GLOBAL ANALYSIS OF ONE-DIMENSIONAL VARIATIONAL PROBLEMS

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From the global analytical point of view a one-dimensional variational problem consists in extremising a differentiable action/cost function $f : X \to \mathbb{R}$, where $X$ is an infinite-dimensional manifold of paths in a manifold $M$, over a subset $\Omega \subset X$ of admissible paths, for example those satisfying some regularity conditions, boundary conditions or other constraints. Thus, a solution to the variational problem is a critical point of the restriction $f | \Omega$.

A standard criterion for existence of critical points is the Palais–Smale condition. If this condition is satisfied then the gradient flow associated with $f$ is well behaved, and we are guaranteed not only existence of critical points but also existence of a minimum. Moreover it is then possible to relate the total number of critical points to topological properties of $\Omega$.

This thesis is about methods for proving that a one-dimensional variational problem satisfies the Palais–Smale condition. The methods are demonstrated with examples motivated by interpolation, approximation, geometric and optimal control problems in Riemannian manifolds. To begin with we consider conditional extremals: the critical points of $\frac{1}{2} \int_I \| \dot{x} - A \|^2 \, dt$, where $I$ is the unit interval, $x : I \to M$ is a path on $M$, $\dot{x}$ is the tangent vector along $x$, $A$ is an arbitrary vector field on $M$ and the admissible paths satisfy fixed boundary conditions. Our results on this topic have appeared in [3]. Next we treat problems with higher order covariant derivatives in the action, such as Riemannian cubics in tension: the critical points of $\frac{1}{2} \int_I \| \nabla_t \dot{x} \|^2 - \tau^2 \| \dot{x} \|^2 \, dt$ with $\tau \in \mathbb{R}$ constant. These results have appeared in [1]. This is followed by an investigation of curves with minimum total squared curvature $\int_I k^2 \, ds$ subject to a fixed length constraint (see [2]). Such curves are known as elastica and this is the first example we encounter with a constraint that is not a boundary condition. Finally, we consider a

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class of problems known as sub-Riemannian, where the admissible paths are required to be tangent to a nonintegrable distribution on $M$.


References


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