ON FUNCTIONS WITH DERIVATIVE OF BOUNDED VARIATION: AN ANALOGUE OF BANACH'S INDICATRIX THEOREM

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1. Statement of the result

A simple, but nice theorem of Banach states that the variation of a continuous function $F: [a,b] \rightarrow \mathbb{R}$ is given by $\int_a^b t(y) dy$, where $t(y)$ is defined as the number of $x \in [a,b]$ for which $F(x) = y$ (see, e.g., [1], VIII.5, Th. 3). In this paper we essentially derive a similar representation for the variation of $F'$ which also yields a criterion for a function to be an integral of a function of bounded variation. The proof given here is quite elementary, though long and somewhat intricate.

Let $-\infty < a < b < \infty$, $F: [a,b] \rightarrow \mathbb{R}$ be continuous.

For any real function $G$ on $[a,b]$ we denote by $v(G)$ its variation and by $l(G)$ the length of its graph; further $||G|| := \sup \{|G(x)|, x \in [a,b]\}$.

Let $a_n := a + (b - a) j 2^{-n}$, $D_n := \{a_n, j = 1, \ldots, 2^n - 1\}$.

For $\alpha > 0$ we define $\mathcal{F}_\alpha(F) := \{G: [a,b] \rightarrow \mathbb{R}, ||F - G|| \leq \alpha, G(a) = F(a), G(b) = F(b)\}$.

We consider $F^*_\alpha: [a,b] \rightarrow \mathbb{R}$ which is defined to be that function $H$ satisfying $H(a) = F(a)$, $H(b) = F(b)$, $F(a_n) - \alpha \leq H(a_n) \leq F(a_n) + \alpha$ ($j = 1, \ldots, 2^n - 1$) which has minimal length. Clearly $F^*_\alpha$ is piecewise linear and continuous (see Fig. 1). We shall show that
$F_a := \lim_{n \to \infty} F_n^a$ exists (pointwise), and $F_a$ is uniquely determined by

$$l(F_a) = \inf \{l(G) \mid G \in \mathcal{F}_a(F)\}. \quad (1)$$

$F_a$ can be visualized as a thread fastened to the point $(a, F(a))$ and drawn as tautly as possible in the region

$$\{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], F(x) - a \leq y \leq F(x) + a\}$$

such that it passes through $(b, F(b))$.

Let $F^0(x)$ be the straight line joining $(a, F(a))$ and $(b, F(b))$.

$$\alpha_0 := \sup \{|F(x) - F^0(x)| \mid x \in [a, b]\}. \quad (2)$$

Suppose $\alpha_0 > 0$. It will be proved that for $\alpha \in (0, \alpha_0)$ there is a finite number of open intervals $J_{x_1}, \ldots, J_{x_k} \subset [a, b]$ (ordered from left to right) with the following properties:

(i) $|F_a(x) - F(x)| < \alpha$ for all $x \in \bigcup_{i=1}^{k} J_{x_i}$

(ii) Let $J_{x_i} = (x_{i,a}, x_{i,a})$. Then for $i = 2, \ldots, k - 1$ either

$$F_a(x_{i,a}) - F(x_{i,a}) = \alpha \text{ and } F_a(x_{i,a}^+) - F(x_{i,a}^+) = -\alpha \text{ or}$$

$$F_a(x_{i,a}) - F(x_{i,a}) = -\alpha \text{ and } F_a(x_{i,a}^+) - F(x_{i,a}^+) = \alpha; \text{ further}$$

$$x_{1,a} = a, F_a(x_{1,a}) - F(x_{1,a}) = \pm \alpha \text{ and } x_{k,a} = b, F_a(x_{k,a}) - F(x_{k,a}) = \pm \alpha.$$

Let for $\alpha \geq \alpha_0$, $s(\alpha) = 0$ and for $\alpha \in (0, \alpha_0)$

$$s(\alpha) := \left[4(x_1^+ - a)^{-1} + (x_2^+ - x_2^{-})^{-1} + \cdots + (x_{k-1}^+ - x_{k-1}^{-})^{-1} \right]^{-1}$$

$$+ \left[4(b - x_{k,a})^{-1} \right]^{-1}. \quad (2)$$

$s(\alpha)$ will be seen to be monotone decreasing. It measures how often $F_a$ varies from $F + \alpha$ to $F - \alpha$ and vice versa and how fast this happens.

Now we can formulate the result.

**Theorem.** If $\int_0^s s(\alpha) \, d\alpha < \infty$ for some $\varepsilon > 0$, there is a $f : [a, b] \to \mathbb{R}$ such that

$$F(x) = F(a) + \int_a^x f(t) \, dt \quad \text{for all } x \in [a, b] \quad (3)$$

$$v(f) = 4 \int_0^\infty s(\alpha) \, d\alpha. \quad (4)$$
If there is a $f : [a, b] \to \mathbb{R}$ of bounded variation satisfying (3), there exists a $f^* : [a, b] \to \mathbb{R}$ such that $f = f^*$ almost everywhere and $\nu(f^*) = 4 \int_0^\infty s(x) \, dx < \infty$.

2. Proof of the Theorem

We have subdivided the proof into a number of separate steps.

(a) It is clear that $\sup_n l(F_n) < \infty$ so that also $\sup_n \nu(F_n) < \infty$. By Helly's extraction theorem ([1], p. 250), there is a pointwise convergent subsequence $F_n \to F_\alpha$, and we have $l(F_\alpha) \leq \liminf_{j \to \infty} l(F_j^n)$. Each $G : [a, b] \to \mathbb{R}$ for which $\|F - G\| \leq \alpha$ and $G(a) = F(a)$, $G(b) = F(b)$ satisfies $l(G) \geq l(F_\alpha)$ for all $n \in \mathbb{N}$ so that $l(G) \geq l(F_\alpha)$. Thus (1) holds. We shall show that (1) uniquely determines $F_\alpha$ thus getting $F_\alpha \to F_\alpha$.

(b) $F_\alpha$ is continuous on $[a, b]$. Indeed, since $\nu(F_\alpha) < \infty$, $F_\alpha(x +)$ and $F_\alpha(x -)$ exist for all $x \in (a, b)$. Suppose, e.g., $F_\alpha(x_1 -) < F_\alpha(x_1 +)$ for some $x_1 \in (a, b)$. Define $\bar{F}_\alpha^e(x) := F_\alpha(x)$ for $x \notin [x_1, x_1 + \varepsilon)$, $\bar{F}_\alpha^e(x_1 + ) := \frac{1}{2} F_\alpha(x_1 - ) + \frac{1}{2} F_\alpha(x_1 + )$, $\bar{F}_\alpha^e$ linear on $[x_1, x_1 + \varepsilon)$ and $\lim_{x \to x_1^+} \varepsilon \to \bar{F}_\alpha^e(x) = F_\alpha(x_1 + \varepsilon - )$.

Then $\bar{F}_\alpha^e \in \mathcal{F}_\alpha(F)$ for small $\varepsilon > 0$ and $l(\bar{F}_\alpha^e) < l(F_\alpha)$, a contradiction. The continuity of $F_\alpha$ in $a$ and $b$ is proved similarly.

(c) By (b), $A_\varepsilon := \{x \in [a, b] | F_\alpha(x) - F(x) < \varepsilon\}$ is open. $F_\alpha$ is concave on each component of $A_\varepsilon$. To see this, let $x_1, x_2 \in D_\alpha$ for some $N$ with the properties $U = (x_1, x_2) \subset A_\varepsilon$ and $\sup y_\alpha(F(x) - x) < \inf y_\alpha(F(x) + x)$. For $n \geq N$ let $G_n$ be the smallest concave function on $U$ satisfying $G_n(x) \geq F(x) - \alpha$ for all $x \in D_\alpha \cap (x_1, x_2)$, $G_n(x_1) = F(x_1)$, $G_n(x_2) = F(x_2)$. Obviously we have $G_n \leq F + \alpha$ and $G_n \leq F_\alpha$ (in $U$); $G_n$ is an increasing sequence, and the limit $G := \lim_{n \to \infty} G_n$ is a concave function on $U$ for which $G \leq F_\alpha, F - \alpha \leq G \leq F + \alpha, G(x_1) = F_\alpha(x_1), G(x_2) = F_\alpha(x_2)$. So we must have $G = F_\alpha$ on $U$.

(d) Similarly as in (c) it is seen that $F_\alpha$ is convex on each component of the (by (a)) open set $B_\varepsilon := \{x \in [a, b] | F_\alpha(x) - F(x) > -\varepsilon\}$. Let $\mathcal{J}_\alpha(F_\alpha)$ be the set of all intervals $[x, y] \subset [a, b]$ such that $x, y \in A_\varepsilon$ and $[x, y] \cap B_\varepsilon = \emptyset$ and $[x, y] \cap A_\varepsilon = \emptyset$. Let $I_\alpha := \bigcup \{[I] \in \mathcal{J}_\alpha\}$, $I_\alpha := \bigcup \{[I] \in \mathcal{J}_\alpha\}$, $J_\alpha := \{[a, b]\} \cap (I_\alpha \cup J_\alpha)$. On the components of $I_\alpha \cup J_\alpha$, resp. $I_\alpha \cup J_\alpha$, resp. $I_\alpha \cup J_\alpha$, $F_\alpha$ is convex resp. concave resp. linear. The number of components of $I_\alpha \cup J_\alpha \cup J_\alpha$ is finite (otherwise there exist sequences $x_i \in \partial I_\alpha, x_i \in \partial J_\alpha$ such that $x_i \to 0$ as $i \to \infty$ so that $F_\alpha(x_i) - F_\alpha(x_i) \to 0$; this yields a contradiction, because $F_\alpha(x_i) = F(x_i) - \alpha, F(x_i) = F_\alpha(x_i) + \alpha$).

(e) By (d), $F_\alpha$ is absolutely continuous. If $F_\alpha$ is a solution of (1) for which $F_\alpha \equiv F_\alpha, F_\alpha$ is also absolutely continuous, and

$$l(\frac{1}{2} F_\alpha + \frac{1}{2} F_\alpha) = \int_a^b \left[1 + \left(\frac{1}{2} F_\alpha(x) + \frac{1}{2} F_\alpha(x)\right)^2\right]^{1/2} \, dx$$

$$< \int_a^b \left[\frac{1}{2} + F_\alpha(x)^2\right]^{1/2} + \frac{1}{2} (1 + F_\alpha(x)^2)^{1/2} \, dx$$

$$= \frac{1}{2} l(F_\alpha) + \frac{1}{2} l(F_\alpha) = l(F_\alpha).$$

(5)
Since \( \frac{1}{2} F_a + \frac{1}{2} F_b \in \mathcal{F}_d(F) \), (5) contradicts (1). So \( F_a \) is uniquely determined by (1), and \( F_a \to F_a (n \to \infty) \).

(f) \( J_a \) is increasing, \( I_a \) and \( T_a \) are decreasing with respect to \( \alpha \). Consider for example \( A' \). Suppose on the contrary that for some \( \alpha < \beta \) and \( x_0 \in (a, b) \) we have \( F_b(x_0) = F(x_0) + \beta \) and \( F_a(x_0) < F(x_0) + \alpha \). Let \( U = (x_1, x_2) \) be the maximal interval in \([a, b]\) containing \( x_0 \) such that \( F_b(x) - \beta > F_a(x) - \alpha \) for \( x \in U \). Clearly \( a < x_1 < x_0 < x_2 < b \), and \( F_a \) is concave on \( U \) (otherwise there is a \( x_3 \in U \) for which \( F_a(x_3) = F(x_3) + \alpha \), but then \( F_b(x_3) - \beta > F(x_3) \), a contradiction). Note that for \( x \in U \)

\[
F_b(x) - \beta > F_a(x) - \alpha \geq F(x) - 2\alpha
\]

so that \( F_b(x) > F(x) - \beta \). Consequently \( U \subset B_\beta \) and \( F_b \) is convex on \( U \). The convex \( F_b \) coincides with the concave \( F_a + \beta - \alpha \) at \( x_1 \) and at \( x_2 \), so \( F_b = F_a + \beta - \alpha \) on \( U \). This is a contradiction to the definition of \( U \).

(g) It follows from (f) that \( s \), as defined by (2), is a monotone decreasing function (note that the construction of \( F_a \) shows that \( J_a \neq \emptyset \) for \( \alpha < \alpha_0 \)).

(h) For the rest of the proof we assume without restriction of generality that \( \inf I_a < \inf T_a \).

Thus \( F_a \) "has a convex start".

(i) \( J_a \) is a finite disjoint union of open intervals \( J_1, a \), \( J_k, a \) (ordered from left to right). Denote their lengths by \( l_{1, a} \), \( l_{2, a} \), \ldots \, l_{k, a} \) and set

\[
\begin{align*}
\ell_a := & -l_{1, a}^{-1} J_{1, a} + 2l_{2, a}^{-1} J_{2, a} - \cdots + (-1)^{k-1} 2l_{k-1, a}^{-1} J_{k-1, a} \\
& + (-1)^k l_{k, a}^{-1} J_{k, a}, \\
\ell_a := & 0, \quad \ell \geq \ell_0.
\end{align*}
\]

If \( s \) is defined by (2), it is easy to verify that

\[
4 s(\alpha) = v(\ell_a).
\]

Further, for all \( \varepsilon > 0 \),

\[
4 \int_{\varepsilon}^{\infty} s(\alpha) d\alpha = \int_{\varepsilon}^{\infty} v(\ell_a) d\alpha = v \left( \int_{\varepsilon}^{\infty} \ell_a d\alpha \right).
\]
We notice that for each \( \alpha \geq \varepsilon \) the set of points at which \( h_\varepsilon \) jumps upwards (resp. downwards), is contained in \( I_\varepsilon \) (resp. \( \tilde{I}_\varepsilon \)). Thus the sign of \( \frac{d}{dx} h_\varepsilon(x_j) - h_\varepsilon(x_{j-1}) \) only depends on \( j \) (not on \( \alpha \geq \varepsilon \)). Therefore the approximating sums for \( \int_0^\infty \nu(h_\varepsilon) \, d\alpha \) and \( \nu(\int_0^\infty h_\varepsilon \, d\alpha) \) belonging to the above partition are equal:

\[
\sum_{j=1}^N \int_\varepsilon^\infty h_\varepsilon(x_j) \, d\alpha - \int_\varepsilon^\infty h_\varepsilon(x_{j-1}) \, d\alpha = \sum_{j=1}^N \int_\varepsilon^\infty h_\varepsilon(x_j) - h_\varepsilon(x_{j-1}) \, d\alpha \\
= \int_\varepsilon^\infty \left( \sum_{j=1}^N \left| h_\varepsilon(x_j) - h_\varepsilon(x_{j-1}) \right| \right) \, d\alpha.
\]

This proves the second equation of (9).

(j) We next derive the equation

\[
F_\varepsilon(x) = \int_a^x h_\varepsilon(u) \, du + F_0(x) + F'_0(x),
\]

First note that \( F_\varepsilon \) is absolutely continuous with respect to \( \alpha \). Indeed, it follows from the proof in (f) that \( F_\varepsilon(x) - \beta \leq F_\varepsilon(x) - \alpha \) for \( \alpha \leq \beta \), and a similar argument shows that \( F_\varepsilon(x) + \alpha \leq F_\varepsilon(x) + \beta \) for \( \alpha \leq \beta \).

Thus the limit of

\[
H_{\varepsilon,0}(x) := \varepsilon^{-1}(F_\varepsilon(x) - F_{\alpha - \varepsilon}(x)),
\]

as \( \varepsilon \to 0^+ \), exists almost everywhere. By the Lipschitz continuity of \( \alpha \to F_\varepsilon(x) \) we have \( |H_{\varepsilon,0}| \leq 1 \). Further for \( x \in A_\varepsilon^* \cup B_\varepsilon^* \)

\[
H_{\varepsilon,0}(x) = \begin{cases} 
 1, & x \in A_\varepsilon^* \\
 -1, & x \in B_\varepsilon^*.
\end{cases}
\]

For if \( x \in A_\varepsilon^* \), \( F_\varepsilon(x) = F(x) + \alpha \) and, by (f), \( F_{\alpha - \varepsilon}(x) = F(x) + \alpha - \varepsilon \); the assertion for \( h_\varepsilon \) follows from the definition (7) and (h). As \( F_\varepsilon \) is linear on the components of \( A_\alpha \cap B_\alpha \), \( F_{\alpha - \varepsilon} \) is concave on the components of \( A_\alpha \cap B_\alpha \) and (by (f)) \( F_\varepsilon \leq F_{\alpha - \varepsilon} \), we can conclude that \( H_\varepsilon \) is convex on the components of \( (A_\alpha \cap B_\alpha) \cap A_\alpha = A_\alpha \). On \( B_\alpha \) we have \( H_\varepsilon \equiv -1 \), so that \( H_\varepsilon \) is convex on the components of \( A_\alpha \). Similarly it is seen that \( H_\varepsilon \) is concave on the components of \( B_\alpha \).

It is easily seen that \( A_\alpha \cap B_\alpha \) is a component of \( A_\alpha \cap B_\alpha \) for \( \varepsilon > 0 \). If \( x \in I_\alpha \), there are \( x_1, x_2 \in A_\varepsilon^* \) such that \( x \in [x_1, x_2] \). Since \( A_\varepsilon^* \subset B_\varepsilon^* \), there is a \( \varepsilon_0 > 0 \) such that \( x_1, x_2 \in B_\varepsilon^* \). As \( H_\varepsilon \) is concave on \( B_\varepsilon^* \), we have \( H_\varepsilon(x_1) = H_\varepsilon(x_2) = 1 \) for \( \varepsilon \in (0, \varepsilon_0] \). Thus \( H_\varepsilon(x) \to 1 \) for \( x \in I_\alpha \). Similarly we get \( H_\varepsilon(x) \to 1 \) for \( x \in \tilde{I}_\alpha \).

Now let \( (x_0, x_1) \) be a component of \( J_{\alpha \varepsilon} \), \( x_0 \in A_\varepsilon, x_1 \in B_\varepsilon \), so that \( H_\varepsilon(x_0) = 1, H_\varepsilon(x_1) = -1 \). Then for small \( \varepsilon > 0 \) there are \( \delta(\varepsilon) > 0, \eta(\varepsilon) > 0 \) such that \( H_\varepsilon \) is concave and decreasing on \( [x_0, x_1 - \eta(\varepsilon)] \), convex and decreasing on \( [x_1 + \delta(\varepsilon), x_1] \) and linear on \( [x_0 + \delta(\varepsilon), x_1 - \eta(\varepsilon)] \), and

\[
\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = \lim_{\varepsilon \to 0^+} \eta(\varepsilon) = 0.
\]
Hence \( \lim_{\epsilon \to 0^+} H_\epsilon^x(x) \) exists for \( x \in [x_0, x_1] \) and is linear and continuous. Thus

\[
\lim_{\epsilon \to 0^+} H_\epsilon^x(x) = -\int_a^b h_\epsilon(u) \, du
\]

(12)

by (11) and the fact that the right-hand side is piecewise linear and continuous.

(k) By (10) and Fubini’s theorem (note that \( |h_\beta| \leq (2/\min_1 l_i, x) \) for \( \beta \geq x \)),

\[
F'_x(x) = \int_a^b h_\beta(x) \, d\beta + \frac{F(b) - F(a)}{b-a} \quad \text{a.e.}
\]

If \( F_a \) is not differentiable at \( x \), \( F'_a \) denotes the right derivative which exists because of the concavity and convexity properties of \( F_a \). As \( v(F'_a) \) and \( v(\int_a^b h_\beta(.) \, d\beta) \) can be computed by only considering partitions of \([a, b]\) contained in a countable dense set, we get

\[
v(F'_a) = v\left( \int_a^b h_\beta(.) \, d\beta \right).
\]

(13)

By (9), (13) and the assumption \( \int_0^\infty s(x) \, dx < \infty \), it follows that

\[
\sup_{a>0} v(F'_a) < \infty.
\]

(14)

By Helly’s selection principle either there exists a function \( f: [a, b] \to \mathbb{R} \) such that \( F'_a \to f \) pointwise for some sequence \( \alpha_j \to 0^+ \) or there is a \( x_0 \in [a, b] \) such that \( |F'_a(x_0)| \to \infty \) for some sequence \( \alpha_j \to 0^+ \). In the second case we have without restriction of generality \( F'_a(x) \to \infty \) for all \( x \in [a, b] \) (use (14)) so that \( F'_a(x) = \int_a^x F'_a(u) \, du \to \infty \) for \( x \in (a, b] \). Thus this possibility is excluded. In the first case however,

\[
F(x) - F(a) = \lim_{j \to \infty} F_a(x) - F(a) = \lim_{j \to \infty} \int_a^x F'_a(t) \, dt = \int_a^x f(t) \, dt.
\]

(15)

(l) We shall now prove

\[
v(f) = 4 \int_0^\infty s(x) \, dx.
\]

(16)

Firstly, by (k) and (9),

\[
v(f) \leq \liminf_{j \to \infty} v(F'_a) = 4 \int_0^\infty s(x) \, dx.
\]

(17)

Next we show that

\[
v(F'_a) \leq \sum |D^+ F(x) - D^- F(x')|,
\]

(18)
where $D^+$ and $D^-$ denote right and left derivative and the sum is taken over all components $[x', x]$ of $I_a \cup \bar{I}_a$. Note that because of $u(f) < \infty$,

$$
\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_x^{x+\varepsilon} f(t) \, dt \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_x^{x-\varepsilon} f(t) \, dt
$$

exist for all $x \in (a, b)$, are continuous from the right resp. left and are both equal to $F'(x)$ almost everywhere (see [2]). Especially, the right-hand sum in (18) is well-defined.

Let us consider an arbitrary component $[x_1, x_2]$ of $I_a$. Then $F_a(x_1) = F(x_1) + \alpha$, $F_a(x_2) = F(x_2) + \alpha$, and $F_a$ is convex on $[x_1 - \varepsilon, x_2 + \varepsilon]$ for $\varepsilon > 0$ so small that $(x_1 - \varepsilon, x_2 + \varepsilon) \subset I_a \cup J_a$. Thus the total variation of $F'_a$ in $[x_1 - \varepsilon, x_2 + \varepsilon]$ is equal to $F'_a(x_2 + \varepsilon) - F'_a(x_1 - \varepsilon)$. On the other hand, by definition of $I_a$ and $\varepsilon$ it is clear that for all $\delta \in (0, \varepsilon)$

$$
F_a(x_2 + \delta) < F(x_2 + \delta) + \alpha, \quad F_a(x_1 - \delta) < F(x_1 - \delta) + \alpha.
$$

Hence,

$$
D^+ F(x_2) = \lim_{\delta \to 0^+} \delta^{-1} \int_{x_2}^{x_2 + \delta} f(t) \, dt
$$

$$
= \lim_{\delta \to 0^+} \delta^{-1} [F(x_2 + \delta) + \alpha - (F(x_2) + \alpha)]
$$

$$
\geq \lim_{\delta \to 0^+} \delta^{-1} [F_a(x_2 + \delta) - F_a(x_2)]
$$

$$
= F'_a(x_2 + \varepsilon)
$$

(19)

(the last equation follows, because $F_a$ is linear on a set containing $[x_2, x_2 + \varepsilon]$). Similarly it is seen that $D^- F(x_1) \leq F'_a(x_1 - \varepsilon)$. Therefore the total variation of $F'_a$ in $[x_1 - \varepsilon, x_2 + \varepsilon]$
is at most as large as $|D^+F(x_2) - D^-F(x_1)|$. An analogous argument applies to the
components of $F'_a$. $v(F'_a)$ is the sum of the above estimated total variations.

This yields (18).

Since $D^+F(x)$ ($D^-F(x)$) is continuous from the right (left) and almost everywhere
equal to $f(x)$, we obtain from (18) that

$$v(F'_a) \leq v(f) \quad \text{for all } \alpha > 0. \quad (19)$$

(17) and (19) together imply (16).

(m) Finally suppose that there is a $f: [a,b] \to \mathbb{R}$ with bounded variation for which
$F(x) = \int_a^x f(t) \, dt$ for all $x \in [a,b]$. As in (1) it is shown that $v(F'_a) \leq v(f)$ for all $\alpha > 0$. As in
(k) it is then proved that there is a $\tilde{f}$ of bounded variation coinciding with $f$ almost
everywhere such that $\tilde{f}(x) = \lim_{j \to \infty} F'_{a,j}(x)$ for all $x \in [a,b]$. Note that also

$$\int_a^\infty s(\beta) \, d\beta \leq v(F'_a) \leq v(\tilde{f}) \quad \text{for all } \alpha > 0. \quad (20)$$

Thus

$$\int_0^\infty s(\beta) \, d\beta < \infty, \quad (21)$$

and the first part of the theorem applies.

REFERENCES

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