mu\) satisfies \(0 \leq \mu \leq \inf_{i \geq 1} d_i\) and \(Q\) defined in (32) satisfies \(\sum_{i=1}^{\infty} \pi_i < \infty\) and \(\sum_{i=1}^{\infty} \pi_i d_i = \infty,\) then there exists a \(Q\)-process for which \(m,\) defined by (33), is a \(\mu\)-invariant probability measure.

**Proof.** The condition \(\sum_{i=1}^{\infty} \pi_i < \infty\) implies that \(m\) is a finite measure, and the facts that \(\sum_{i=1}^{\infty} \pi_i d_i = \infty\) and \(\sum_{i=1}^{\infty} \pi_i < \infty\) together imply that \(\sum_{j=1}^{\infty} h_j = \infty.\) Hence, the result follows from Theorem 3.

### 6. Necessary conditions

In both of the examples above, our finite measure \(m\) satisfied
\[ \sum_{i \neq b} m_i q_{ib} = \infty \quad (34) \]

and, hence, was invariant for \(Q\) (that is, (5) holds for all \(j \in S\)). We have established that only almost \(\mu\)-invariance is needed for the existence of a \(Q\)-process for which the given (finite)
measure is \( \mu \)-invariant. It would therefore be of interest to know whether (34) is actually necessary for a (finite or infinite) measure \( m \) to be \( \mu \)-invariant for \( P \). We shall content ourselves with the following result, which shows that (34) is necessary in the \( \mu = 0 \) case under the condition that \( P \) is reversible.

**Theorem 4.** Let \( Q \) be a uni-instantaneous \( q \)-matrix with instantaneous state \( b \) and let \( P \) be a \( Q \)-process with invariant measure \( m \). If \( P \) is reversible with respect to \( m \), that is if

\[
m_i p_{ij}(t) = m_j p_{ji}(t), \quad i, j \in S,
\]

then (34) holds.

**Proof.** On dividing (35) by \( t \) and letting \( t \downarrow 0 \), we obtain \( m_i q_{ij} = m_j q_{ji}, \ j \neq i \). Hence,

\[
\sum_{i \neq j} m_i q_{ij} = m_j \sum_{i \neq j} q_{ji}, \quad j \in S,
\]

meaning that, in particular,

\[
\sum_{i \neq b} m_i q_{ib} = m_b \sum_{i \neq b} q_{bi} = \infty,
\]

since \( Q \) is conservative.

We gain some insight into the general case from the following simple result, which follows directly from the proof of Theorem 1.

**Theorem 5.** Let \( Q \) be a uni-instantaneous \( q \)-matrix with instantaneous state \( b \) and let \( P \) be a \( Q \)-process with \( \mu \)-invariant measure \( m \). Let \( P^* \) and \( Q^* \) be, respectively, the \( \mu \)-reverse of \( P \) with respect to \( m \) and the \( \mu \)-reverse of \( Q \) with respect to \( m \). Then \( P^* \) is honest. In particular, \( b \) is an honest state for \( P^* \), while being instantaneous for \( Q^* \). Moreover,

\[
m_b \sum_{j \neq b} q_{bj} = m_j q_{jb},
\]

meaning that, in particular, \( b \) is a conservative state for \( Q^* \) if and only if (34) holds.

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**References**


Uni-instantaneous $Q$-processes  


