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Block idempotents and the Brauer correspondence

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Let H be a subgroup of a finite group G. In this paper the Brauer correspondence between blocks of H and blocks of G is characterized in terms of a relationship among the block idempotents.

Let R be an integral domain (commutative with unit element) satisfying the ascending chain condition on ideals. Suppose that N, the radical of R, is a principal ideal $(N = (\pi))$ and that $R^* = R/N$ satisfies the descending chain condition on ideals. Let N contain a rational prime $p = 1 + 1 + \ldots + 1$. Assume that R is complete with respect to the topology induced by N. Then either $\pi = 0$ in which case $R = R^*$ is a field of characteristic p, or R is a complete discrete valuation ring.

Let G be a finite group with a subgroup H. Let $R(G)$ denote the group algebra of G with coefficients in R. Then $R(H)$ is a subalgebra of $R(G)$. Let $Z(G)$ and $Z(H)$ denote the centers of $R(G)$ and $R(H)$, respectively. If E is a primitive idempotent in $Z(G)$ the block $B = B(E)$ is the collection of all (right) $R(G)$-modules V with $VE = V$. To each primitive idempotent E in $Z(G)$ there is associated a linear character $\psi : Z(G) \to R^*$ with $\psi(E) = 1^*$ and $\psi(F) = 0^*$ for any primitive idempotent $F \neq E$ in $Z(G)$. We shall assume that R^* is a splitting field for $R^*(G)$ and for $R^*(H)$. Then the linear characters on $Z(G)$ and $Z(H)$ will be in one-to-one correspondence with the primitive idempotents in $Z(G)$ and $Z(H)$ respectively (see [2]). We shall write

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$B \leftrightarrow E \leftrightarrow \psi$ where B is the block of $R(G)$ corresponding to the primitive idempotent in E in $Z(G)$ and ψ is the character of $Z(G)$ associated to E.

For any $\alpha = \sum_{g \in G} a_g g$ in $Z(G)$, let

$$\theta(\alpha) = \sum_{g \in H} a_g g .$$

Then $\theta : Z(G) \to Z(H)$ is an R-module homomorphism.

DEFINITION. Let $b \leftrightarrow e \leftrightarrow \lambda$ be a block of $R(H)$. If the map $\psi = \lambda \circ \theta : Z(G) \to R^*$ is a linear character of $Z(G)$ corresponding to a block $B \leftrightarrow E \leftrightarrow \psi$ of $R(G)$, then we say that $b^G = B$ is defined. The correspondence sending b to b^G is called the Brauer correspondence.

Let $\mathrm{rad}Z(H)$ denote the radical of $Z(H)$. We can now state the main result of this paper.

THEOREM. *Let $B \leftrightarrow E \leftrightarrow \psi$ and $b \leftrightarrow e \leftrightarrow \lambda$ be blocks of $R(G)$ and $R(H)$ respectively. Then $b^G = B$ if and only if $\theta(F)e$ is in the radical of $Z(H)$ for every primitive idempotent $F \neq E$ in $Z(G)$.*

Proof. Let $K_1, \ldots, K_s$ be the conjugate classes of G. The ring $Z(G)$ has an R-basis consisting of the elements $\hat{K}_1, \ldots, \hat{K}_s$ where

$$\hat{K}_i = \sum g \quad (g \in K_i) .$$

In the same way if $L_1, \ldots, L_t$ are the conjugate classes in H then $\hat{L}_1, \ldots, \hat{L}_t$ span $Z(H)$. Let $A = A(G : H)$ be the R-algebra generated by the class sums $\hat{K}_1, \ldots, \hat{K}_s$ and $\hat{L}_1, \ldots, \hat{L}_t$. That is, A is the minimal subalgebra of $R(G)$ which contains both $Z(G)$ and $Z(H)$. The products $\hat{K}_i\hat{L}_j$ $(i = 1, \ldots, s, j = 1, \ldots, t)$ span A as an R-algebra.

The map θ can be easily extended to a map $\theta' : A \to Z(H)$ given by

$$\theta'(\hat{K}_i\hat{L}_j) = \theta(\hat{K}_i)\hat{L}_j = \left(\sum \hat{L}_k\right)\hat{L}_j \quad (i = 1, \ldots, s; j = 1, \ldots, t) ,$$

where the sum is over those k with $L_k \subseteq K_i$. θ' is the projection of

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A onto $Z(H)$.

Suppose $b^G = B$. Let F be any primitive idempotent in $Z(G)$ with $F \neq E$. If $f \leftrightarrow \phi$ is any primitive idempotent in $Z(H)$ with $f \neq e$ then $fe = 0$ and $\phi(\theta(F)e) = 0$. Also $\lambda\{\theta(F)e\} = \lambda\circ\theta(F) = \psi(F) = 0$. Since $\theta(F)e$ is in the kernel of every linear character on $Z(H)$ it is in the radical of $Z(H)$.

Conversely suppose that $\theta(F)e$ is in $\mathrm{rad}Z(H)$ for every primitive idempotent F in $Z(G)$ with $F \neq E$. Let $E = E_1, E_2, \ldots, E_m$ be all of the primitive idempotents in $Z(G)$. Then $E_1^*, E_2^*, \ldots, E_m^*$ are all of the primitive idempotents in $Z^*(G) = Z(G)/(\pi)Z(G)$. Since $\sum_{i=1}^{m} E_i^* = 1^*$, we have that $e^* = \sum_{i=1}^{m} E_i^* e^*$. Now for any $j = 1, \ldots, s$,

$$\hat{K}_j^* e^* = \sum_{i=1}^{m} K_j^* E_i^* e^* = \sum_{i=1}^{m} \psi_i(\hat{K}_j) E_i^* e^* ,$$

where $E_i \leftrightarrow \psi_i$. Then

$$\begin{aligned} \lambda\left\{\theta(\hat{K}_j)\right\}e^* &\equiv \theta(\hat{K}_j^*)e^* \equiv \theta'(\hat{K}_j^* e^*) \\ &\equiv \psi(\hat{K}_j)\theta'(E^* e^*) \\ &\equiv \psi(\hat{K}_j)e^* \mod\{\mathrm{rad}Z^*(H)\} , \end{aligned}$$

since $e^* = \theta'(e^*) = \theta'\left\{\sum E_i^* e^*\right\} \equiv \theta'(E^* e^*) \mod\{\mathrm{rad}Z^*(H)\}$. Hence $\lambda \circ \theta = \psi$ on $Z(G)$.

COROLLARY. *Let* $B \leftrightarrow E \leftrightarrow \psi$ *and* $b \leftrightarrow e \leftrightarrow \lambda$ *be blocks of* $R(G)$ *and* $R(H)$ *respectively. If* b^G *is defined then* $b^G = B$ *if and only if* $\theta(E)e \equiv e \mod\{\mathrm{rad}Z(H)\}$.

This follows directly from the theorem since if $E_1, \ldots, E_s$ are all of the primitive idempotents in $Z(G)$ then

$$e = \theta'(e) = \theta'\left\{\sum E_i e\right\} = \sum \theta(E_i)e .$$

REMARK 1. The condition $\theta(E)e \equiv e \mod(\mathrm{rad} Z(H))$ is not sufficient to guarantee that b^G is defined. For example let $G = \langle x \| x^3 = 1 \rangle$, $H = \{1\}$, and $R = Z_2[\alpha]$, the algebraic extension of the field with two elements by an element α with $\alpha^2 + \alpha + 1 = 0$. Then the elements $E_1 = 1 + x + x^2$, $E_2 = 1 + \alpha x + \alpha^2 x^2$, and $E_3 = 1 + \alpha^2 x + \alpha x^2$ are all of the idempotents in $Z(G)$. But $e = 1$ is the only idempotent in $Z(H)$ and $\theta(E_1)e = \theta(E_2)e = \theta(E_3)e = e$.

REMARK 2. If H is a normal subgroup of G then $B = b^G$ if and only if $Ee = e$ (see [4]).

REMARK 3. Using these results it is easy to prove the well known result that if $b^G = B$ then any defect group of b is conjugate to a subgroup of some defect group of B (see [1]). For suppose $Z(G : H)$ denotes the centralizer of H in $R(G)$. If D is a defect group of b then by [3] the R-subalgebra $Z_D(G : H)$, spanned by those class sums of conjugate-in-H elements of G with defect groups conjugate to subgroups of D, is an ideal in $Z(G : H)$. Since $\theta(E)e \equiv e \mod(\mathrm{rad} Z(H))$ it is clear that some defect group of B must contain D.

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