THE ROLE OF ROTATION IN STELLAR EVOLUTION

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1. Introduction

During the seventeenth century, in the wake of the discovery of the solar differential rotation, some scientists argued that stellar variability was a direct consequence of axial rotation, the spinning body showing alternately its bright (unspotted) and dark (spotted) hemispheres to the observer (Brunet 1931). Although this idea did not withstand the passage of time, it is nevertheless an interesting one because it is clearly indicative of the kind of fascination stellar rotation has aroused since its inception. And yet, at this writing there is no longer any doubt that spherically symmetric models do explain the major observed properties of stars. Moreover, if one excepts the very early and very late moments of a star's lifetime, the effects of rotation on stellar structure are apparently dynamically unimportant (e.g., Tassoul 1978, hereafter T.R.S.; Moss and Smith 1981, and references therein). What is the purpose, then, to discuss the role of rotation on the main-sequence and post-main-sequence phases of stellar evolution?

As we shall see below, perhaps the most interesting effect of rotation is to generate small-scale, eddy-like motions as well as large-scale meridional currents wherever radiative equilibrium prevails. The importance of these motions lies in the fact that, under certain conditions, they may lead to some degree of mixing of the stellar material, or that they may prevent the gravitational sorting of the elements in the surface layers of most (but not all!) early-type stars. However, since I disagree with the current habit of placing the cart in front of the horse, I shall not discuss these practical implications without making first a thorough discussion of the state of motion in a stellar radiative zone. This is definitely not an exhaustive review of the recent literature. Rather, I shall try to summarize in nontechnical terms the theoretical work I'm pursuing in collaboration with my wife (Tassoul and Tassoul 1982a, b, c, 1983a, b, c, 1984a, b; hereafter Papers I, II, ..., VIII, respectively). Our personal approach, which we borrowed from geophysics, resolves in a very simple manner the many contradictions and inconsistencies that have beset the theory of rotating stars.

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2. Stability Considerations

In principle, by making use of the basic equations of astrophysical fluid dynamics, one should be able to obtain at every instant the angular momentum distribution within a star. For evident reasons, this is an impossible task, even were the initial conditions known. The standard procedure is to calculate in an approximate manner an equilibrium structure that corresponds to some prescribed rotation law, ruling out (in principle) those configurations that are dynamically or thermally unstable with respect to axisymmetric disturbances. Although such an *ad hoc* approach may be used to estimate the gross effects of rotation on stellar evolution, it is totally inadequate for the following two reasons: (i) the instantaneous angular momentum distribution does not follow from the equations of motion, and (ii) no dynamically stable model can possibly exist when non-axisymmetric disturbances are taken into account. Because the role of these *ever-present* dynamical instabilities has been overlooked in astrophysics, a brief review of the relevant concepts is thus in order.

If a rotating star was a barotrope, then its angular velocity Ω would be constant on cylinders centered on the rotation axis, i.e., $\Omega = \Omega\ (\varpi, t)$, where ϖ is the distance from the rotation axis. In this case, the dynamical stability of the configuration depends on the specific angular momentum $j = \Omega\varpi^2$ and the Richardson number $Ri = N^2/S^2$, where N is the buoyancy frequency and S is the shear in the linear velocity $\Omega\varpi$. (Since we consider radiative zones only, we have $N^2>0$ everywhere.) Instability with respect to axisymmetric disturbances occurs wherever the j-distribution decreases outward, that is, $dj^2/d\varpi<0$. In geophysics, it is called the condition for *symmetric instability*, and it merely generalizes the Rayleigh criterion for an incompressible fluid (e.g., T.R.S., p. 167). As was shown by Lorimer and Monaghan (1980), the symmetric instability is a *violent* one in the sense that, given an initially unstable j-distribution, a slowly rotating barotrope will at once generate meridional currents and non-axisymmetric motions in the nonlinear regime, the resulting flow becoming chaotic with a very slow trend to equilibrium. Equally well known is the fact that a rotating barotrope may also become dynamically unstable with respect to non-axisymmetric disturbances. It is the so-called *shear-flow instability* that occurs wherever $Ri \lesssim 1/4$. This instability is a *mild* one in the sense that it merely generates turbulence when the shear is large enough to overcome the stabilizing influence of the density stratification. No dynamical instability occurs in a rotating barotrope when $Ri \gtrsim 1/4$ and $dj^2/d\varpi>0$ (e.g., Turner 1973; see also Chimonas 1979).

As we shall see below, a real rotating star is not a barotrope but a barocline; that is, the isothermal surfaces are in general inclined over the isobaric surfaces so that, in cylindrical coordinates (ϖ, ϕ, z), one has $\Omega = \Omega\ (\varpi, z, t)$ or $\partial\Omega/\partial z \neq 0$. In this case, then, the violent *symmetric instability* occurs wherever the j-distribution decreases outward on the surfaces of constant specific entropy (e.g., T.R.S., p. 168), and the mild *shear-flow instability* sets in wherever

Ri $\lesssim$ O(1). In sharp contrast to the ideal case of a barotrope, however, a barocline is also dynamically unstable with respect to non-axisymmetric disturbances wherever Ri $\gtrsim$ O(1). It is the so-called *baroclinic instability*, depending essentially upon the difference between the isothermal and isobaric surfaces. This instability, which draws its kinetic energy mainly from the potential energy of the unperturbed rotational motion, has received the most attention in geophysics because of its importance to the dynamics of the Earth's atmosphere and oceans (e.g., Charney 1973; see also Paper VI, n. 3, and references therein). It is a *mild* instability in the sense that it continuously generates small-scale, time-dependent motions that propagate in the azimuthal direction. As it is done in geophysics, we shall merely assume that the ever-present baroclinic instability produces anisotropic turbulence in the radiative zone of a rotating star. Lacking any better description of these irregular motions, we shall further assume that the eddy flux of momentum can be represented parametrically by means of suitable coefficients of eddy viscosity. Evidently, in this representation the ever-present *thermal instabilities* (such as the *feeble* GSF instability) play no important role because they are overshadowed by the ever-present shear-flow and baroclinic instabilities, which have time scales of the order of the rotation period (Paper I, pp. 337-338).

3. Rotation and Circulation

All theoretical speculations about the angular momentum distribution within a rotating star have their roots in von Zeipel's paradox, which states that the conditions of hydrostatic and radiative equilibrium are in general incompatible in a rotating barotrope. This paradox can be solved in two different ways: *either* one allows for a slight departure from barotropy and choose the angular velocity $\Omega = \Omega\ (\omega, z, t)$ so that strict radiative equilibrium prevails at every point, *or* one allows for large-scale motions in meridian planes. The first alternative (originally developed by M. Schwarzschild) is mainly of academic interest because, as we pointed out in Paper I (p. 341), these circulation-free models are dynamically unstable with respect to non-axisymmetric motions; hence, the slightest disturbance will generate three-dimensional motions and, as a result, a large-scale meridional circulation will commence. The second alternative was independently suggested by Vogt and by Eddington, who pointed out that the breakdown of strict radiative equilibrium tends to set up slight rises in temperature and pressure over some areas of any given level surface and slight falls over other areas, the ensuing pressure gradient between the poles and the equator causing a flow of matter along the level surfaces. In other words, it is the small departures from spherical symmetry in a rotating star that lead to unequal heating along the polar and equatorial radii, which in turn causes large-scale circulatory currents in meridian planes. Eddington's (1925) farsightedness is particularly apparent from the following remark: "The coefficient of viscosity in a star is rather high, and it seems likely that when the currents attain a moderate speed a steady rate will result; of course, the fundamental equations of equi-

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librium are then modified by the addition of viscous stresses to the pressure-system, and the star is relieved from von Zeipel's condition." It is my purpose to show that this sentence contains the essence of our global solution, although obviously the nature of this "rather high viscosity" has to be specified more clearly.

I have already referred to the many inconsistencies that can be found in the literature on meridional streaming in a stellar radiative zone (T.R.S., pp. 198-207; Paper IV, p. 301). As is well known, in most papers the main consensus is that viscous friction can be neglected in a radiative zone because: (i) the microscopic (molecular and radiative) viscosity is negligibly small in a star, and (ii) laminar motions always prevail outside a (turbulent) convective zone. The first statement is correct, as we showed in Paper I. The second one, which most probably stems from man's propensity to idealize what he cannot see, is an unacceptable oversimplification in the present context. Our global solution rests essentially on a dynamical linkage between the ever-present eddy-like motions (which we called "anisotropic turbulence" because they are predominantly two-dimensional) and the mean flow (that is, the differential rotation and concomitant meridional currents). To be more specific, strict radiative equilibrium prevents a rotating star from being a barotrope; hence, a radiative zone is necessarily filled with small-scale transient motions that are caused by the non-axisymmetric instabilities. This anisotropic turbulence, in turn, generates viscous boundary layers so that the circulation velocities do not become infinite near the boundaries of the radiative zone. (The presence of unwanted singularities was the main defect of Sweet's [1950] laminar, inviscid solution.) Moreover, the turbulent friction acting on the differential rotation can be made to balance (in part or in toto) the transport of angular momentum by the large-scale meridional currents (Papers IV and VI).

By making use of the eddy-mean flow interaction which takes place continuously in a radiative zone, one can obtain a simple but adequate description of the mean state of motion in a rotating star. For example, in Papers I, IV, and VI we have considered the case of a chemically homogeneous, nonmagnetic, early-type star, assuming that departures from spherical symmetry are not too large. As we have shown, if one excludes the most rapid rotators on the verge of equatorial break-up, one can rightfully expand about hydrostatic equilibrium in powers of the small parameter $\varepsilon = \Omega_0^2 R^3/GM$, neglecting all terms of $O(\varepsilon^2)$ or smaller. Here Ω_0 is the (constant) overall angular velocity, R the radius, G the constant of gravitation, and M the mass. (Note that $\varepsilon \approx 0.4$ at the centrifugal limit.) To make a long story short, let us say that the overall rotation, of $O(\varepsilon^{1/2})$, generates a large-scale meridional circulation, which is of $O(\varepsilon)$ because the centrifugal force is of that order; these currents, in turn, react back on the overall rotation so that, correct to $O(\varepsilon^{3/2})$, the actual angular velocity takes the form

$$\Omega = \Omega_0 [1 + \varepsilon (A + B \sin^2 \theta)], \qquad (1)$$

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in spherical coordinates (r, θ, ϕ). The problem of evaluating the mean circulation velocities is thus neatly separated from that of evaluating the mean departure from solid-body rotation, i.e., the functions $A(r)$ and $B(r)$ in equation (1). This fact explains why some of the early results remain partially valid, even though their meanings are changed because of the essential eddy-mean flow interaction.

In Paper I we have illustrated the meridional currents in the turbulent radiative envelope of a $3M_\odot$ early-type star. Clearly, to $O(\varepsilon)$ the circulation pattern does not depend on the overall rotation rate; it consists of a single cell extending from the core-envelope interface to the surface, with rising motions at the poles and sinking motions at the equator. Because of the presence of viscous boundary layers, there are no singularities in the mean flow; hence, the circulation velocities remain uniformly small everywhere in the radiative zone. The typical speed of these currents is of the order of $\varepsilon\, LR^2/GM^2$, where L is the total luminosity. This result confirms the belief that the time scale of the large-scale circulatory currents established by rotation in a chemically homogeneous radiative zone is everywhere of the order or the Eddington-Sweet time, $t_{ES} = t_{KH}/\varepsilon$, where t_{KH} $(= GM^2/RL)$ is the Kelvin-Helmholtz time.

The nicest feature of our mean circulation pattern is that the meridional velocities do not depend on the (largely unknown) eddy viscosities in the bulk of a radiative zone. Of course, these velocities depend on the radial (i.e., along gravity) coefficient of eddy viscosity (μ_t, say) in the core and surface boundary layers. Fortunately, they depend respectively on $\mu_t^{1/7}$ and $\mu_t^{1/10}$ in these layers. Hence, because of the presence of the small exponents 1/7 and 1/10, the dependence on the poorly understood coefficient μ_t is appreciably reduced near the boundaries. To be specific, an uncertainty of $O(10^3)$, say, on μ_t near the surface leads to an error of $O(10^{0.3}) \approx 2$ on the meridional velocities. The same accuracy was achieved in our discussions of meridional streaming in tidally distorted stars (Papers II and III) and cooling white dwarfs (Paper V). In all cases, there are no singularities in the meridional flow, the circulation velocities remaining uniformly small everywhere in the star. Owing to the extreme smallness of these currents, however, they are without any doubt dynamically unimportant. Yet, because they continuously transport angular momentum, they play a crucial role in establishing the actual amount of differential rotation in a star.

The derivation of the rotation law is a much more intricate problem (see eq. [1]). In principle, the function Ω may be obtained by merely stating that the transport of angular momentum by the meridional flow is balanced by the turbulent friction acting on the differential rotation and a change in time of the mean azimuthal motion. (This is the meaning of the ϕ-component of the equations for the mean motion, see eq. [4] of our Paper VI.) Even assuming that a mean steady state has been reached $(\partial\Omega/\partial t \equiv 0)$, one is still faced with the inescapable fact that the functions A and B are roughly proportional to $1/\mu_t$. Since it is impos-

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sible at this time to perform a meaningful evaluation of the eddy coefficients in a radiative zone, there is thus no hope to calculate the departure from solid-body rotation with any accuracy. (A similar difficulty occurs in the theory of the solar rotation in the outer convective envelope; but, then, it is at least possible to adjust the theoretical rotation law to the observed surface rotation; see, e.g., Monin and Simuni 1982, and references therein.) The problem is even more severe when one considers a radiative zone in which the mean rotation is not steady in time (because the star is very young or rapidly evolving, or because it loses angular momentum). In this case, as we showed in Papers VI and VIII, the (non-uniform) overall angular velocity, of $0(\varepsilon^2)$, must be derived also from the ϕ-component of the Reynolds equations. Hence, until such time as the anisotropic turbulence in a radiative zone has been thoroughly investigated, it is impossible to derive from first principles only the instantaneous rotation law in a star. Perhaps the only sure thing is that, although the departure from uniform rotation may be quite small (if μ_t is quite large), there is no longer any reason to believe that a radiative zone is spinning exactly like a solid body, no more than there is to claim that strict uniform rotation is forced upon a star by the simple presence of a magnetic field. Uniform rotation may be a convenient (and reassuring) assumption in some cases; in no case is it compatible with the conservation principles of astrophysical fluid dynamics.

4. The Effects of a μ-gradient

As is well known, Mestel (e.g., 1965) has convincingly argued that the nonspherical distribution of mean molecular weight μ set up by meridional streaming in a rotating star tends to choke back the motion in meridian planes. According to Mestel, then, there should be no tendency for spontaneous mixing of matter between a chemically inhomogeneous region and the rest of a star (except perhaps for rapidly rotating stars on the verge of equatorial break-up). Since Mestel's analysis is mainly a qualitative one, in Paper VII we have discussed the effects of a μ-gradient on the meridional currents that pervade the turbulent radiative zone of a single, nonmagnetic, main-sequence star. To the best of my knowledge, this is the first attempt to actually solve the time-dependent differential equations that govern the meridional flow in an evolving star.

To illustrate the main features of meridional streaming in a chemically inhomogeneous star, we have considered the simple problem of a hydrogen-burning core for which the μ-gradient develops quasi-statically from the center outwards. In (almost) agreement with Mestel's finding, we found that the meridional flow is the superposition of "Ω-currents" (i.e., rotationally-driven currents that are nevertheless modified by the radial component of the μ-gradient) and "μ-currents" (i.e., currents that try to restore spherical symmetry to the μ-distribution). When the departure from spherical symmetry is not too large, the corresponding speeds-v (Ω) and v (μ), say - are of the order of $\varepsilon LR^2/GM^2$

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and $(\varepsilon LR^2/GM^2)$ $(\Delta\mu/\mu)$, respectively, where $\varepsilon\Delta\mu/\mu$ is a measure of the latitudinal μ-variations over the isothermal surfaces. As was expected, both speeds are proportional to the ratio ε because both currents are caused by the nonsphericity of the rotating star. Hence, to O (ε) the streamlines of the meridional flow do not depend on the overall rotation rate.

The numerical work indicates further that, almost from the start, the μ-currents oppose the Ω-currents, the large-scale currents dying out as the μ-gradient spreads through the core of a solar-type star. Within the numerical accuracy of our calculations, then, a μ-gradient virtually kills off the meridional flow. Yet, although no substantial mixing of matter may take place between the inner (inhomogeneous) and outer (homogeneous) regions, it is found that the μ-gradient does not perfectly insulate these regions. Evidently, to O (ε) we were unable to substantiate Mestel's claim that, in spite of a μ-gradient, Ω-currents do exist in stars on the verge of equatorial break-up. This conjecture is plausible but, short of a second-order analysis in ε, it remains unproven.

How do we explain, then, that Kippenhahn (1974) was able to derive a simple, first-order criterion for rotational mixing in chemically inhomogeneous zones? His argument is as follows. To O (ε), the speed of the Ω-currents is $v\ (\Omega) = \varepsilon LR^2/GM^2$, which is Sweet's original result. As for the μ-currents, Kippenhahn evaluates $v\ (\mu)$ by means of a local analysis, claiming that $v\ (\mu)$ is about equal to (LR^2/GM^2) $(D\mu/\mu)$, where $D\mu$ is the difference in molecular weight between a blob of matter and its surroundings. In this crude picture, a μ-gradient kills off the meridional currents if $v\ (\Omega) \lesssim v\ (\mu)$; hence, it follows at once that no mixing will occur if $\varepsilon \lesssim D\mu/\mu$. The fallacy of this criterion lies in the fact that, as I said, to O (ε) one has $v\ (\mu) = (\varepsilon LR^2/GM^2)$ $(\Delta\mu/\mu)$. Accordingly, because both speeds are proportional to ε in a first-order calculation, no critical value of ε can possibly be found to this order. In other words, Kippenhahn's derivation is incorrect because his local analysis does not include the fact that departure from spherical geometry is the ultimate cause of the global μ-currents. This is also the reason why Huppert and Spiegel (1977)'s analysis did not confirm Kippenhahn's.

To sum up, in Paper VII we have found that the large-scale meridional currents virtually die out from the center outwards as the μ-gradient steadily grows in a late-type star. Correct to order ε, our quantitative discussion thus amply confirms Mestel's picture of μ-barriers that prevent substantial mixing rates in slowly rotating stars, although these barriers may be penetrated to some extent. We have not yet calculated the circulatory currents in the radiative envelope of an evolving early-type star. Technically speaking, however, the two problems are quite similar. Accordingly, one readily sees from the appropriate equations that, to O (ε), the pattern of meridional streaming in an early-type star does not depend on the overall rotation rate; that is, to this order there is no critical rotation rate above which unimpeded mixing may take place in its turbulent radiative envelope. (This obvious fact was overlooked by Mestel and Kippenhahn because they did not make the proper scaling and ordering in powers of ε.) Moreover, the μ-gradient should inhibit the large-scale currents in the bulk of the radiative shell interior to the initial core boundary. However, in view of our numerical results, there is no reason to believe that this region cannot be penetrated to some degree. Hence, one must allow for some rotational mixing between the homogeneous and inhomogeneous parts of the radiative envelope. Of course, the ability of the circulatory currents to bring CNO-processed material to the outer surface will depend primarily on their speed, which is proportional to the squared overall angular velocity Ω_0^2. Hence, _if_ the radiative envelope of an early-type star is rotating almost uniformly, abnormal abundances could be observed at some stage, when the mixing time t_{ES} (= $G^2M^3/LR^2v_e^2$) is shorter than the star's main-sequence lifetime, the mixing efficiency of the meridional circulation increasing as the _squared_ equatorial velocity v_e^2 (= $\Omega_0^2R^2$). As I pointed out in Section 3, however there is absolutely no guarantee that the observed equatorial velocity v_e (or, for that matter, $v_e \sin i$) is a correct measure of the star's _inner_ rotation, which cannot be predicted on pure theoretical grounds only.

5. Applications

Quite understandingly, an observer would like to receive clear-cut answers to the following two questions: (i) how does a star rotate? and (ii) what are the effects of this rotation on the observable parameters?

As far as the first question is concerned, it has long been accepted that the detailed rotation law in a convective core is much uncertain, because our description of turbulent convection in a rotating fluid necessarily involves some free parameters that cannot be calculated from first principles alone. Our knowledge of the state of motion in an outer convective envelope is also incomplete; but then, as it is done in geophysics, the free parameters may be selected so that the theoretical rotation rate agrees with the observed surface motions.

In the case of a radiative core or envelope, it was implicitly assumed that the rotation law would not involve free parameters because, as it was thought, motions are always laminar outside a convective zone. In this lecture I have advocated the idea that the prevalent baroclinic instability in a stellar radiative zone continuously generates anisotropic turbulence, which interacts with both the rotation law and the concomitant meridional currents. (A similar eddy-mean flow interaction has been known to the geophysicists since the late 1940s!) Because the study of transient, eddy-like motions in a radiative zone is a new development, it is not yet possible to obtain the eddy viscosities and, thence, the mean rotation law. (As we pointed out in Paper VI, the geophysicists themselves cannot evaluate their eddy viscosities from first principles alone; unless one is willing to rely upon doubtful hand-waving arguments, there is thus no hope to evaluate ours at this time!)

Fortunately, thanks to a caprice of nature, the slow but inexorable meridional currents in a turbulent radiative zone are almost independent of the smaller-scale motions. Hence, the circulation velocities presented in Papers I and II are amply adequate to discuss the problem of circulation versus diffusion in (single and double) early-type stars. This was done with great clarity by Michaud and his collaborators; I shall not repeat their arguments that point in favor of both the diffusion model and large-scale meridional streaming in turbulent radiative envelopes (Michaud 1982, Michaud et al. 1983). Of equal importance is the problem of rotational mixing in evolving stars, which I will now briefly consider.

Although meridional circulation has the potentially important effect of mixing the composition of the stellar material, it is known that most stars do not mix because the assumption of homogeneous stellar evolution does not explain the existence of the giant branch in the HR diagram. Mestel's idea that μ-barriers prevent the mixing of matter (except perhaps for stars on the verge of equatorial break-up) was thus all the more plausible, and the ill-formulated problem of rotationally-driven meridional currents could be swept under the rug. By necessity, however, the concept of slow rotational mixing was evoked by Paczyński (1973) and again by Sweigart and Mengel (1979) to explain CNO anomalies in some early-type stars and red giants (see also, e.g., Norris 1981, and references therein). What can we add to these phenomenological discussions in the light of the quantitative results presented in Papers I-VIII?

First of all, there is some confusion about the fact that, supposedly, there should exist a simple recipe which indicates when rotational mixing will or will not occur. As I have explained in Section 4, for all stars rotating well below the break-up velocities, the streamlines of meridional circulation do not depend on the overall rotation rate. Furthermore, our detailed numerical calculations clearly show that, although a μ-gradient virtually kills off large-scale currents, it does not perfectly insulate a chemically inhomogeneous region from its surroundings. In other words, some mass exchange may take place, after some lapse of time, whenever the Eddington-Sweet time in the homogeneous region is smaller than the evolutionary time scale.

The extreme slowness of the circulatory currents is the main reason why rotational mixing is utterly inefficient when it comes to bringing a sizable amount of hydrogen-rich material into the burning core of a solar-type star (unless, of course, its inner rotation period is no more than a fraction of a day!) Accordingly, rotational mixing cannot possibly explain the solar neutrino problem. But then, as I have shown in Section 2, one must keep in mind that rotation always generates a whole spectrum of eddy-like motions in a radiative zone. Hence, turbulent diffusion mixing (as originally suggested by Schatzman) remains a plausible explanation to this problem, even though the numerical modelling of Schatzman et al. (1981) might appear inadequate, as was shown by Ulrich and Rhodes (1983). In my opinion, because the (non-uniform) turbulent transport coefficients in the Sun are still too much uncertain, no unequivocal solar model can be made at this time; and there is not a single good reason why the corresponding Reynolds number should take a universal value in all main-sequence stars.

In early-type stars one has in general $\varepsilon \approx 0.01-0.1$, whereas in solar-type stars ε may be two or three orders of magnitude smaller. This is the main reason why even a moderate rotation rate can bring CNO-processed material to the outer surface in an evolving early-type star, in spite of the presence of a μ-gradient near the core-envelope interface. The idea that slow rotationally-driven currents may be the ultimate cause of abnormal atmospheric abundances in some stars is a most plausible one, therefore. Not unexpectedly, we found it impossible to derive a precise criterion stating under what condition(s) and to what extent meridional streaming may indeed change atmospheric CNO abundances, either on the main sequence or on the giant branch. It is my firm belief that no simple, black-or-white criterion exists, because this problem involves too many (unknown) parameters. (For example, the inner rotation rate of a star may be larger than that implied by its observed $v_e \sin i$ value, so that the mixing currents may be much faster than those expected on the basis of the results presented in Paper I.) Yet, I'm looking forward to the time when CNO surface abundances (and perhaps other seemingly unrelated observations) will be used to probe the inner state of motion in a stellar radiative zone, which always consists of highly anisotropic turbulence, rotation, and much slower circulatory currents.

Since mixing has also been mentioned in connection with the blue stragglers, I shall conclude this lecture by commenting briefly on these puzzling objects. As we know, Saio and Wheeler (1980) and Maeder and Mermilliod (1981) have shown empirically that partial ad hoc mixing is a viable assumption to account for their existence. On the basis of our self-consistent results about meridional streaming, it would appear most unlikely that slow rotational mixing can indeed play a role in this problem, being overshadowed by convective overshooting, which is a much more efficient stirring mechanism. However, because there is no compelling reason why overshooting should occur in some stars but not in others, one should not exclude the possibility that blue stragglers have rapidly rotating cores. If so, I do not think that rotational mixing would be of much importance either, extended main-sequence lifetime being then caused (in part or in toto) by a much reduced temperature in the central region (as compared to their slowly rotating counterparts). Of course, observation indicates that the blue stragglers do not appear to be abnormally rapid rotators (e.g., Smith and Hesser 1983). But then, as I have already said, these measurements refer to the surface rotation rates, which may not be indicative in all cases of the inner rotation rates. Since it is most likely that there exists a spectrum of core rotational velocities in stars, the assumption of rapidly rotating cores in the blue stragglers is therefore not unreasonable. And, again, it could be used as an indirect way to evaluate unobservable inner motions.

To sum up, the main leitmotiv of this lecture was that neither an observer nor a theoretician can give a firm, direct information about the state of motion deep inside an evolving star. The main difficulty for the theoretician is that the motions within a rotating star are always turbulent; hence, given the present state of knowledge about turbulence in a compressible fluid, the mean rotation rate necessarily depends on poorly known parameters (such as the eddy viscosities). Accordingly, instead of studying the gross effects of ad hoc rotation laws on stellar evolution, one should perhaps accept rotation as a plausible explanation of some observations and, thence, use the data to evaluate the mean and fluctuating motions within a star. In other words, one should not overlook the fact that some unrelated phenomena (such as the solar 5-minute oscillations, the CNO abundances, or perhaps the blue stragglers) might provide a clue as to what is the specific angular momentum within an evolving star. Moreover, because direct measurements of turbulence and large-scale motions can now be made in the Earth's atmosphere and oceans, the theoretician who has a personal interest in stellar rotation should avail himself of the much more rapid related advances that are being made in geophysics. A genuine background in fluid mechanics may be of some use too.

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DISCUSSION

Tayler: I agree that in principle you need to make the careful time-dependent treatment of μ-currents. However, is it not true that it is only really essential when $t_{ES} \approx t_{nuclear}$; if $t_{ES} << t_{nuclear}$ the star is kept homogeneous, if $t_{ES} >> t_{nuclear}$ the μ distribution will evolve independent of the ES circulation and you can then ask what effect the prescribed μ has on the circulation?

J.-L. Tassoul: Let me rephrase our results in another manner. There are three time scales: $t_{nuclear}$, $t_{ES} = t_{KH}/\varepsilon$, and $t_1 = t_{KH}/\varepsilon_1$, where ε_1 (≈ 0.1, say) is the value of ε above which our first-order expansions break down. We have made a careful discussion of the simultaneous interaction between the effects of a μ-distribution on the meridional currents and the effects of this circulation on the μ-distribution in an evolving 1 $M_\odot$ star. We have found that, when $t_{ES} \gtrsim t_1$, the unsteady circulation pattern is the same in all cases ($t_{ES} << t_{nuclear}$ or $t_{ES} \gtrsim t_{nuclear}$). We found almost no penetration across the outwardly moving μ-barrier because, when $t_{ES} \gtrsim t_1$, the radial part μ_o of the μ-distribution produces a large effect on the circulation pattern whereas the back reaction $\varepsilon\mu_{1,2}$ of the currents on the μ-distribution is a small effect only. (Incidentally, this is why our first-order expansions are self-consistent) In other words, to order ε it takes a quite small departure from spherical symmetry in the μ-distribution to kill off the meridional currents in the central inhomogeneous region, where nuclear burning takes place. (McDonald reached exactly the same conclusions in his correct, first-order discussion of the μ-distribution that is needed to obtain steady, circulation-free solutions; see Ap. Sp. Sci., 19, 309, 1972). Now, if $t_{ES} < t_1$, second-order series in ε are needed. In this more complex initial-value problem, one does not expect to find almost perfect insulation in all cases, because the streamlines of meridional circulation then depend on ε. This is another time-dependent problem that has not yet been solved. On physical grounds, new behaviours are previsible when $t_{ES} << t_1$. However, unless one is willing to make use of unsound first-order series in ε, these more complex circulation patterns can be described by second-order expansions only. Our calculation is strictly first-order in ε. I do not want to claim more than one can achieve on the basis of a consistent first-order analysis (see also our Paper VII).

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Spruit: In the beginning you said that for ordinary microscopic viscosity "there is no solution". Does that mean that the mathematical problem stated has no solution, or does it mean that the solution just looks unappealing?

J.L. Tassoul: I should have said: "there is no acceptable steady solution when ordinary microscopic viscosity is taken into account." The reason why we did not want to retain our laminar solutions is that we wanted to restrict ourselves to mean steady motions in our Paper I. Later on, we realized that one can also obtain consistent solutions with a much smaller viscosity, provided one allows for unsteadiness in the mean azimuthal motion. This is explained at length in our Papers IV and VI.

Spruit: From the fact that the solution for low ν has large differential rotation one cannot conclude that there is turbulence. This has to come from a different argument.

J.L. Tassoul: Our argument that the motions are turbulent rather than laminar stems from the fact that, no matter what kind of viscosity one assumes, the rotation law necessarily depends on ϖ and z. Accordingly, one must consider baroclinic models, which may easily sustain a wide range of small-scale, eddy-like motions. This is shown in our Papers I (App. B) and VII (App.).

Spruit: Your statement that baroclinic instability will always be present in a barocline is not justified. Also, when it is present it need not produce a significant turbulence in the bulk of the star. See Spruit and Knobloch (A & A, 1983, "Baroclinic instability in stars"). This paper also describes why baroclinic instability in stars is so much less important than in the Earth's atmosphere and oceans. Only when Ω is of the order N can baroclinic instability become important in stars.

J.L. Tassoul: In the Earth's atmosphere, baroclinicity is caused by the pole-equator temperature difference due to solar heating. As we explained in our paper I (App. B), the thermal-wind relation and the geostrophic approximation do not apply in a stellar radiative zone. Indeed, in such a system, baroclinicity is a consequence of strict radiative equilibrium at every point, which forces large-scale meridional currents and a mean rotation law of the form $\Omega = \Omega(\varpi, z, t)$. By virtue of the Poincaré-Wavre theorem, this condition implies that the isothermal surfaces are in general inclined over the isobaric surfaces, so that there always exist some unstable baroclinic modes in a stellar radiative zone. In other words, because the basic mechanisms that lead to baroclinicity (and, hence, to baroclinic instability) are not the same in geophysics and in astrophysics, one should not make use of approximations that are valid in the former case to discuss baroclinic instability in a rotating star. The fact that this instability may be less effective in a star is of no concern to us either, because we do not want to describe stellar weather waves. In fact, all what we need are some small-scale disturbances superimposed on the large-scale flow, no matter how large or small these eddy-like motions may be (see our Papers IV and VI).

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Zahn: What I highly appreciate in your model of a rotating star (Tassoul and Tassoul, 1982) is its mathematical consistency, achieved for the first time. But one has also to worry about its physical consistency, as mentioned by I. Roxburgh. I came to the conclusion that the conservation of angular momentum, in the stationary state you have described, requires the turbulent Prandtl number to be of order one (13th Advanced Course of Saas-Fee, 1983). This would have dramatic consequences on the structure of the star, since the transport of heat by turbulent motions would be of the same order as the radiation transfer. How can you overcome this difficulty?

J.L. Tassoul: During the 13th Saas-Fee Course, by making use of an order-of-magnitude argument, Zahn has explained why he did not believe that our solutions were physically consistent. The big flaw in his argument can readily be seen as follows. By virtue of equations (121) and (122) of our Paper I, one has $\beta_i = O(ur/\nu_t)$, where $\nu_t = \mu_t/\rho$ and j = 1,3. According to Zahn, then, one can write $ur = O(R^2/t_{KH}) = O(\chi_r/\rho c_v)$, so that $\beta_i = O(P_t^{-1})$, where $P_t = c_v\mu_t/\chi_r$. His conclusion follows at once from this formula because, for consistency, one must have $\beta_i = O(1)$. Now, by making use of the numerical results of our Paper I, one readily sees that ur/ν_t is a bounded function that vanishes at $r = R_c$ and $r = R$; on the contrary, in our models one has $P_t \propto T$, so that P_t^{-1} varies as the inverse of the temperature in the bulk of a stellar radiative zone! Since T drops by many orders of magnitude from the core-envelope interface to the surface, it is now pretty much obvious that Zahn's relation $\beta_i = O(P_t^{-1})$ is largely in error in a realistic stellar radiative zone. In other words, Zahn has routinely made an order-of-magnitude discussion that applies to a Boussinesq fluid, without noticing that such an argument does not apply to our solutions because we have considered highly non-Boussinesq fluids. The following conclusions can thus be made: (i) the solutions reported in our Papers I-VIII are physically and mathematically consistent, (ii) because we have shown that the turbulent transport of energy is everywhere much smaller than the transport of energy by radiation, Sweet's solution adequately describes the mean meridional flow in the bulk of a nonmagnetic, chemically homogeneous radiative zone (see our Paper VI), (iii) viscous boundary layers that depend weakly on the magnitude of the prevailing eddy viscosity prevent the mean circulation velocities from having unwanted singularities at the boundaries (see our Paper I), and (iv) the back reaction of meridional streaming on the overall rotation rate can be obtained in a consistent manner in a nonmagnetic star, no matter how large or how small the prevailing eddy viscosity is (see our Paper IV).

Frogel: CNO abundance anomalies have been observed in giants that evolve from solar-type stars. You said core rotation is too slow in the sun to cause mixing. How then can it cause mixing in stars which have solar-type stars as their predecessors?

J.L. Tassoul: My comment applies only to rotational mixing in the radiative core of a main-sequence, solar-type star. Once such a star leaves the main sequence, however, the core contraction modifies its (unobservable) inner rotation rate, so that rotational mixing may then no longer be negligible during some phases of post-main-sequence evolution. (Remember that the circulation speeds are proportional to the squared overall inner rotation rate).

R. Cayrel: 1) I was confused about the value of the meridional circulation for the sun: was it 10^{-5} cm s^{-1} or 10^{-9} cm s^{-1}?
2) Can you tell us the order of magnitude of the turbulent velocity you expect to be associated with your solution?

J.L. Tassoul: 1) The circulation speed is less than 10^{-9} cm s^{-1} because, in our solar models, $v(\Omega) = \varepsilon|u|$, where $\varepsilon \simeq 10^{-4}$ and $|u| \lesssim 10^{-5}$ cm s^{-1} ;
2) At this time, no firm statement can be made about the speed of the turbulent eddies (see also our Paper IV, p. 300).

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