DIFFUSION CHARACTER IN FOUR-DIMENSIONAL VOLUME-PRESERVING MAP

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Abstract. Due to the existence of invariant tori, chaotic sea and hyperbolic structures in higher dimensional phase space of a volume-preserving map, the diffusion route of chaotic orbits will be complicated. The velocity of diffusion will be very slow if the orbits are near an invariant torus. In order to realize this complicated diffusion phenomenon, in this paper we study the diffusion characters in the different regions, i.e., chaotic, hyperbolic and invariant tori's regions. We find that for the three different regions, the diffusion velocities are different. The diffusion velocity in the vicinity of an invariant torus is the slowest one.

Key words: Volume-Preserving map – Invariant tori – Diffusion

1. Introduction

According to the KAM theorem for Hamiltonian systems with two degrees of freedom, the two dimensional invariant tori will prevent the escape of orbits on a three-dimensional energy surface. When the degrees of freedom, n, exceeds two, the n-dimensional invariant tori cannot divide the (2n-1)-dimensional energy surface into two disconnected parts, so the escape will appear. Because of the existence of n-dimensional invariant tori, the escape route across a net of invariant tori will be complicated and the velocity of escape will be very slow, this is called Arnold diffusion. This kind of diffusion can hardly be detected by numerical methods, as pointed out by Laskar with a four-dimensional symplectic map similar to the so-called Froeschlé map (Laskar 1993). Efthymiopoulos et al. (1998) also conclude that the diffusion can be practically ignored.

The problems related to invariant manifolds and diffusion orbits in four-dimensional symplectic maps have been carefully studied (e.g. Froeschlé 1971, Froeschlé 1972, Ding et al. 1990, Laskar 1993, Efthymiopoulos et al. 1998, etc.). In this paper, we study a similar problem in a more general kind of maps, i.e., four-dimensional volume-preserving maps. In order to realize the complicated diffusion phenomenon, it's worth to study at first the diffusion characters in chaotic, hyperbolic and invariant tori's regions, respectively. We investigate the above problem using a four-dimensional volume-preserving map obtained by coupling two area-preserving maps with a perturbation parameter, which possesses the hyperbolic and parabolic fixed point, respectively. In terms of the results on the three-dimensional volume-preserving map, the ordered region near the unstable fixed points will be changed into a chaotic one by perturbation, but for the ordered region distant from it the invariant tori will survive (Sun et al. 1988, Zhang et al. 1989). We suspect

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that we might find some invariant tori, chains of islands and hyperbolic structure in the above four-dimensional volume-preserving map.

2. The Map

We study the following volume-preserving map with four dimension.

$$T_{hp}\begin{cases} x_{n+1}= s\,(x_n \,cos\,\varphi_n - y_n \,sin\,\varphi_n), \\ y_{n+1}= s^{-1}\,(x_n \,sin\,\varphi_n + y_n \,cos\,\varphi_n) + c\,sin(x_{n+1}+z_{n+1}), \\ z_{n+1}= z_n - b\,t_n^3, \\ t_{n+1}= z_n + t_n - b\,t_n^3 + c\,sin(x_{n+1}+z_{n+1}), \end{cases} \tag{1}$$

$$\varphi_n = (x_n^2 + y_n^2)^k. \tag{2}$$

The map T_{hp} is the coupling of the following two area-preserving maps

$$T_h\begin{cases} x_{n+1}= s\,(x_n \,cos\,\varphi_n - y_n \,sin\,\varphi_n), \\ y_{n+1}= s^{-1}\,(x_n \,sin\,\varphi_n + y_n \,cos\,\varphi_n), \end{cases} \tag{3}$$

$$\varphi_n = (x_n^2 + y_n^2)^k. \tag{4}$$

and

$$T_p\begin{cases} z_{n+1}= z_n - b\,t_n^3, \\ t_{n+1}= z_n + t_n - b\,t_n^3. \end{cases} \tag{5}$$

Here b, s, k are parameters ($s \neq 1$) and c the perturbation parameter (coupling parameter). We take $s = 1.05, k = -1.5, b = 1.5$ and $c = 0.03$, in which case the map T_{hp} is not defined at the origin. Obviously, the map T_{hp} is symmetric with respect to the origin. As can be easily verified, the map T_{hp} is volume-preserving but not symplectic.

At first, we explore the structure of phase space for the map T_{hp}, i.e., to look for ordered and chaotic regions and the hyperbolic structure. We have investigated the structure of phase space of the map T_{hp} by computing the LCIs (Lyapunov Characteristic Indicators, the approximate values of Lyapunov Characteristic Exponents up to the finite time of computation) and exploring fixed points and checking their stability. Fig.1(a) and Fig.1(b) are the projections of some invariant tori (roughly approximated by quasi-periodic orbits) and chaotic sea onto the planes (x, y) and (z, t), respectively. These figures display the global structure of phase space. We know that the dimensionality of the invariant tori is two and these tori can not divide the four-dimensional phase space into two disconnected parts. As a result, the chaotic orbits diffuse around the invariant tori, as easily seen in the Fig.1(a) and Fig.1(b).

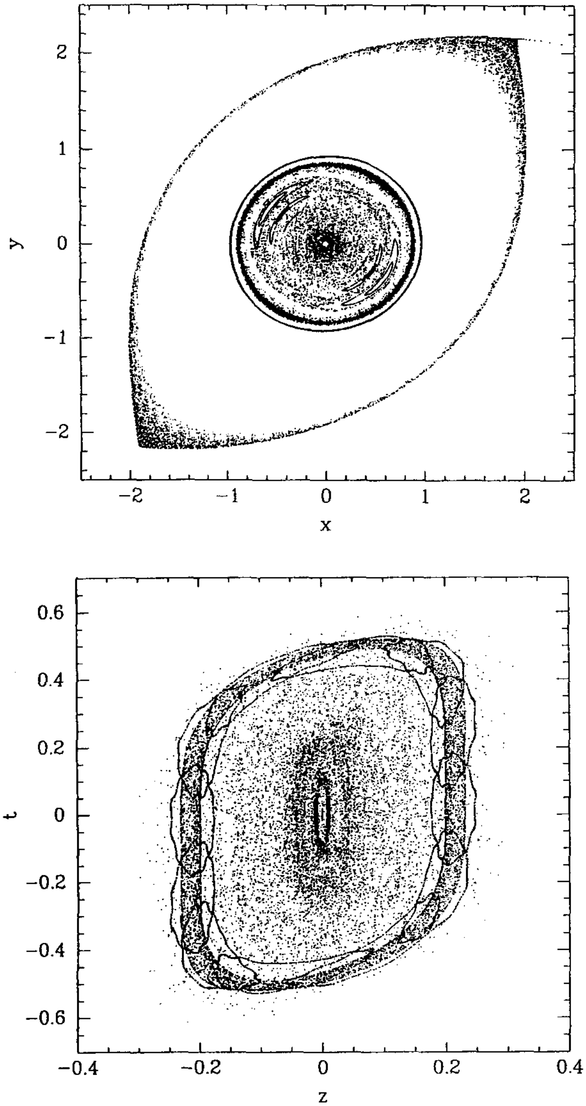

Fig. 1. The projections on the two coordinate planes of the quasi-periodic orbits starting from the following initial points: (0.376, 0.582, -0.0002, 0.008), (0.232, 0.658, 0.00019, 0.0038), (0.68, 0.68, 0.2125, 0.4250), (0.356, -0.246, -0.012, 0.003), (0.392, -0.268, 0, 0.0058), (0.2, -0.5, -0.0127, 0.0016), (0.40, -0.37, -0.0127 ,0.0016), (0.16, -0.65 ,0, 0.009), (0.40 ,-0.55, 0 ,0.009), (0.61 ,0.61, 0.190625 ,0.38125) and their symmetric points with respect to the origin for the map T_{hp}, and the chaotic orbits starting from the following initial points: (0.261101993556, 0.341296119830, -0.007526015390, 0.001646639192), (1.918131071904, 2.156373097694 -0.028487926171, -0.000415023486), (0.68 ,0.53 ,0.190625, 0.381250). ($s = 1.05, k = -1.5, b = 1.5$ and $c = 0.03$).

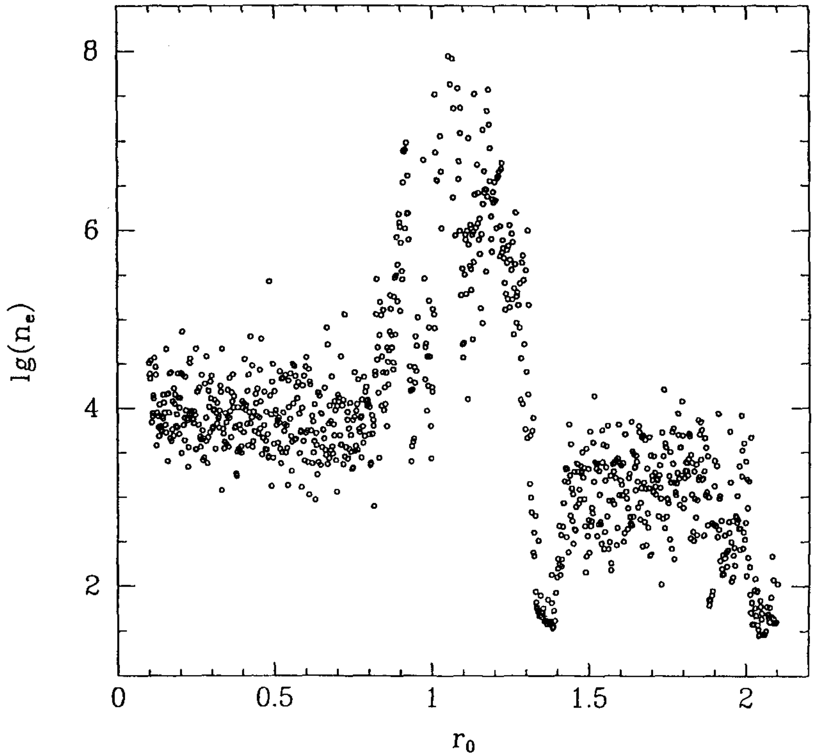

Fig. 2. Distribution of escape time of orbits with respect to the distance d_0 along the line (6).

3. Diffusion Characters in Different Regions

In order to clarify the diffusion of orbits in the phase space globally, we have made a transversal exploration of escape time along the line passing through the origin and the point of which the projections on the (x,y) and (z,t) planes are, respectively, the farthest boundary point of the ordered region for the maps T_h and T_p. The equations of this line is as follows

$$5x = 5y = 16z = 8t. \tag{6}$$

We take 1001 initial points on the line (6), of which the distances to the origin are as follows

$$r_0 = 0.1 + 0.002i, \quad i = 0, 1, ..., 1000. \tag{7}$$

Defining the escape of orbits as $r = \sqrt{x^2 + y^2 + z^2 + t^2} > 2.5$, we calculate the escape time for each orbit. Fig.2 shows the result of this exploration (not including the orbits on the invariant tori). We find a slow escape region near $r = 1.05$, where exist most of the explored initial points on invariant tori. It implies that the escape

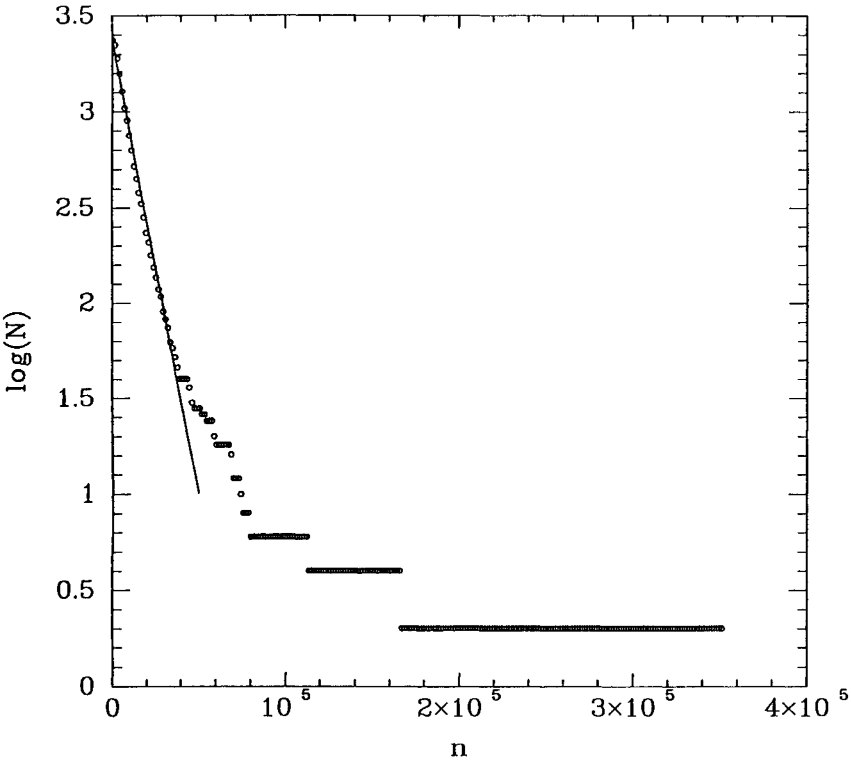

Fig. 3. Diagrams of the number of orbits having not escaped before the iteration number n, for the case (a). circle: computational values, curve: fitted values.

velocity of orbits near the invariant tori is much slower than that in the chaotic region, and this is due to the "stickiness" effect of invariant tori as can be seen below. From Fig.2 we can also see that even in the "dense" region of invariant tori, there exist chaotic zones with faster escape. In the following we will investigate the diffusion characters in different regions.

(a) Chaotic sea

We choose 2352 initial points (x_0, y_0, z_0, t_0) in the chaotic region near the origin as follows

$$\begin{aligned} &(x_0, y_0, z_0, t_0) = (i, j, l, m)s, \\ &s = 0.1, i, j, l, m = 0, \pm 1, \pm 1.5, \pm 2 \ (excluding\, i = j = 0) \end{aligned} \tag{8}$$

Defining the escape of orbits as above, i.e., $r = \sqrt{x^2 + y^2 + z^2 + t^2} > 2.5$, we count the number, N, of orbits which have not escaped yet before the number of iterations n. In Fig.3 the circles stand for the numerical results of $\log(N)$ versus n. Because there are positive LCEs for the chaotic orbits, we try to fit the numerical

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results by an exponential law, i.e.,

$$N = N_0 \times 10^{-\alpha n} \tag{9}$$

However, this law can not be applied for large n, as can be easily seen from Fig.3. Actually, there should exist some "island" tori which exert the "stickiness" effect on the orbits near them (Meiss and Ott 1985, Meiss and Ott 1986, Lee 1988, Ding et al. 1990, Lai 1992). Therefore, we fit the numerical results only for $n < 40000$ (including about 99% of the explored orbits) and obtain an analytic curve with $N_0 = 2532$ and $a = 4.735163 \times 10^{-5}$ (see Fig.3). This analytic curve is in good agreement with the numerical values for $n < 40000$. Accordingly, we suspect that the diffusion of orbits in a "complete" chaotic sea, i.e., the region where the area occupied by islands can be neglected, should possess the exponential law. As it is difficult to find out small tori in four-dimensional phase space, the above point is just a conjecture.

(b) Vicinity of invariant torus

There exists a slow escape region near the point on the line (6), of which the distance to the origin is $r = 1.05$ (see Fig.2).

By computing the LCIs and drawing the projective figures of the orbit starting from this point, we find that the point is on an invariant torus. In the following, we study the diffusion character of orbits in the vicinity of this torus by taking 2401 initial points (x_0, y_0, z_0, t_0) as follows

$$\begin{aligned} &(x_0, y_0, z_0, t_0) = (x_c, y_c, z_c, t_c) + (i, j, l, m)s, \\ &s = 0.005, i, j, l, m = 0, \pm1, \pm2, \pm3. \end{aligned} \tag{10}$$

At first, we look for a region in which the orbits will stay for quite a long time ($n \geq 4096$) and we choose this region as small as possible so that it contains as less points far from the torus as possible. According to our testing calculation, we define the escape of orbits as they leave the following zone,

$$\begin{aligned} G = \{ \ &(x, y, z, t) : \\ &x \in (-0.98, 0.98), y \in (-0.94, 0.94), z \in (-0.26, 0.26), t \in (-0.55, 0.55), \\ \sqrt{x^2 + y^2} \ &\in (0.86, 0.99) \} \end{aligned} \tag{11}$$

(Note: $\sqrt{x^2 + y^2}$ is an invariant of map T_{hp} with $s = 1$ and $c = 0$.)

Fig.4 indicates the diagram of $\log(N)$ versus n with 605 orbits escaped before $n = 10^8$ iterations, where N is the number of orbits in the region G at the iteration number n. We note that only a small fraction of the orbits escape before $n = 10^7$ iterations, which implies that the diffusion in the vicinity of invariant tori is very slow. This is due to the "stickiness" effect of invariant tori, but here it is different from the usual one which has an algebraic decay law of diffusion (Meiss and Ott 1985, Ding et al. 1990). According to the method used to study the usual "stickiness" effect in related papers, we suspect that the algebraic decay law

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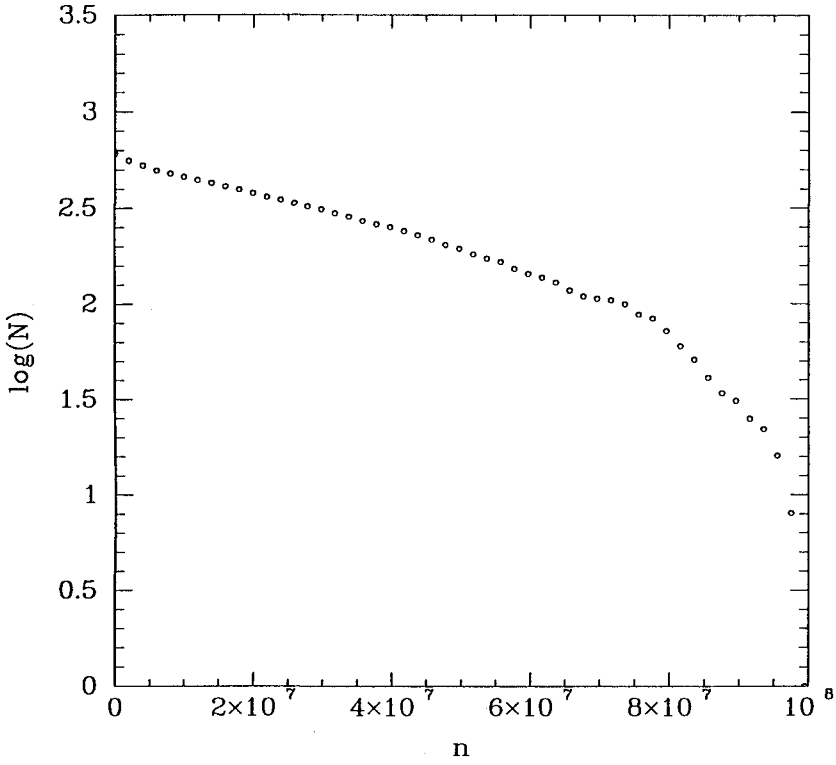

Fig. 4. Diagram of the number of orbits having not escaped before the iteration number n, for the case (b).

should be the "averaged" law of diffusion over the chaotic region with invariant islands and cantori. The orbits diffuse sometimes "freely" through chaotic sea, and sometimes by crossing cantori and passing around islands, so the usual "stickiness" effect of invariant tori is actually modified by the diffusion process in chaotic sea. If a region of higher dimensional phase space is filled "densely" by invariant tori, the "stickiness" effect would appear to be that in our case.

(c) Vicinity of hyperbolic-elliptic fixed point

According to the exploration about the fixed points, we find that

$$(x_h, y_h, z_h, t_h) = (1.918131071904, 2.156373097694, -0.028487926171, -0.000415023486)$$

is a hyperbolic-elliptic fixed point. Choosing 2401 initial points (x_0, y_0, z_0, t_0) in its vicinity as follows

$$\begin{aligned} &(x_0, y_0, z_0, t_0) = (x_h, y_h, z_h, t_h) + (i, j, l, m)s \\ &s = 10^{-5}, i, j, l, m = 0, \pm 1, \pm 2, \pm 3, \end{aligned} \tag{12}$$

we study the diffusion character in the vicinity of the above fixed point. The orbit with initial point sufficiently close to the hyperbolic-elliptic fixed point should

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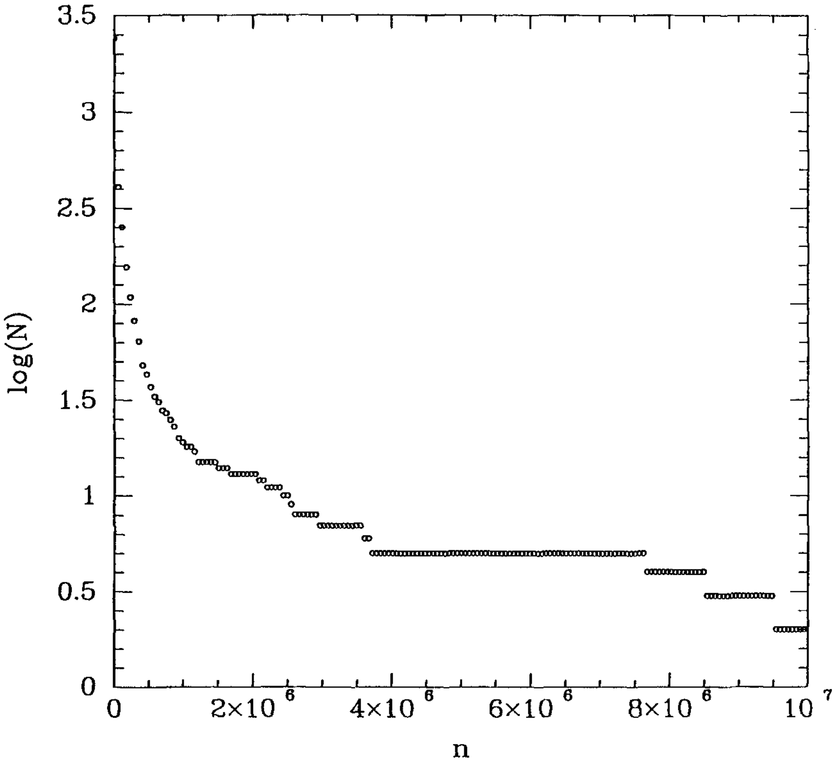

Fig. 5. Diagram of the number of orbits N having not escaped before the iteration number n, for the case (c).

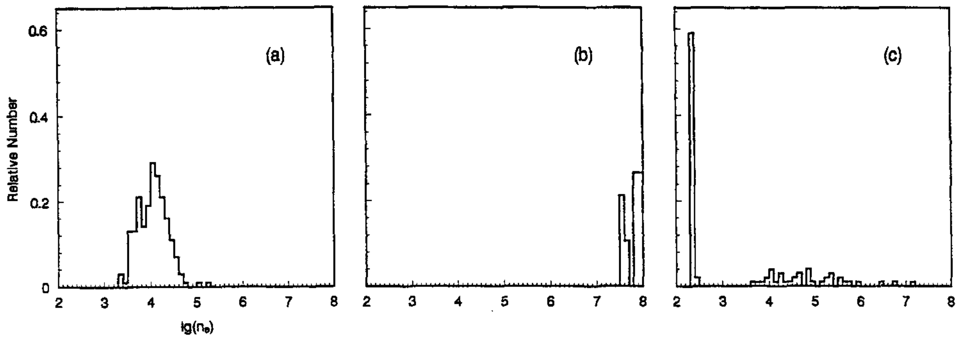

Fig. 6. Distribution of the relative number of orbits escaping at the iteration number n for (a) chaotic sea, (b) vicinity of invariant torus and (c) vicinity of hyperbolic-elliptic fixed point.

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wander near the hyperbolic structure on the plane (x, y) and near the origin on the plane (z, t) for quite a long time. After testing calculation, we find that, as soon as the distance d between the fixed point and the returning point of orbit to the vicinity of the fixed point exceeds $\bar{d} = 0.670213829709$ or the distance d' of the orbit to the origin exceeds 10, the orbit will diffuse to very distant point from the origin ($d' > 10^5$) with no return, so we regard $d > \bar{d}$ or $d' > 10$ as escape. Fig.5 shows the variation of N, number of the orbits having not escaped yet, with the number of iteration n. We find that this result is similar to that in chaotic sea, except that there are some orbits with much slower diffusion, of which the initial points are very close to the hyperbolic-elliptic fixed point. This implies that the hyperbolic-elliptic fixed point also has "stickiness" effect on some nearby orbits. Another interesting feature is the appearance of fast escape orbits starting near the hyperbolic-elliptic fixed point, which causes fast decreasing of N for small n (about 49% of the explored orbits escaping before $n = 145$). Contopoulos et al. (1997) have found the "hole" of escape embedded in stickiness region near an invariant curve of two-dimensional area-preserving map. The above-mentioned feature shows that the "holes" of escape also exist in the vicinity of the hyperbolic-elliptic fixed point of four-dimensional volume-preserving map.

In the above studies, we discuss separately the diffusion characters in three different regions, and now we will compare the diffusion velocities for the different cases with the same definition of escape. For this purpose, we choose randomly 250 initial points in each region as above, and regard an orbit as escape if the distance of an orbit point to the origin exceeds 100. In the chaotic region and the vicinity of hyperbolic-elliptic fixed point, all of the 250 orbits escape, but in the vicinity of invariant torus, there are only 45 orbits escaping before $n = 10^8$ iterations. Fig.6 exhibits the results. We can see that the diffusion velocities in the different region are different. The diffusion in chaotic sea is the fastest, while the slowest one is in the vicinity of the invariant torus. In the vicinity of hyperbolic- elliptic fixed point, the diffusion velocity is between the above two cases, because of the existence of narrow weak "stickiness" zone nearby the perturbed hyperbolic-elliptic structure.

4. Conclusions

In a higher dimensional volume-preserving map, due to the existence of invariant tori, chaotic sea and hyperbolic structure, the diffusion of orbits will be very complicated. In this paper, we study the diffusion characters in the different regions, respectively, and compare the diffusion velocities with each other. This kind of work is the base of realizing the global diffusion in phase space. We find that for the three explored regions, the diffusion velocities are different and have different characters. The diffusion velocity in the vicinity of an invariant torus is the slowest one, the reason for this is the "stickiness" effect. We believe that the "stickiness" effect would take the main responsibility for very slow diffusion in higher dimensional phase space.

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References

Contopoulos, G., Voglis, N., Efthymiopoulos, C., Froeschlé, C., Gonczi R., Lega E., Dvorak R. and Lohinger E.: 1997, Transition spectra of dynamical systems, *Celest. Mech. and Dyn. Astro.*, **67**, 293

Ding, M.Z., Bountis, T. and Ott, E.: 1990, Algebraic escape in higher dimensional Hamiltonian systems, *Physics Letter A*, **151**, 395

Efthymiopoulos, C., Voglis, N. and Contopoulos, G.: 1998, Diffusion and transient spectra in a 4-d symplectic mapping, in "Advances in Discrete Mathematics and Applications, Volume 1, Analysis and Modelling of Discrete Dynamical Systems", D. Benest and C. Froeschlé (eds), Gordon and Breach Science Publishers, 91

Froeschlé, C.: 1971, On the number of isolating integrals in systems with three degrees of freedom, *Astrophys. Space Sci.*, **14**, 110

Froeschlé, C.: 1972, Numerical study of a four-dimensional mapping, *Astron. Astrophys.*, **16**, 172

Lai, Y.C., Ding. M.C. and Blümel, R.: 1992, Algebraic decay and fluctuations of the decay exponent in Hamiltonian systems, *Phy. Rev. A*, **46**, 4661

Laskar, J.: 1993, Frequency analysis for multi-dimensional systems. Global dynamics and diffusion, *Physica D*, **67**, 257

Lee, K.C.: 1988, Long-time tails in a chaotic system, *Phy. Rev. Lett.*, **60**, 1991

Meiss, J.D. and Ott, E.: 1985, Markov-tree model of intrinsic transport in Hamiltonian systems, *Phy. Rev. Lett.*, **55**, 2741

Meiss, J.D and Ott, E.: 1986, Markov-tree model of transport in area-preserving maps, *Physica D*, **20**, 387

Morbidelli, A. and Giorgilli, A.: 1995, Superexponential stability of KAM tori, *J. Stat. Phys*, **78**,1607

Sun, Y.S. and Yan, Z.M.: 1988, A perturbed extension of hyperbolic twist mapping, *Celest. Mech. and Dyn. Astro.*, **42**, 369

Zhang, T.L. and Sun, Y.S.: 1989, Behaviour of a class of perturbed measure-preserving mappings. *J. Nanjing Uni.*, **25**, 187 (In Chinese).