SOME FURTHER PROPERTIES OF A q-ANALOGUE OF MACROBERT'S E-FUNCTION

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(Received 7 November, 1961)

1. Introduction. Recently, I gave an analogue [1] of the MacRobert's E-function [4] in the form

$$E_q(\alpha, \beta :: z) = \sum_{\alpha,\beta} \frac{G(\alpha)G(\beta-\alpha)}{G(1)} \prod_{n=0}^{\infty} \frac{(1+z^{-1}q^{\alpha+n})(1+zq^{1-\alpha+n})}{(1+z^{-1}q^{n})(1+zq^{1+n})}\, {}_1\Phi_1(\alpha;\ \alpha-\beta+1;\ zq^{2-\beta}),$$

where the symbol $\sum_{\alpha,\beta}$ denotes that a similar expression with α and β interchanged is to be added to the expression following it. It has since then been generalized by N. Agarwal [2], who defined and studied the q-analogue of the generalized E-function. In this paper I give some further properties of the E_q-function.

2. Notation and definitions. The following notation is used throughout the paper. Let $|q|<1$,

$$(q^a)_n \equiv (a)_n = (1-q^a)(1-q^{a+1}) \dots (1-q^{a+n-1}),$$
$$(q^a)_0 = 1, \quad (q^a)_{-n} = (-)^n q^{\frac{1}{2}n(n+1)} q^{-na}/(q^{1-a})_n;$$

then we define the generalized basic hypergeometric function as

$${}_{r+1}\Phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1};\ z \\ b_1, b_2, \dots, b_r \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_{r+1})_n}{(q)_n (b_1)_n \dots (b_r)_n} z^n,$$

and the " confluent " basic hypergeometric function as

$${}_1\Phi_1(a;\ b;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n (b)_n} q^{\frac{1}{2}n(n-1)} z^n.$$

Also

$$E_q(x) = \prod_{n=0}^{\infty} (1-xq^n) = \sum_{n=0}^{\infty} \frac{(-)^n q^{\frac{1}{2}n(n-1)}}{(q)_n} x^n,$$

$$e_q(x) = 1\Big/\prod_{n=0}^{\infty}(1-xq^n),$$

$$(x+y)_\alpha = x^\alpha\left(1+\frac{y}{x}\right)_\alpha = x^\alpha \prod_{n=0}^{\infty}\left(\frac{1+yx^{-1}q^n}{1+yx^{-1}q^{\alpha+n}}\right),$$

and

$$G(\alpha) = \left\{\prod_{n=0}^{\infty}(1-q^{\alpha+n})\right\}^{-1}.$$

The basic-differential operator q^θ is defined by $q^\theta z(x) = q^{(x\,d/dx)} z(x) = z(qx)$, where $q^{(x\,d/dx)}$ means $\exp\left(x\frac{d}{dx}\log q\right)$.

Further, following Hahn [3], the basic integral of a function, under suitable conditions, is defined as

$$\int_0^x f(y)\, d(qy) = x(1-q) \sum_{i=0}^{\infty} q^i f(q^i x),$$

$$\int_x^{\infty} f(y)\, d(qy) = x(1-q) \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x),$$

and thus

$$\int_0^{\infty} f(y)\, d(qy) = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j).$$

All products occurring are infinite products so that, for example, $\prod_{n=0}^{\infty} (1-xq^n)$ is written as $\prod(1-xq^n)$.

3. A difference-equation satisfied by $E_q(\alpha, \beta :: z)$. Let

$$S = \prod \frac{(1+z^{-1}q^{\alpha+n})(1+zq^{1-\alpha+n})}{(1+z^{-1}q^n)(1+zq^{1+n})}.$$

Then it is easy to see that

$$(q^\theta - q^\alpha)S = 0. \tag{1}$$

Also,

$$\omega = {}_1\Phi_1\,(\alpha;\ \alpha-\beta+1;\ zq^{2-\beta})$$

satisfies the q-difference equation

$$zq^{2-\beta}(1-q^{\theta+\alpha})q^\theta \omega = (1-q^\theta)(1-q^{\theta+\alpha-\beta})\omega. \tag{2}$$

Putting $v = \omega S$, we have, after slight calculations,

$$zq^{2-\alpha-\beta}(1-q^\theta)q^\theta v = (1-q^{\theta-\alpha})(1-q^{\theta-\beta})v. \tag{3}$$

Hence

$$v = \prod \frac{(1+z^{-1}q^{\alpha+n})(1+zq^{1-\alpha+n})}{(1+z^{-1}q^n)(1+zq^{1+n})}\ {}_1\Phi_1\ (\alpha;\ \alpha-\beta+1, zq^{2-\beta})$$

satisfies the difference equation (3). Since (3) is symmetrical in α and β,

$$\prod \frac{(1+z^{-1}q^{\beta+n})(1+zq^{1-\beta+n})}{(1+z^{-1}q^n)(1+zq^{1+n})}\ {}_1\Phi_1\,(\beta;\ \beta-\alpha+1;\ zq^{2-\alpha})$$

also satisfies (3). Hence (3) gives the q-difference equation satisfied by $E_q(\alpha, \beta :: z)$.

4. A contour integral representation for $E_q(\alpha, \beta :: z)$. Consider the contour integral

$$\frac{t}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \prod \left[\frac{(1+q^{\alpha-z+s+n})(1+q^{1+z-\alpha-s+n})}{(1-q^{\alpha+s+n})(1-q^{\beta-\alpha-s+n})(1-q^{-s+n})} \right] ds, \tag{4}$$

where $q = e^{-t}$, $t > 0$.

https://doi.org/10.1017/S2040618500034675 Published online by Cambridge University Press

Evaluating the integral by the calculus of residues,† we find that (4) is equal to

$$\prod\left[\frac{(1+q^{n-z})(1+q^{1+n+z})}{(1-q^{n+1})^2}\right]E_q(q^\alpha, q^\beta :: q^z).$$

Writing z, α, β for q^z, q^α, q^β, respectively, we get the required result.

5. Two definite integral representations for $E_q(\alpha, \beta :: z)$. Consider the known integral [3, p. 290]

$$\underset{0}{\overset{\infty}{\mathrm{S}}} e_q(-sx)x^{\beta-1}{}_1\Phi_1(q^\alpha;\ q^\gamma;\ -tx)\,d(qx)$$
$$=(1-q)_{\beta-1}\prod\frac{(1+q^{\beta+n}s)(1+s^{-1}q^{1-\beta+n})}{(1+sq^n)(1+s^{-1}q^{1+n})}\,{}_2\Phi_1(q^\alpha, q^\beta;\ q^\gamma;\ -tq^\beta/s). \tag{5}$$

Letting $\gamma\to\infty$, $s\to 1$ and $t=1/(zq^\beta)$ in (5), we have

$$\underset{0}{\overset{\infty}{\mathrm{S}}} e_q(-x)x^{\beta-1}{}_1\Phi_1(q^\alpha;\ 0;\ -xq^{-\beta}/z)\,d(qx)$$
$$=(1-q)_{\beta-1}\prod\frac{(1+q^{\beta+n})(1+q^{1-\beta+n})}{(1+q^n)(1+q^{1+n})}\,{}_2\Phi_0(q^\alpha, q^\beta;\ -1/z).$$

Now, since we know that,‡ when $|z|>1$,

$${}_2\Phi_0(q^\alpha, q^\beta;\ -1/z)=\sum_{\alpha,\beta}\frac{G(\beta-\alpha)}{G(\beta)}\prod\frac{(1+z^{-1}q^{\alpha+n})(1+zq^{1-\alpha+n})}{(1+zq^{1+n})(1+z^{-1}q^n)}\,{}_1\Phi_1(q^\alpha;\ q^{1+\alpha-\beta};\ zq^{2-\beta}),$$

we have, for $R(\beta)>0$ and $|z|>1$,

$$\prod\frac{(1+q^{\beta+n})(1+q^{1-\beta+n})(1-q^{\alpha+n})}{(1+q^n)(1+q^{1+n})}E_q(q^\alpha, q^\beta :: z)=\underset{0}{\overset{\infty}{\mathrm{S}}} e_q(-x)x^{\beta-1}{}_1\Phi_1(q^\alpha;\ 0;\ -xq^{-\beta}/z)\,d(qx). \tag{6}$$

This integral representation is interesting in the sense that it gives an alternative definition of the E_q-function corresponding to the alternative analogue $e_q(x)$ of the exponential function.

Next we deduce another definite integral involving in the integrand a basic analogue of the ${}_1F_1$-function. In particular, let us consider the integral

$$\frac{1}{1-q}\underset{0}{\overset{1/b}{\mathrm{S}}} E_q(qb\lambda)E_q(\lambda aq^{\alpha-m-1})\lambda^{\alpha+m-1}\{[1+\lambda]_{\alpha-m}\}^{-1}{}_2\Phi_1(-q^{1+m-\alpha}/\lambda, 0; q^{2m+1};\ ab\lambda^2q^{\alpha-m-1})\,d(q\lambda). \tag{7}$$

Expanding $E_q(\lambda aq^{\alpha-m-1})$ and ${}_2\Phi_1$, we have

$$\frac{1}{1-q}\underset{0}{\overset{1/b}{\mathrm{S}}} E_q(qb\lambda)\lambda^{\alpha+m-1}\{[1+\lambda]_{\alpha-m}\}^{-1}$$
$$\times\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-)^s\lambda^s a^s q^{s(\alpha-m-1)+\frac{1}{2}s(s-1)}(-q^{1-\alpha+m}/\lambda)_t}{(q)_s(q)_t(q^{2m+1})_t}a^tb^t\lambda^{2t}q^{t(\alpha-m-1)}d(q\lambda).$$

† For details see L. J. Slater, *Proc. Cambridge Phil. Soc.* **48** (1952), 578–82.
‡ Slater, *ibid.* equation (13).

Putting $t=r-s$, and changing the order of summation and integration, which is justified by absolute convergence of the series involved, for $Rb>1$, $|abq^{\alpha-m-1}|<1$, we have, after some simplification, that

$$\frac{1}{1-q}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-)^s a^r b^{r-s} q^{s(\alpha-m+s-r-2)+\frac{1}{2}r(r-1)}}{(q)_s(q)_{r-s}(q^{2m+1})_{r-s}}\int_0^{1/b} E_q(qb\lambda)\lambda^{\alpha+m+r-1}{}_1\Phi_0(\alpha-m+s-r;\ -\lambda)\ d(q\lambda).$$

Changing the variable through the transformation $\lambda b=v$ and evaluating the integral by [**1**, (3)], we get on simplification that the above is equal to

$$\frac{b^{-\alpha-m}}{G(\alpha-m)}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-)^{r+s}b^{-s}a^r(q^{m-\alpha+1})_r(q^{-r})_s(q^{-2m-r})_s}{(q)_s(q^{\alpha-m-r})_s(q)_r(q^{2m+1})_r}$$
$$\times q^{(\alpha-m-1)r+s(\alpha+m+r)}E_q(\alpha-m+s-r,\alpha+m+r::b).$$

Summing the s-series by [**1**, (10)], we have finally that (7) is equal to

$$\frac{b^{-\alpha-m}}{G(\alpha-m)}E_q(\alpha-m,\alpha+m::b){}_1\Phi_1(\alpha+m;\ 2m+1;\ a),$$

which gives the required result.

6. An E_q-function with negative argument. Consider the function

$$E_q(1-\alpha,1-\beta::z/q)=\sum_{\alpha,\beta}\frac{G(1-\alpha)G(\alpha-\beta)}{G(1)}\prod\frac{(1+z^{-1}q^{2-\alpha+n})(1+zq^{\alpha+n-1})}{(1+z^{-1}q^{1+n})(1+zq^n)}$$
$$\times{}_1\Phi_1(1-\alpha;\ 1+\beta-\alpha;\ zq^\beta).$$

Using the basic analogue of Kummer's formula, namely

$${}_1\Phi_1(q^a;\ q^b;\ x)=\prod(1+xq^{a-b+n}){}_2\Phi_1(q^{b-a},0;\ q^b;\ -xq^{a-b}),$$

we have, on simplification and transposition, that

$$E_q(z^{-1}q)E_q(1-\alpha,1-\beta::-z/q)$$
$$=\sum_{\alpha,\beta}\frac{G(1-\beta)G(\beta-\alpha)}{G(1)}\prod(1-z^{-1}q^{2-\beta+n})(1-zq^{\beta+n-1}){}_2\Phi_1(\alpha,0;\ 1+\alpha-\beta;\ z). \qquad (8)$$

The formula (8) suggests the consideration of another q-analogue of the E-function in the form

$$A{}_2\Phi_1(\alpha,0;\ 1+\alpha-\beta;\ z)+B{}_2\Phi_1(\beta,0;\ 1+\beta-\alpha;\ z),$$

where A and B are suitable functions of α, β and z. This is natural to expect also, since, corresponding to the ${}_1F_1$ function there can be two q-analogues, one with a quadratic power $q^{\frac{1}{2}n(n-1)}$ in the argument and the other without it. Such a definition forms the subject matter of a subsequent communication.

It may also be interesting to note that the function $f(\alpha,\beta)\equiv E_q(z^{-1}q)E_q(1-\alpha,1-\beta::-z/q)$ has properties very similar to the $E_q(\alpha,\beta::z)$ function. For instance, it is easy to see that corresponding to (7) of [**1**] we have the recurrence relation

$$(1-q^{1-\alpha})f(\alpha,\beta)-f(\alpha-1,\beta)+(q^{2-\alpha}/z)f(\alpha-1,\beta-1)=0.$$

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LUCKNOW UNIVERSITY, LUCKNOW,

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https://doi.org/10.1017/S2040618500034675 Published online by Cambridge University Press