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ON GENERALIZATION OF NAKAYAMA'S LEMMA

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Abstract. Let R be a commutative ring with identity. We will say that an R-module M has Nakayama property, if $IM = M$, where I is an ideal of R, implies that there exists $a \in R$ such that $aM = 0$ and $a - 1 \in I$. Nakayama's Lemma is a well-known result, which states that every finitely generated R-module has Nakayama property. In this paper, we will study Nakayama property for modules. It is proved that R is a perfect ring if and only if every R-module has Nakayama property (Theorem 4.9).

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1. Introduction. Throughout this paper all rings are commutative with identity, and all modules are unitary. Also we consider R to be a ring, $J(R)$ the intersection of all maximal ideals of R and M a unitary R-module. By $N \leq M$, we mean N is a submodule of M. If $N \leq M$, then $(N : M) = \{t \in R \mid tM \subseteq N\}$.

The set of maximal submodules (resp. ideals) of M (resp. R) is denoted by $Max(M)$ (resp. $Max(R)$). Also we consider

$$Maxx(M) = \{N \leq M \mid (N : M) \in Max(R)\}.$$

DEFINITION. We will say that an R-module M has Nakayama property, if $IM = M$, where I is an ideal of R, implies that there exists $a \in R$ such that $aM = 0$ and $a - 1 \in I$.

Nakayama's Lemma is a well-known result, which states that every finitely generated R-module has Nakayama property (see [**9**, Theorem 2.2]).

We will try to substitute the condition finitely generated for M with weaker or different conditions, and we will study the modules having Nakayama property.

Recall that a module M is said to be *finitely annihilated* if there exists a finite subset T of M with $Ann\ T = Ann\ M$. The finitely annihilated concept is believed to be due to P. Gabriel [**7**]. This subject has been studied by some authors under the name H-condition (see e.g. [**10**]). Evidently, every finitely generated module is finitely annihilated. However, the converse is not correct. For example, let F be a non-zero free module. Then for any element x of a basis of F, we have $Ann\ F = 0 = Ann\ \{x\}$. Thus every (infinite rank) free module is finitely annihilated. Also the $\mathbb{Z}$-module Q is finitely annihilated, but not finitely generated.

A ring over which every non-zero module has a maximal submodule is called a *Max ring*. These rings have been characterized in [**6**]. Also a ring R is called a *perfect ring* if R has DCC property on principal ideals (see [**1**, Theorem 28.4 (Bass)]).

In this paper, we prove the following result:
[Theorem 4.5 and Theorem 4.9]. *Let R be a ring.*

(i) *Consider the following statements:*
   (a) *R is a Max ring;*
   (b) *For any finitely annihilated R-module M and every $m \in Max(R)$, the R_m-module M_m has Nakayama property;*
   (c) *For any finitely annihilated R-module M and every $m \in Max(R)$ containing $Ann\ M$, there exists $N \in Max(M)$ with $(N : M) = m$;*
   (d) *Every finitely annihilated R-module has Nakayama property;*
   (e) *$dim\ R = 0$.*

   Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e).

(ii) *R is a perfect ring if and only if every R-module has Nakayama property.*

2. Some preliminary results. Let M be an R-module and S a multiplicatively closed subset of R. For any $N \leq M_S$, we consider $N^c = \{x \in M \mid x/1 \in N\}$.

A proper submodule N of M is a *prime* submodule of M, if for each $r \in R$ and $a \in M$, the condition $ra \in N$ implies that $a \in N$ or $rM \subseteq N$. In this case, $P = (N : M)$ is a prime ideal of R, and we say N is a *P-prime* submodule of M (see e.g. [**5, 3, 4, 8, 11**]).

Lemma 2.1. *Let M be an R-module and S a multiplicatively closed subset of R.*

(i) *If N is a P-prime submodule of M with $P \cap S = \emptyset$, then N_S is a P_S-prime submodule of M_S as an R_S-module.*
(ii) *If T is a Q-prime submodule of M_S as an R_S-module, then T^c is a Q^c-prime submodule of M.*
(iii) *If $L \in Maxx(M)$, then L is a prime submodule of M.*
(iv) *If M is a flat module and P a prime ideal of R with $PM \neq M$, then PM is a P-prime submodule of M.*

Proof. (i) and (ii) See [**8**, Proposition 1].
(iii) The proof is easy and it is left to the reader.
(iv) The assertion is given by [**3**, Corollary 2.6(i)] and [**5**, Corollary 2.9(i)]. □

The following lemma gives us some information about $Max(M)$ and $Maxx(M)$.

Lemma 2.2. *Let M be a non-zero R-module. Then*

(i) *$Max(M) \subseteq Maxx(M)$.*
(ii) *$Maxx(M) \neq \emptyset$, for every faithfully flat R-module M.*
(iii) *Let M be a free R-module. Then $Max(M) = Maxx(M)$, if and only if $M \cong R$.*
(iv) *If $N \in Maxx(M)$ with $(N : M) = m$, then $N_m \in Maxx(M_m)$.*
(v) *If $m \in Max(R)$ and $L \in Maxx(M_m)$, then $L^c \in Maxx(M)$ with $(L^c : M) = m$.*
(vi) *If M is a projective module, then $Maxx(M) \neq \emptyset$.*
(vii) *$Max(M) \neq \emptyset$ if and only if $Maxx(M) \neq \emptyset$.*

Proof. (i) Suppose that $N \in Max(M)$. Since M/N is a simple module, $M/N \cong R/m$, where m is a maximal ideal of R and $m = Ann(M/N) = (N : M)$. Hence $N \in Maxx(M)$.

(ii) Let $m \in Max(R)$. According to [**9**, Theorem 7.2], for every faithfully flat module M, we have $mM \neq M$. Then $mM \in Maxx(M)$.

(iii) Let $M = \oplus_{j \in J} R$, and m a maximal ideal of R. Consider $N = mM = \oplus_{j \in J} m$. By part (ii), $N \in Maxx(M)$. Now if $|J| > 1$, consider $j_0 \in J$ and $L = \oplus_{j \in J} I_j$, where

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$I_{j_0} = m$ and $I_j = R$, for each $j \in J \setminus \{j_0\}$. Then evidently $N \subset L \subset M$. This shows that $N \notin Max(M)$.

(iv) By Lemma 2.1(iii), N is an m-prime submodule of M. So by Lemma 2.1(ii), N_m is an m_m-prime submodule of M_m. Now since $(N_m : M_m) = m_m \in Max(R_m)$, we have $N_m \in Maxx(M_m)$.

(v) Note that $(L : M_m) \in Max(R_m) = \{m_m\}$. Then Lemma 2.1(iii) implies that L is an m_m-prime submodule of M_m. So according to Lemma 2.1(ii), L^c is an m-prime submodule of M. Thus $L^c \in Maxx(M)$ and $(L^c : M) = m$.

(vi) Let m be a maximal ideal of R such that $M_m \neq 0$. Then M_m is a projective R_m-module. According to [**9**, Theorem2.5], every projective module over a local ring is a free module. Then M_m is a free R_m-module. Now by part (ii), $Maxx(M_m) \neq \emptyset$, and so by part (v), $Maxx(M) \neq \emptyset$.

(vii) Let $N \in Maxx(M)$ and suppose that $(N : M) = P$. Then M/PM is a non-zero vector space over the field R/P. So M/PM has a maximal subspace L/PM. It is easy to see that L is a maximal submodule of M as an R-module. □

We will consider $J_M(R) = \cap\{m \mid m \in Max(R),\ mM \neq M\}$. If $\{m \mid m \in Max(R),\ mM \neq M\} = \emptyset$, then we define $J_M(R) = R$.

Evidently, $J(R) \subseteq J_M(R) = \cap\{(N : M) \mid N \in Maxx(M)\}$.

EXAMPLE 2.3. Let R be a non-local ring and suppose that $m \in Max(R)$. Consider $M = R/m$. Then M is cyclic and $J(R) \subset m = J_M(R)$. Hence, even for a cyclic module, it is not necessary that $J(R) = J_M(R)$.

LEMMA 2.4. *Let M be an R-module and I a proper ideal of R with $IM = M$. Then $Maxx(M_S) = \emptyset$, for $S = \{1 + x \mid x \in I\}$.*

Proof. It is easy to see that $I_S M_S = M_S$ and $I_S \subseteq J(R_S) \subseteq J_{M_S}(R_S)$. On the contrary let $N \in Maxx(M_S)$. From $J_{M_S}(R_S) \subseteq (N : M_S)$, we have $M_S = I_S M_S \subseteq J_{M_S}(R_S)M_S \subseteq (N : M_S)M_S \subseteq N$, and then $(N : M_S) = R$, which is a contradiction. Hence $Maxx(M_S) = \emptyset$. □

LEMMA 2.5. *Let M be a finitely annihilated R-module and I an ideal of R. Then the following are equivalent:*

(i) *There exists $a \in R$ such that $aM = 0$ and $a - 1 \in I$;*

(ii) *$Ann\ M \not\subseteq m$, for each maximal ideal m of R containing I.*

Proof. (i) $\Longrightarrow$ (ii) Evidently, $a \in Ann\ M \setminus m$, for each maximal ideal m of R containing I.

(ii) $\Longrightarrow$ (i) Suppose that $T = \{t_1, t_2, t_3, \ldots, t_n\}$ is a finite subset of M with $Ann\ T = Ann\ M$. Consider A to be the submodule of M generated by T. According to our assumption for each prime ideal P of R containing I, we have $Ann\ M \not\subseteq P$, which implies that $(Ann\ M)_P = R_P$. Then $R_P = (Ann\ M)_P = (0 : A)_P = (0 : A_P)$. Hence $A_P = 0$, for each prime ideal P of R containing I.

Now put $S = \{1 + x \mid x \in I\}$. For each maximal ideal $\mathbf{m}$ of R_S, we have $I_S \subseteq \mathbf{m}$, and so $I \subseteq \mathbf{m}^c$. Thus $0 = A_{\mathbf{m}^c} \cong (A_S)_{\mathbf{m}^c_S} = (A_S)_{\mathbf{m}}$. Consequently $A_S = 0$.

Then for each $t_i \in T$, $1 \le i \le n$, we have $t_i/1 = 0/1$, in M_S, that is, there exists $s_i \in S$ with $s_i t_i = 0$. Thus $s_1 s_2 s_3 \cdots s_n \in Ann\ T = Ann\ M$. So $s_1 s_2 s_3 \cdots s_n$ is the desired element of R. □

3. Nakayama property. Clearly, M has Nakayama property if and only if $IM = M$ for an ideal I of R implies that $Ann\ M + I = R$. So the zero module has Nakayama

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property, because $Ann\ M = R$. Also if $I = R$, then evidently $Ann\ M + I = R$. Hence for studying the Nakayama property, we assume that M is a non-zero module and I is a proper ideal of R.

LEMMA 3.1. *Let M be an R-module. Consider the following statements:*

(i) *For each maximal ideal m of R, the R_m-module M_m has Nakayama property;*

(ii) *M has Nakayama property;*

(iii) *If I is an ideal of R with $IM = M$ and $S = \{1 + x \mid x \in I\}$, then $M_S = 0$.*

Then (ii) $\Longrightarrow$ (iii), *and if M is finitely annihilated, then* (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Proof. (ii) $\Longrightarrow$ (iii) According to our assumption there exists $a \in R$ with $aM = 0$ and $a - 1 \in I$. Evidently, $(a/1)M_S = 0$, and $(a/1) - 1 \in I_S$. Now, since $I_S \subseteq m$, for each maximal ideal m of R_S, $(a/1) - 1 \in J(R)$. Then $a/1$ is a unit in R_S, and so $M_S = 0$.

Now suppose that M is finitely annihilated and assume that $T = \{t_1, t_2, \ldots, t_n\}$ is a subset of M with $Ann\ T = Ann\ M$.

(i) $\Longrightarrow$ (ii) Suppose that M does not have Nakayama property. Then according to Lemma 2.5[(ii) $\Longrightarrow$ (i)] there exists a maximal ideal m containing an ideal I such that $IM = M$ and $Ann\ M \subseteq m$.

Consider A to be the submodule of M generated by T. Then $(Ann\ M)_m \subseteq Ann\ M_m \subseteq (0 : A_m) = (0 : A)_m = (Ann\ M)_m$, that is, $Ann\ M_m = (Ann\ M)_m$. Since $I_m M_m = M_m$, according to our assumption there exist $a \in R$ and $s \in (R \setminus m)$ such that $a/s \in Ann\ M_m$ and $(a/s) - 1 \in I_m$. Then $a/1 \in Ann\ M_m = (Ann\ M)_m \subseteq m_m$, and so $a \in (m_m)^c = m$. Also from $(a - s)/s \in I_m$, we get $(a - s)/1 \in I_m \subseteq m_m$, and thus $a - s \in (m_m)^c = m$. Consequently, $s \in m$, which is a contradiction.

(iii) $\Longrightarrow$ (ii) Let I be a proper ideal of R with $IM = M$, and put $S = \{1 + x \mid x \in I\}$. By our assumption $M_S = 0$. So for each $t_i \in T$, $1 \leq i \leq n$, there exists $s_i \in S$ with $s_i t_i = 0$. Then $s_1 s_2 s_3 \cdots s_n \in Ann\ T = Ann\ M$, and thus $s_1 s_2 s_3 \cdots s_n$ is the desired element of R. □

PROPOSITION 3.2. *A projective R-module M has Nakayama property, if one of the following holds:*

(i) *M is finitely annihilated;*

(ii) *$Ann\ M$ is a prime ideal.*

Proof. (i) Let $IM = M$, where I is a proper ideal of R. Put $S = \{1 + x \mid x \in I\}$. Then M_S is also a projective R_S-module. If $M_S \neq 0$, then according to Lemma 2.2(vi), $Maxx(M_s) \neq \emptyset$, which is a contradiction by Lemma 2.4. Hence $M_s = 0$. Now the proof follows from Lemma 3.1[(iii) $\Longrightarrow$ (ii)].

(ii) According to Lemma 2.1(iv), $(Ann\ M)M = 0$ is a prime submodule of M. Now if $0 \neq x_0 \in M$ and $rx_0 = 0$, where $r \in R$, then since the zero submodule is a prime submodule, $r \in Ann\ M$. Thus $Ann\ x_0 = Ann\ M$, that is M is finitely annihilated. Now the proof is given by part (i). □

COROLLARY 3.3.

(i) *If R is an integral domain and M is non-zero projective, then $IM \neq M$, for each proper ideal I of R.*

(ii) *Every non-zero projective module over an integral domain is faithfully flat.*

Proof. (i) By Lemma 2.1(iv), $0M = 0$ is a 0-prime submodule of M. Then $Ann\ M = (0 : M) = 0$ is a prime ideal of R. Now if for an ideal I of R, $IM = M$, then by Proposition 3.2(ii), there exists $a \in R$ such that $aM = 0$ and $a - 1 \in I$. Thus $a \in Ann\ M = 0$, and so $1 \in I$.

(ii) By part (i), $mM \neq M$, for each maximal ideal m of R. So M is faithfully flat, by [**9**, Theorem 7.2]. □

PROPOSITION 3.4. *Let $\{M_i\}_{i\in\alpha}$ be a family of R-modules such that M_i has Nakayama property, for each $i \in \alpha$. Then $M = \oplus_{i\in\alpha} M_i$ has Nakayama property, if one of the following holds:*

(i) *$\{\cap_{i\in F} Ann\ M_i \mid F$ is a finite subset of $\alpha\}$ has a minimal element;*
(ii) *$\{Ann\ M_i \mid i \in \alpha\}$ is a finite set. In particular, if $|\alpha| < \infty$;*
(iii) *M is finitely annihilated;*
(iv) *M has DCC on the submodules of the form rM, $r \in R$.*

Proof. Suppose that $IM = M$, where I is an ideal of R. Then $\oplus_{i\in\alpha}(IM_i) = IM = M = \oplus_{i\in\alpha} M_i$, and so $IM_i = M_i$, for each $i \in \alpha$. According to our assumption M_i has Nakayama property for each $i \in \alpha$, then there exists $a_i \in R$ with $a_i M_i = 0$ and $a_i - 1 \in I$.

(i) Consider

$$\mathcal{A} = \{\cap_{i\in F} Ann\ M_i \mid F \text{ is a finite subset of } \alpha\},$$

and assume that $\cap_{i\in F_0} Ann\ M_i$ is a minimal element of $\mathcal{A}$.

Put $a = \Pi_{i\in F_0} a_i$. Evidently, $a - 1 \in I$. Let $j \in \alpha$. Note that $\cap_{i\in F_0} Ann\ M_i$ is a minimal element of $\mathcal{A}$, then $a \in \cap_{i\in F_0} Ann\ M_i = (\cap_{i\in F_0} Ann\ M_i) \cap Ann\ M_j \subseteq Ann\ M_j$. Therefore $a \in Ann\ M$.

(ii) The proof is clear by part (i).

(iii) Consider $S = \{1 + x \mid x \in I\}$. According to Lemma 3.1[(ii) $\Longrightarrow$ (iii)], $(M_i)_S = 0$, for each $i \in \alpha$. Then $M_S \cong \oplus_{i\in\alpha}(M_i)_S = 0$. Hence by Lemma 3.1[(iii) $\Longrightarrow$ (ii)], M has Nakayama property.

(iv) Consider the set

$$\mathcal{C} = \{(\Pi_{i\in F} a_i)M \mid F \text{ is a finite subset of } \alpha\}.$$

Define the partially ordered relation $(\mathcal{C}, \leq)$ as follows:

$$c_1 \leq c_2 \Longleftrightarrow c_1 \supseteq c_2 \qquad (c_1, c_2 \in \mathcal{C}).$$

We show that every chain $\mathcal{D}$ in $\mathcal{C}$ has an upper bound. Suppose not, and let $c_1 \in \mathcal{D}$. Since c_1 is not an upper bound of the chain $\mathcal{D}$, there exists $c_1 \neq c_2 \in \mathcal{D}$ with $c_1 \leq c_2$, that is $c_2 \subset c_1$. The same argument shows that there exists $c_3 \in \mathcal{D}$ such that $c_3 \subset c_2$. Now we can construct inductively a descending chain $\cdots \subset c_3 \subset c_2 \subset c_1$ of submodules of the form $\{rM \mid r \in R\}$, which does not stop, and this is against our assumption.

Hence every chain in $\mathcal{C}$ has an upper bound, and so by Zorn's Lemma, $(\mathcal{C}, \leq)$ has a maximal element, i.e., $\mathcal{C}$ has a minimal element $(\Pi_{i\in F_0} a_i)M$ with the relation $\subseteq$. Then for each $j \in \alpha$, $(\Pi_{i\in F_0} a_i)M_j = (\Pi_{i\in(F_0\cup\{j\})} a_i)M_j = 0$. Thus $\Pi_{i\in F_0} a_i \in \cap_{j\in\alpha} Ann\ M_j = Ann\ M$, and $(\Pi_{i\in F_0} a_i) - 1 \in I$. □

The following corollary introduces a method for making non-finitely generated modules, which have Nakayama property.

COROLLARY 3.5. *Let M be a finitely annihilated R-module. Then $M' = \oplus_{x\in M} Rx$ as an R-module has Nakayama property.*

Proof. It is easy to see that the condition (i) of Proposition 3.4 is satisfied. □

EXAMPLE 3.6.

(1) Let $\mathcal{P}$ be the set of odd prime numbers and consider $M' = \oplus_{p\in\mathcal{P}}\mathbb{Z}_p$ as a $\mathbb{Z}$-module. Then $IM' = M'$, where $I = 2\mathbb{Z}$. But M' does not have Nakayama property, since $Ann\, M' = 0$. However $\mathbb{Z}_p$ has Nakayama property, for each $p \in \mathcal{P}$, as it is cyclic (compare with Proposition 3.4).

(2) Let M_1 be an R-module and suppose M_2 is an R-module with the property that $IM_2 = M_2$ just for $I = R$. So if $I(M_1 \oplus M_2) = M_1 \oplus M_2$, then from $IM_1 \oplus IM_2 = I(M_1 \oplus M_2)$, we get $IM_2 = M_2$, and thus $I = R$. This shows that the R-module $M_1 \oplus M_2$ has Nakayama property.

(3) By part (2), the $\mathbb{Z}$-modules, $M'' = Q \oplus \mathbb{Z}$ and $K'' = 0 \oplus \mathbb{Z}$ have Nakayama property, but $M''/K'' \cong Q$ does not have Nakayama property, because $(2\mathbb{Z})Q = Q$ and $Ann\, Q = 0$. Moreover this shows that the converse of Proposition 3.4, parts (i), (ii) and (iii) are not correct.

(4) Consider $M' = \oplus_{p\in\mathcal{P}}\mathbb{Z}_p$ from part (1), and suppose $K' = 0 \oplus (\oplus_{3<p\in\mathcal{P}}\mathbb{Z}_p)$. Then $M'/K' \cong \mathbb{Z}_3$ has Nakayama property, because it is cyclic, but M' does not have Nakayama property.

(5) By part (2), if M_2 is a non-zero faithfully flat R-module, then $M_1 \oplus M_2$ has Nakayama property.

4. Rings for which certain modules over them have Nakayama property. We know that every non-zero finitely generated module has a maximal submodule. In the following, we are looking for a similar result for modules with Nakayama property.

DEFINITION. A proper submodule N of an R-module M will be called almost maximal, if $(N : M) = (L : M)$, for each proper submodule L of M containing N.

In the following, some properties of almost maximal submodules are given. The proof of the following lemma is easy and it is left to the reader.

LEMMA 4.1. *Let M be an R-module.*

(i) *If $(L : M) = 0$ for each proper submodule L of M, then every proper submodule of M is almost maximal. Particularly, if M is a divisible module over an integral domain, then every proper submodule of M is almost maximal.*

(ii) *A proper submodule N of M is almost maximal, if and only if $N + rM = M$ for each $r \in R \setminus (N : M)$.*

(iii) *A submodule N of M is almost maximal, if and only if $(N : M)$ is a prime ideal of R and M/N is a divisible $R/(N : M)$-module.*

(iv) *A submodule N of M is almost maximal, if and only if $(N : M)$ is a prime ideal of R and M/N is a secondary R-module.*

(v) *If $N \in Maxx(M)$, then N is almost maximal. In particular, if $N \in Max(M)$, then N is almost maximal.*

(vi) *If N is almost maximal in M, then every proper submodule of M containing N is also almost maximal in M.*

THEOREM 4.2. *Let M be a non-zero R-module.*

(i) *If M has Nakayama property, then M has an almost maximal submodule N. Moreover $N = (N : M)M$ and $(N : M)$ is a prime ideal of R.*

(ii) *If R is a Noetherian ring, then M has an almost maximal submodule.*

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Proof. (i) Consider the set $\mathcal{T}$ as follows:

$$\mathcal{T} = \{I \mid I \text{ is an ideal of } R, IM \neq M\}.$$

Evidently, $0 \in \mathcal{T}$, then $\mathcal{T} \neq \emptyset$. Let $\{I_j \mid I_j \in \mathcal{T}\}_{j\in\alpha}$ be a chain in $\mathcal{T}$. If $(\cup_{j\in\alpha} I_j)M = M$, then there exists $a \in M$ with $aM = 0$ and $1 - a \in \cup_{j\in\alpha} I_j$. Suppose that $1 - a \in I_{j_0}$, where $j_0 \in \alpha$. Then $M = (1 - a)M \subseteq I_{j_0}M$, and consequently $I_{j_0}M = M$, which is a contradiction. Hence $\cup_{j\in\alpha} I_j \in \mathcal{T}$. Now by Zorn's Lemma $\mathcal{T}$ has a maximal element P. We show that PM is an almost maximal submodule of M.

Let L be a proper submodule of M containing PM. Clearly, $P \subseteq (PM : M) \subseteq (L : M)$. So if $(PM : M) \neq (L : M)$, then $P \subset (L : M)$ and so $M = (L : M)M \subseteq L \subseteq M$, which is a contradiction. Therefore PM is an almost maximal submodule of M.

Evidently, $P \subseteq (PM : M)$. So if $P \neq (PM : M)$, then $M = (PM : M)M \subseteq PM \subseteq M$, which is impossible. Hence $P = (PM : M)$.

To prove that P is a prime ideal, let $bc \in P$, where $b, c \in R \setminus P$. Then from $(P + Rb)M = M$ and $(P + Rc)M = M$, we get $M = (P + Rb)(P + Rc)M \subseteq PM \subseteq M$, which is a contradiction.

(ii) Consider the set $\mathcal{T}$ as follows:

$$\mathcal{T} = \{(N : M) \mid N \text{ is a proper submodule of } M\}.$$

Suppose that $(N_0 : M)$ is a maximal element of $\mathcal{T}$. Then evidently N_0 is an almost maximal submodule of M. □

Let M be an R-module and I an ideal of R with $IM = M$ such that $I \subseteq J(R)$. Then obviously for each maximal ideal P of R, $PM = M$. Now If M is finitely generated, then $M = 0$. Compare this result with the following corollary.

COROLLARY 4.3. *Let M be an R-module such that $PM = M$ for each prime ideal P of R. If M has Nakayama property, then $M = 0$. In particular, if M is finitely generated, then $M = 0$.*

Proof. Let $0 \neq M$. Then Theorem 4.2(i) implies that M has an almost maximal submodule of the form PM, where P is a prime ideal of R. Hence $PM \neq M$, which is a contradiction. □

An ideal I of a ring R is called *T-nilpotent* in case for every sequence $a_1, a_2, \cdots \in I$, there is a positive integer n such that $a_1a_2a_3 \cdots a_n = 0$ (see [**1**, p. 314]).

A ring R is a *Von Neumann regular* ring, if $Ra = Ra^2$, for each $a \in R$.

LEMMA 4.4. *[**6**, The main theorem] A ring R is a Max ring if and only if $J(R)$ is T-nilpotent and $R/J(R)$ is a Von Neumann regular ring.*

THEOREM 4.5. *Let R be a ring. Consider the following statements:*

(a) *R is a Max ring;*
(b) *For any finitely annihilated R-module M and every $m \in Max(R)$, the R_m-module M_m has Nakayama property;*
(c) *For any finitely annihilated R-module M and every $m \in Max(R)$ containing $Ann\ M$, there exists $N \in Max(M)$ with $(N : M) = m$;*
(d) *Every finitely annihilated R-module has Nakayama property;*
(e) *$dim\ R = 0$.*

Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e).

Proof. (a) $\Longrightarrow$ (b) Let R be a Max ring. We prove the result in the following two steps:

Step 1. For any multiplicatively closed subset S of R, the ring R_S is a Max ring.

Proof of Step 1. First note that $dim\ R = 0$. To prove that let P be an ideal of R and m a maximal ideal of R containing P. Consider $a \in m$. By Lemma 4.4, $R/J(R)$ is a Von Neumann regular ring, then there exists $t \in R$ with $a(1 - ta) \in J(R)$. Again by Lemma 4.4, $J(R)$ is T-nilpotent, then $a(1 - ta)$ is nilpotent. Let $a^n(1 - ta)^n = 0$, where n is a positive integer. Then $a^n(1 - ta)^n \in P$ and since $P \subseteq m$, we have $1 - ta \notin P$, so $a \in P$.

From $dim\ R = 0$, we get $J(R) = \mathcal{N}(R)$, where $\mathcal{N}(R)$ is the intersection of all prime ideals of R. By [**2**, Corollary 3.12], $(J(R))_S = (\mathcal{N}(R))_S = \mathcal{N}(R_S)$. Also $dim\ R = 0$ implies that $dim\ R_S = 0$, then $\mathcal{N}(R_S) = J(R_S)$. Therefore $(J(R))_S = J(R_S)$.

According to Lemma 4.4, $J(R)$ is T-nilpotent, then clearly $J(R_S) = (J(R))_S$ is T-nilpotent. Also $R/J(R)$ is a Von Neumann regular ring, then $R_S/J(R_S) = R_S/(J(R))_S \cong (R/J(R))_S$ is a Von Neumann regular ring. Consequently by Lemma 4.4, R_S is a Max ring.

Step 2. If R is a Max ring, then every finitely annihilated R-module has Nakayama property.

Proof of Step 2. Suppose that M is a finitely annihilated R-module and $IM = M$, where I is a proper ideal of R. Consider $S = \{1 + x \mid x \in I\}$.

By Step 1, R_S is also a Max ring and by Lemma 2.4, $Max(M_S) = \emptyset$, hence $M_S = 0$. So by Lemma 3.1[(iii) $\Longrightarrow$ (ii)], M has Nakayama property.

Now for the proof of the result, note that by Step 1, for every $m \in Max(R)$, the ring R_m is a Max ring. Thus by Step 2, for any finitely annihilated R-module M, the R_m-module M_m has Nakayama property.

(d) $\Longrightarrow$ (e) First, we prove that a non-zero divisible module M over an integral domain R has Nakayama property if and only if R is a field.

Evidently, $Ann\ M = 0$. Suppose that M has Nakayama property. For each $0 \neq r \in R$, we have $(Rr)M = M$. Now since M has Nakayama property, $Rr = Ann\ M + Rr = R$, hence R is a field. For the converse, note that in a non-zero vector space M if $IM = M$ for an ideal I of R, then $I = R$.

Now suppose R is a ring such that every finitely annihilated R-module has Nakayama property.

Let P be a prime ideal of R and K the quotient field of R/P. One can easily see that $Ann_R\ K = P = Ann_R\ \{1 + P\}$, and so K is a finitely annihilated R-module. Hence by our assumption K has Nakayama property as an R-module. It is easy to see that K has Nakayama property as an R/P-module and we know that K is a non-zero divisible R/P-module, therefore R/P is a field.

(b) $\Longrightarrow$ (c) First, we show that every finitely annihilated R_m-module M' has Nakayama property, and so by part (d) $\Longrightarrow$ (e), $dim\ R_m = 0$.

Evidently, M' is an R-module by considering the natural homomorphism $R \longrightarrow R_m$, and it is easy to see that M' is finitely annihilated as an R-module. Thus by our assumption $(M')_m$ has Nakayama property as an R_m-module, and one can easily see that $M' \cong (M')_m$ as an R_m-module.

Now we show that $M_m \neq 0$, for any maximal ideal m of R containing $Ann\ M$.

Suppose that $T = \{t_1, t_2, t_3, \ldots, t_n\}$ is a finite subset of M with $Ann\ T = Ann\ M$. If $M_m = 0$, then for each $t_i \in T$, $1 \leq i \leq n$, there exists $s_i \in R \setminus m$ with $s_i t_i = 0$. Thus $s_1 s_2 s_3 \cdots s_n \in Ann\ T = Ann\ M \subseteq m$, which is a contradiction.

By our assumption the R_m-module M_m has Nakayama property, so Theorem 4.2(i) implies that M_m has an almost maximal submodule L, where $(L : M_m)$ is a prime ideal of R_m. As $dim\ R_m = 0$, $L \in Maxx(M_m)$ and $(L : M_m) = m_m$. By Lemma 2.2(v), $L^c \in Maxx(M)$ with $(L^c : M) = m$. Note that $mM \subseteq L^c$, so $mM \neq M$. Then M/mM is a non-zero vector space over the field R/m. So M/mM has a maximal subspace N/mM. Thus N is a maximal submodule of M and since $m \subseteq (N : M)$, we have $(N : M) = m$. This completes the proof.

(c) $\Longrightarrow$ (d) Suppose that M is a finitely annihilated R-module and $IM = M$, where I is an ideal of R. According to Lemma 2.5[(ii) $\Longrightarrow$ (i)] it is enough to show that $Ann\ M \nsubseteq m$, for each maximal ideal m of R containing I. On the contrary, let $Ann\ M \subseteq m$, where m is a maximal ideal of R containing I. By our assumption, there exists a maximal submodule N of M with $(N : M) = m$. Since $I \subseteq m$, we have $M = IM \subseteq mM \subseteq N$, which is a contradiction. □

EXAMPLE 4.6. Let M be a non-zero divisible R-modules, where R is an integral domain, which is not a field. Then M does not have Nakayama property, by the proof of Theorem 4.5[(d) $\Longrightarrow$ (e)]. Particularly the $\mathbb{Z}$-modules Q and $\mathbb{Z}_{P^\infty}$ do not have Nakayama property. However if M is torsion-free divisible, then M is finitely annihilated, for example Q is finitely annihilated.

A submodule K of a module M is said to be *small (or superfluous)* in case for every $L \leq M$, the equality $K + L = M$ implies that $L = M$. It is said that a module M has a *projective cover* if there exists an epimorphism $f : P \longrightarrow M$ such that P is a projective module and $Ker\, f$ is small in P (see [**1**, p. 199]).

According to [**1**, Theorem 28.4 (Bass)], a ring R is a perfect ring, if and only if every R-module has a projective cover.

LEMMA 4.7. *[**6**, Corollary on page 1136] and [**1**, Theorem 28.4] Let R be a ring. Then the following are equivalent:*
(i) *R is a perfect ring;*
(ii) *R is a Max ring and R has no infinite set of orthogonal idempotents;*
(iii) *$R/J(R)$ is a semi-simple ring and $J(R)$ is T-nilpotent.*

Evidently, any Artinian ring is a perfect ring, and Lemma 4.7 implies that every perfect ring is a Max ring.

EXAMPLE 4.8.
(1) Let K be a field and $\{x_i \mid i \in \mathbb{N}\}$ a set of infinite independent indeterminates, and suppose $\mathfrak{M} = \langle x_1, x_2, x_3, \ldots \rangle$. Then for each $1 < n \in \mathbb{N}$, the ring $R = K[x_1, x_2, x_3, \ldots]/\mathfrak{M}^n$, is a perfect ring, but it is not an Artinian ring.
(2) Let F be a field and consider $R = \prod_{n\in\mathbb{N}} F$. Then R is a Max ring, but it is not a perfect ring.

Proof. (1) Evidently, $J(R) = \mathfrak{M}/\mathfrak{M}^n$. Then $(J(R))^n = 0$, and this shows that $J(R)$ is T-nilpotent. Also clearly $R/J(R) \cong K$ and thus $R/J(R)$ is a semi-simple (indeed a simple) ring. Now according to Lemma 4.7((iii)$\Longrightarrow$(i)), R is a perfect ring.

Note that the following chain of ideals of R does not stop:

$$\frac{\langle x_1 \rangle}{\mathfrak{M}^n} \subset \frac{\langle x_1, x_2 \rangle}{\mathfrak{M}^n} \subset \frac{\langle x_1, x_2, x_3 \rangle}{\mathfrak{M}^n} \subset \cdots,$$

hence R is not Noetherian and evidently it is not Artinian.

(2) For each $a = \{a_n\}_{n\in\mathbb{N}} \in R$, we have $a = a^2\{a'_n\}_{n\in\mathbb{N}} \in Ra^2$, where for each $n \in \mathbb{N}$, $a'_n = a_n^{-1}$ if $0 \neq a_n$, otherwise $a'_n = 0$. This shows that R and consequently $R/J(R)$ is a Von Neumann regular ring.

Now we prove that $J(R) = 0$. Put $I_k = \prod_{n\in\mathbb{N}}(1-\delta_{nk})F$. Then I_k is a maximal ideal of R, for each $k \in \mathbb{N}$. For proof, let J be an ideal of R with $I_k \subset J$. Then there exists $x = \{x_n\}_{n\in\mathbb{N}} \in J$ such that $0 \neq x_k$. Note that $y = \{(1-\delta_{nk})(1-x_n)\}_{n\in\mathbb{N}} \in I_k \subset J$. So $x + y = \{z_n\}_{n\in\mathbb{N}} \in J$, where $z_k = x_k$ and $z_n = 1$, for each $n \neq k$. Thus $1 = \{z_n\}_{n\in\mathbb{N}}.\{z_n^{-1}\}_{n\in\mathbb{N}} \in J$. Hence I_k is a maximal ideal of R, for each $k \in \mathbb{N}$, and evidently $J(R) \subseteq \cap_{k\in\mathbb{N}} I_k = 0$.

Therefore $J(R)$ is T-nilpotent and $R/J(R)$ is a Von Neumann regular ring and so by Lemma 4.4, R is a Max ring.

Note that the set $\{e_k \mid k \in \mathbb{N}\}$, where $e_k = \{\delta_{nk}\}_{n\in\mathbb{N}}$ is an infinite set of orthogonal idempotents of R, so by Lemma 4.7, R is not a perfect ring. □

THEOREM 4.9. *A ring R is a perfect ring if and only if every R-module has Nakayama property.*

Proof. ($\Longrightarrow$) Let R be a perfect ring. We prove that every R-module has Nakayama property in the following three steps:

Step 1. Let $\{M_i\}_{i\in\alpha}$ be a family of R-modules such that M_i has Nakayama property, for each $i \in \alpha$. Then $\oplus_{i\in\alpha} M_i$ has Nakayama property.

Proof of Step 1. Put $M = \oplus_{i\in\alpha} M_i$. Suppose that $IM = M$, where I is an ideal of R. Then $\oplus_{i\in\alpha}(IM_i) = IM = M = \oplus_{i\in\alpha} M_i$, and so $IM_i = M_i$, for each $i \in \alpha$. According to our assumption, M_i has Nakayama property for each $i \in \alpha$, then there exists $a_i \in R$ with $a_i M_i = 0$ and $a_i - 1 \in I$.

Let

$$\mathcal{B} = \{R(\Pi_{i\in F} a_i) \mid F \text{ is a finite subset of } \alpha\}.$$

As R has DCC on principal ideals, Zorn's Lemma implies that $\mathcal{B}$ has a minimal element. Let $R(\Pi_{i\in F_0} a_i)$ be a minimal element of $\mathcal{B}$. Then for each $j \in \alpha$, we have $R(\Pi_{i\in F_0} a_i) = R(\Pi_{i\in(F_0\cup\{j\})} a_i)$. Hence $\Pi_{i\in F_0} a_i \in Ann\ M_j$ for each $j \in \alpha$, and so $\Pi_{i\in F_0} a_i \in \cap_{j\in\alpha} Ann\ M_j = Ann\ M$, and clearly $(\Pi_{i\in F_0} a_i) - 1 \in I$.

Step 2. Every projective R-module has Nakayama property.

Proof of Step 2. Let P be a projective R-module. According to [**1**, Theorem 27.11], P is isomorphic to a direct sum of cyclic submodules. Hence P has Nakayama property, by Step 1.

Step 3. Every R-module has Nakayama property.

Proof of Step 3. Let M be an R-module. Note that R is a perfect ring, then every R-module has a projective cover ([**1**, Theorem 28.4 (Bass)]). Let $M \cong P/K$, where P is a projective module and K is a small submodule of P. Suppose that $I(P/K) = P/K$, where I is an ideal of R. Then $K + IP = P$, and since K is a small submodule of P, $IP = P$. Now by Step 2, $Ann\ P + I = R$. Evidently, $Ann\ P \subseteq Ann(P/K)$, thus $Ann(P/K) + I = R$, which completes the assertion.

($\Longleftarrow$) We prove that R is a Max ring and R has no infinite set of orthogonal idempotents. Hence by Lemma 4.7, R is a perfect ring.

Let M be an arbitrary R-module. By Theorem 4.2(i), M has an almost maximal submodule N and $(N : M)$ is a prime ideal of R. By Theorem 4.5[(d) $\Longrightarrow$ (e)], $dim\ R = 0$, then $N \in Maxx(M)$, and consequently by Lemma 2.2(vii), $Max(M) \neq \emptyset$.

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Now let $\{e_i\}_{i=1}^{+\infty}$ be an infinite set of orthogonal idempotents of R. Consider I to be the ideal of R generated by $\{e_i\}_{i=1}^{+\infty}$ and suppose $M' = I/Re_1$. Since $I^2 = I$, we have $IM' = M'$ and we know that M' has Nakayama property, so $Ann\ M' + I = R$.

Note that $Ann\ M' = \cap_{i=2}^{+\infty} Ann\ e_i$. To prove that let $r \in Ann\ M'$. Thus $re_i \in Re_1$ for each $i > 1$, and so there exists $s_i \in R$ with $re_i = s_ie_1$. Then $re_i = re_i.e_i = s_ie_1e_i = 0$. The proof of the converse inclusion is evident.

Now from $Ann\ M' + I = R$, and $Ann\ M' = \cap_{i=2}^{+\infty} Ann\ e_i$, we get $(\cap_{i=2}^{+\infty} Ann\ e_i) + I = R$. Let $s + \sum_{i=1}^{n} r_ie_i = 1$, where $s \in \cap_{i=2}^{+\infty} Ann\ e_i$ and n is a positive integer and $r_i \in R$ for each $1 \le i \le n$. Then for each $j > n$, we have $e_j = e_j.1 = se_j + \sum_{i=1}^{n} r_ie_ie_j = 0$, which completes the proof. □

COROLLARY 4.10. *Let R be a Noetherian ring. Then the following are equivalent:*
(i) *Every R-module has Nakayama property;*
(ii) *Every finitely annihilated R-module has Nakayama property;*
(iii) *R is an Artinian ring.*

Proof. (ii) $\Longrightarrow$ (iii) $dim\ R = 0$, by Theorem 4.5[(d) $\Longrightarrow$ (e)].
(iii) $\Longrightarrow$ (i) The proof follows from Theorem 4.9. □

Note. If M is an R-module such that $Ann\ M$ is a maximal ideal, then M has Nakayama property. For the proof, note that M has Nakayama property as an R-module if and only if M has Nakayama property as an $R/Ann\ M$-module. Thus by Corollary 4.10, M has Nakayama property. □

Let K be a proper submodule an R-module M. Example 3.6(3),(4) shows that the Nakayama property for M does not imply the Nakayama property for M/K, and conversely.

COROLLARY 4.11. *Let K be a proper submodule an R-module M.*
(i) *If M/K has Nakayama property, then M has an almost maximal submodule containing K.*
(ii) *If M has Nakayama property, then M has an almost maximal submodule of the form PM, where P is a prime ideal of R containing $(K : M)$.*

Proof. (i) By Theorem 4.2(i), there exists an almost maximal submodule N/K of M/K,where N is a submodule of M containing K. One can easily see that N is an almost maximal submodule of M.
(ii) Consider the set $\mathcal{T}$ as follows:

$$\mathcal{T} = \{I \mid I \text{ is an ideal of } R,\ IM \neq M,\ \ (K : M) \subseteq I\}.$$

Note that $(K : M) \in \mathcal{T}$. Now follow the proof of Theorem 4.2(i). □

Recall that a non-zero module M is called *sum-irreducible*, in case $L + K \neq M$, for each proper submodules L, K of M (see [**9**, p. 44]).

THEOREM 4.12. *Let M be an Artinian module. Then the following are equivalent:*
(i) *M is a finitely generated module;*
(ii) *M is finitely annihilated;*
(iii) *$R/Ann\ M$ is an Artinian ring;*
(iv) *Every submodule of M is finitely annihilated;*
(v) *Every submodule of M has Nakayama property.*

Proof. (i) $\Longrightarrow$ (ii) The proof is obvious.

(ii) $\Longrightarrow$ (iii) Suppose that $T = \{t_1, t_2, t_3, \dots, t_n\}$ is a finite subset of M with $Ann\ T = Ann\ M$. Then the module $M' = Rt_1 + Rt_2 + \cdots + Rt_n$ is a finitely generated Artinian R-module. Hence $R/Ann\ M' = R/Ann\ M$ is an Artinian ring.

(iii) $\Longrightarrow$ (iv) Note that every module M over an Artinian ring R is finitely annihilated. To prove that, consider

$$\mathcal{A} = \{Ann\ T \mid T \text{ is a finite subset of } M\}.$$

Suppose that $Ann\ T_0$ is a minimal element of $\mathcal{A}$. If $Ann\ M \neq Ann\ T_0$, then let $r \in Ann\ T_0 \setminus Ann\ M$. So there exists $m \in M$ such that $rm \neq 0$. Now since $r \in Ann\ T_0 \setminus Ann(T_0 \cup \{m\})$, we get $Ann(T_0 \cup \{m\}) \subset Ann\ T_0$, which is a contradiction.

Now let N be an arbitrary submodule of M. Since $R/Ann\ M$ is an Artinian ring, N is finitely annihilated as an $R/Ann\ M$-module and consequently as an R-module.

(iv) $\Longrightarrow$ (v) Let N be a submodule of M. According to the proof of (ii) $\Longrightarrow$ (iii), $R/Ann\ N$ is an Artinian ring, and every Artinian ring is a perfect ring, so by Theorem4.9, N has Nakayama property as an $R/Ann\ N$-module. Consequently, N has Nakayama property as an R-module.

(v) $\Longrightarrow$ (i) Suppose that M is not finitely generated. Consider the set $\mathcal{T}$ as follows:

$$\mathcal{T} = \{N \leq M \mid N \text{ is not finitely generated}\}.$$

Let N_1 be a minimal element of $\mathcal{T}$. Then N_1 is a sum-irreducible module. To see the proof, let L, K be proper submodules of N_1 with $L + K = N_1$. So L and K are finitely generated, which implies that N_1 is finitely generated.

According to our assumption N_1 has Nakayama property, then by Theorem 4.2(i), N_1 has an almost maximal submodule N_0. We show that $(N_0 : N_1)$ is a maximal submodule of R. Let J be an ideal of R with $(N_0 : N_1) \subset J$. Consider $r \in J \setminus (N_0 : N_1)$. Since N_0 is almost maximal, $N_0 + rN_1 = N_1$. Note that N_1 is sum-irreducible, then $rN_1 = N_1$, that is $IN_1 = N_1$, where $I = Rr$. By our assumption N_1 has Nakayama property, then there exists $a \in Ann\ N_1$ such that $a - 1 \in I = Rr$. Now $a \in (N_0 : N_1) \subset J$ and $1 - a \in I = Rr \subseteq J$, and so $1 \in J$.

Since N_1 is sum-irreducible, it is easy to see that the vector space N_1/N_0 over the field $R/(N_0 : N_1)$ is also sum-irreducible. Therefore $rank_{R/(N_0:N_1)}\ N_1/N_0 = 1$, that is, N_1/N_0 as an $R/Ann(N_1/N_0)$-module and evidently as an R-module is finitely generated. Also N_0 is finitely generated, since N_0 is a proper submodule of N_1. Consequently N_1 is finitely generated, which is a contradiction. □

Corollary 4.13. *Every Artinian module over an Artinian ring is a Noetherian module.*

Proof. The proof is evident, by Theorem 4.12(iii) and (i). □

Example 4.14. The $\mathbb{Z}$-module $\mathbb{Z}_{P^\infty}$ is an Artinian module and every proper submodule of $\mathbb{Z}_{P^\infty}$ is cyclic. Then obviously every proper submodule of $\mathbb{Z}_{P^\infty}$ has Nakayama property. However, $\mathbb{Z}_{P^\infty}$ does not have Nakayama property , by Example 4.6.

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