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THE ALPERIN WEIGHT CONJECTURE AND DADE'S CONJECTURE FOR THE SIMPLE GROUP J_4

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Abstract

The authors construct faithful permutation representations of maximal 2-local subgroups and classify the radical chains of the Janko simple group J_4; hence the Alperin weight conjecture and the Dade reductive conjecture for J_4 are verified.

1. *Introduction*

The program of deciding Dade's reductive conjecture [10] for the sporadic groups has made very substantial progress: it has been verified for all of the sporadic simple groups except Fi_{24}', J_4, B and M.

The use of computer algebra systems, namely Magma [7] and GAP [12], to study permutation (or in some cases matrix) representations of the groups has been a central step of the program. Since the smallest faithful permutation representation of J_4 has degree 173067389, it is difficult to verify the conjecture directly. However, from the classification of maximal subgroups of J_4 (see [14]), we know that the normalizer of each radical 2- and 3-subgroup of J_4 is a subgroup of one of precisely four maximal 2-local subgroups. Thus we can classify radical chains in these four maximal subgroups without performing any calculation in J_4.

In this paper, we construct faithful permutation representations for each maximal 2-local subgroup, classify radical chains, and hence verify the Alperin weight conjecture and the Dade reductive conjecture for J_4.

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Subsequently, Dade [9] generalized the Knörr–Robinson version of the Alperin weight conjecture, and presented his ordinary conjecture, exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Later, Dade [10] announced that his reductive conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has a trivial Schur multiplier and trivial outer automorphism group, then the ordinary conjecture is equivalent to the reductive conjecture.

The paper is organized as follows. In Section 2, we fix the notation, state the conjectures in detail, and state two lemmas. In Section 3, we explain how to construct faithful permutation representations of the four maximal 2-local subgroups. In Section 4, we recall the modified local strategy [3, 4]; we also explain how we applied it to determine the radical subgroups of each maximal subgroup, and how to fuse the radical subgroups in J_4. In Section 5, we use the list of radical subgroups of J_4 given by [17] to verify the Alperin weight conjecture.

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In Section 6, we do some cancellations in the alternating sum of Dade's conjecture, and then determine radical chains (up to conjugacy) and their local structures. Finally, we verify the ordinary conjecture of Dade for J_4.

2. *Dade's ordinary conjecture*

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-*weight* if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ is the p-defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and let $\ell(B)$ be the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \ldots < P_n \tag{2.1}$$

of G, define $|C| = n$, $C_k : P_0 < P_1 < \ldots < P_k$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \ldots \cap N(P_n). \tag{2.2}$$

The chain C is *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$; and

(b) $P_k = O_p(N(C_k))$ for $1 \leqslant k \leqslant n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Let $\mathrm{k}(N_G(C), B, d)$ be the number of characters ψ in $\mathrm{Irr}(N_G(C))$ such that $\mathrm{d}(\psi) = d$ and $B(\psi)^G = B$. In the notation used above, the Dade ordinary conjecture is stated as follows.

Dade's ordinary conjecture (see [9]). *If $O_p(G) = 1$ and B is a p-block of G with defect group $D(B) \neq 1$, then for any integer $d \geqslant 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let G be the Janko simple group J_4. Then its Schur multiplier and outer automorphism group are both trivial, so by [10], Dade's ordinary conjecture is equivalent to his reductive conjecture.

In Section 6, we shall use the following lemmas.

Lemma 2.1. *Let $\sigma : O_p(G) < P_1 < \ldots < P_{m-1} < Q = P_m < P_{m+1} < \ldots < P_\ell$ be a fixed radical p-chain of a finite group G, where $1 \leqslant m < \ell$. Suppose that*

$$\sigma' : O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell$$

is also a radical p-chain such that $N_G(\sigma) = N_G(\sigma')$. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell - 1)$th subchain $C_{\ell-1}$ is conjugate to σ' in G, and

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$\mathcal{R}^0(\sigma, Q)$ the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓth subchain C_ℓ is conjugate to σ in G. Then the map g, sending any

$$O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell < \ldots$$

in $\mathcal{R}^-(\sigma, Q)$ to

$$O_p(G) < P_1 < \ldots < P_{m-1} < Q < P_{m+1} < \ldots < P_\ell < \ldots,$$

induces a bijection, denoted again by g, from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| - 1$ and $N_G(C) = N_G(g(C))$.

Proof. This is straightforward. □

LEMMA 2.2. *If Q is a p-subgroup of a finite group G, then there is a radical p-subgroup R such that*

$$Q \leqslant R \quad \text{and} \quad N_G(Q) \leqslant N_G(R).$$

Proof. This follows by [2, Lemma 2.1]. □

3. *Construction of permutation representations of maximal* 2*-local subgroups*

We use the notation of [8]. In particular, $p^{1+2\gamma} = p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p or type $+$ according to whether p is odd or even. If X and Y are groups, we use $X{\cdot}Y$ and $X{:}Y$ to denote a nonsplit extension and a split extension of X by Y, respectively. Given a positive integer n, we use p^n to denote the elementary abelian group of order p^n, n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

The four maximal 2-local subgroups of J_4 that we wish to construct are $2^{11}{:}M_{24}$, $2^{10}{:}L_5(2)$, $2^{3+12} \cdot (S_5 \times L_3(2))$ and $2^{1+12} \cdot 3 \cdot M_{22}{:}2$. The first two of these are easy to construct abstractly as affine groups. The first can be written as 12×12 matrices over $GF(2)$, acting either with a fixed vector or with a fixed hyperplane. These representations can be easily obtained from a submodule or quotient module (respectively) of the restriction of the 112-dimensional representation of J_4. These give rise naturally to permutation representations on $759 + 1288 + 2048$ non-zero vectors in the one case, or $1 + 1518 + 2576$ in the second. Any of the three faithful representations, of degrees 2048, 1518, and 2576, can then be used to generate the character table of the group.

Similarly, the group $2^{10}{:}L_5(2)$ has two natural affine representations, in which the orbits of non-zero vectors are either $155 + 868 + 1024$ or $1 + 310 + 1736$ in length, and again any of the three faithful representations, of degree 1024, 310 and 1736, can be used as a starting point for the calculations.

The other two maximal 2-local subgroups of J_4 are harder to construct. They can be obtained as subgroups of J_4 using the words in the standard generators of J_4 given in [5]. This, however, limits us to two particular representations of the groups, given by restricting the representations of J_4 in dimension 112 over $GF(2)$ or dimension 1333 over $GF(11)$. For the purposes of calculating the character tables, we want to obtain faithful permutation representations on a reasonably small number of points.

On the other hand, of course, the point stabilizer in a faithful representation cannot contain any non-trivial subgroup that is normal in the whole group, and this puts severe constraints on which subgroups we can use as a point stabilizer. For example, in the case $2^{1+12}{\cdot}3{\cdot}M_{22}{:}2$, if we are to avoid the central involution, the point stabilizer can have at most

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a subgroup of order 2^6 in the normal 2^{1+12}. Moreover, such a subgroup must be invariant under a reasonable subgroup of $6^{\cdot}M_{22}$, which must again split off the central involution. The biggest subgroup that we could identify as a potential point stabilizer was a group of shape $2^6{:}3^{\cdot}A_6$, which we found as the subgroup generated by certain words in the given generators of $2^{1+12}{\cdot}3^{\cdot}M_{22}{:}2$.

This subgroup has index $2^7.2.77.16 = 630784$. We obtained the corresponding permutation representation on 630784 points by calculating the subgroup explicitly in the 112-dimensional representation over $GF(2)$, and using MAGMA to calculate the permutation action on the cosets. We then found that in fact there is a block system of 315392 blocks of size 2, such that the action on the blocks is still faithful. This representation was then used as input to the next stage of calculation.

In the remaining case, we found a subgroup of shape $2^{1+8}(5 \times S_4)$ as a potential point stabilizer in $2^{3+12}(S_5 \times L_3(2))$. We used the 1333-dimensional $GF(11)$-representation of J_4, and restricted to a 1120-dimensional submodule on which the maximal 2-local subgroup acts faithfully and irreducibly. Restricting further to the potential point stabilizer, we found a 15-dimensional invariant subspace. Taking the permutation action of our maximal 2-local subgroup on the 61440 images of this 15-space gave us the desired faithful permutation representation.

4. *A local subgroup strategy and fusions*

Kleidman and Wilson [14] classified the maximal subgroups of J_4. From this, we know that there are 4 maximal 2-local subgroups up to conjugacy. In addition, each radical 2- and 3-subgroup R of J_4 is radical in one of the subgroups M and, further, $N_{J_4}(R) = N_M(R)$. The radical 2-subgroups of J_4 are classified by [17].

In [3] and [4], a (modified) local strategy was developed to classify the radical p-subgroups R. We review this method here.

Let $Q = O_p(M)$, so that $Q \leqslant R$. Choose a subgroup X of M. We explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

After applying the strategy, we list the radical subgroups of each M, and then we do the fusions as follows.

Suppose that R is a radical p-subgroup of M. Using the local structure, we can determine whether or not $N_M(R)$ is a subgroup of another maximal subgroup M'. Suppose that $N_M(R)$ is a subgroup of M'. By Lemma 2.2, there is a radical subgroup R' of M' such that $R \leqslant R'$ and $N_M(R) \leqslant N_{M'}(R')$. Using the local structure, we can determine whether or not R is radical in M'; if so, we can identify R with a radical subgroup R' of M'. In this case, $N_M(R) =_G N_{M'}(R')$. Some more details are given below.

The computations reported in this paper were carried out using MAGMA V2.9-9 on a Sun UltraSPARC Enterprise 4000 server.

5. *Weights*

Let $\mathcal{R}_0(G, p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \leqslant G$, we write $H \leqslant_G K$ if $x^{-1}Hx \leqslant K$, and we write $H \in_G \mathcal{R}_0(G, p)$ if $x^{-1}Hx \in \mathcal{R}_0(G, p)$ for some $x \in G$.

Let G be the Janko simple group J_4. Then

$$|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43,$$

and we may suppose that $p \in \{2, 3, 11\}$, since both conjectures hold for a block with a cyclic defect group, by [9, Theorem 7.1]. If $p = 11$, then by [6, Proposition 1.3], a Sylow subgroup of G is a trivial intersection group, so that Dade's ordinary conjecture and Alperin's weight conjecture follow, by [11]. Moreover, Uno's ordinary conjecture, which is refinement of Dade's ordinary conjecture, also holds for J_4. Thus we may suppose that $p = 2$ or $p = 3$.

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H, and by $\mathrm{d}(H)$ the number $\log_p(|H|_p)$. Given $R \in \mathcal{R}_0(G, p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (see [3, (4.1)])

$$\mathcal{W}(B_0) = \sum_R |\,\mathrm{Irr}^0(N/C(R)R)|, \tag{5.1}$$

where R runs over the set $\mathcal{R}_0(G, p)$ such that $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find that $|\,\mathrm{Irr}^0(N/C(R)R)|$.

In Table 1, we recall the classification from [17] of the radical 2-subgroups of $G = J_4$.

Suppose that $p = 3$. As shown in [14, Section 3],

$$\mathcal{R}_0(G, 3) = \{1, 3, 3^2, 3_+^{1+2}\},$$

where $3 = Z(3_+^{1+2})$. In addition, $C(3) = 6.M_{22}$, $C(3^2) = 3^2 \times 2^3$, $C(3_+^{1+2}) = 6$ and

$$N(R) = \begin{cases} 6.M_{22}\colon 2 \leqslant 2_+^{1+12}.3.M_{22}\colon 2, & \text{if } R = 3, \\ (3^2\colon 2 \times 2^3).S_4 \leqslant 2^{11}\colon M_{24}, & \text{if } R = 3^2, \\ (2 \times 3_+^{1+2}\colon 8)\colon 2 \leqslant 2_+^{1+12}.3.M_{22}\colon 2, & \text{if } R = 3_+^{1+2}. \end{cases}$$

LEMMA 5.1. *Let* $G = J_4$ *and* $B_0 = B_0(G)$, *let* $\mathrm{Blk}^+(G, p)$ *be the set of p-blocks with a non-trivial defect group, and let* $\mathrm{Irr}^+(G)$ *be the characters of* $\mathrm{Irr}(G)$ *with positive p-defect. If a defect group* $D(B)$ *of* B *is cyclic, then* $\mathrm{Irr}(B)$ *is given by* [13, p. 326].

(a) *If* $p = 3$, *then* $\mathrm{Blk}^+(G, p) = \{B_i \mid 0 \leqslant i \leqslant 6\}$ *such that* $D(B_1) \simeq 3_+^{1+2}$, $D(B_2) \simeq 3^2$ *and* $D(B_i) \simeq 3$ *for* $3 \leqslant i \leqslant 6$. *In the notation of* [8, p. 188],

$$\begin{aligned} \mathrm{Irr}(B) &= \{\chi_2, \chi_3, \chi_{12}, \chi_{13}, \chi_{17}, \chi_{18}, \chi_{22}, \chi_{23}, \chi_{24}, \chi_{26}, \chi_{38}, \chi_{39}, \chi_{44}, \chi_{50}\}, \\ \mathrm{Irr}(B_2) &= \{\chi_{14}, \chi_{21}, \chi_{25}, \chi_{27}, \chi_{28}, \chi_{30}, \chi_{31}, \chi_{35}, \chi_{41}\}, \end{aligned}$$

and

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\cup_{i=1}^{6} \mathrm{Irr}(B_i)).$$

Moreover, $\ell(B_0) = \ell(B_1) = 9$, $\ell(B_2) = 5$, $\ell(B_i) = 2$ *for* $3 \leqslant i \leqslant 5$ *and* $\ell(B_6) = 1$.

(b) *If* $p = 2$, *then* $\mathrm{Blk}^+(G, 2) = \{B_0\}$, *and so* $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)$. *Moreover,* $\ell(B_0) = 22$.

Proof. If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By [13, p. 326], we may suppose that D is non-cyclic. Thus $D = 3^2$ or 3_+^{1+2}. In each case, $N(D)$ has one orbit on the non-trivial character of $\mathrm{Irr}^0(C(D)D/D)$ with $N(\theta)/C(D)D$ a $3'$-group.

Table 1: Non-trivial radical 2-subgroups of J_4

R	$C(R)$	$N(R)$	$\lvert \mathrm{Irr}^0(N/C(R)R) \rvert$
2^{10}	2^{10}	$2^{10}: L_5(2)$	1
2^{11}	2^{11}	$2^{11}: M_{24}$	0
2^{1+12}	2	$2^{1+12}.(3.M_{22}): 2$	1
$2^{10}: 2^4$	2^4	$2^{10}: 2^4.L_4(2)$	1
$2^{11}: 2^4$	2^6	$2^{11}: 2^4.A_8$	1
2^{3+12}	2^3	$2^{3+12}.(S_5 \times L_3(2))$	0
$2^{3+12}.2$	2^3	$2^{3+12}.2(S_3 \times L_3(2))$	1
$2^{1+12}.2^3$	2	$2^{1+12}.2^3.(S_3 \times L_3(2))$	1
$2^3.2^{6+8}$	2^3	$2^3.2^{6+8}.(S_3 \times L_3(2))$	1
$2^2.2^{5+10}$	2^2	$2^2.2^{5+10}.(S_3 \times S_5)$	0
$2^{1+12}.2^4$	2	$2^{1+12}.2^4.3.S_6$	1
$2^{3+12}.2^2$	2	$2^{2+12}.2^2.(S_3 \times S_5)$	0
$2^{10}.2^{3+4}$	2	$2^{10}.2^{3+4}.L_3(2)$	1
$2^{3+12}.D_8$	2^3	$2^{3+12}.D_8.L_3(2)$	1
$2^{3+12}.2^3$	2^2	$2^{3+12}.2^3.(S_3 \times S_3)$	1
$2^{1+12}.2^{2+3}$	2	$2^{1+12}.2^{2+3}.(S_3 \times S_3)$	1
$2^{11}.2^{1+6}$	2^3	$2^{11}.2^{1+6}.L_3(2)$	1
$2^{1+12}.2^5$	2	$2^{1+12}.2^5.S_5$	0
$2^{1+12}.2^2.2^4$	2	$2^{1+12}.2^2.2^4.(S_3 \times S_3)$	1
$2^{1+12}.2.2^{2+3}$	2	$2^{1+12}.2.2^{2+3}.(S_3 \times S_3)$	1
$2^{3+12}.2^4$	2	$2^{3+12}.2^4.(S_3 \times S_3)$	1
$2^{1+12}.2^3.2^3$	2	$2^{1+12}.2^3.2^3.S_3$	1
$2^{3+12}.2^3.2^2$	2^2	$2^{3+12}.2^3.2^2.S_3$	1
$2^{11}.2^2.2^3.2^4$	2	$2^{11}.2^2.2^3.2^4.S_3$	1
$2^{1+12}.2^{2+5}$	2	$2^{1+12}.2^{2+5}.S_3$	1
$2^{1+12}.2.2^3.2^3$	2	$2^{1+12}.2.2^3.2^3.S_3$	1
S	2	S	1

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Using the method of central characters, $\mathrm{Irr}(B)$ is as above. If $D(B)$ is cyclic, then $\ell(B)$ is given by [13, p. 326].

If $p = 3$ and $B = B_1$ or $B = B_2$, then the non-trivial elements of $D(B)$ are of type $3A$, and $C_G(3A) = 6.M_{22}$. It follows by [15, Theorem 5.4.13] that

$$k(B) = \ell(B) + \sum_{b \in \mathrm{Blk}(6.M_{22}, B)} \ell(b),$$

where $\mathrm{Blk}(6.M_{22}, B) = \{b \in \mathrm{Blk}(6.M_{22}) : b^G = B\}$.

If $B = B_2$, then $D(B_2) =_G 3^2$, $\mathrm{Blk}(6.M_{22}, B) = \{b_2, b_2'\}$ and, by [13, p. 78], $\ell(b_2) = \ell(b_2') = 2$, and so $\ell(B_2) = 5$.

If $B = B_1$, then $D(B_1) =_G 3_+^{1+2}$, $\mathrm{Blk}(6.M_{22}, B) = \{b\}$; moreover, $\ell(b) = \ell(b_1)$ for a unique block of $2.M_{22}$ with $D(b_1) \simeq 3^2$. Now $2.M_{22}$ has exactly one class x of elements of order 3 and $C_{2.M_{22}}(x) = 3^2 \times 2^3$. It follows, by [15, Theorem 5.4.13] again, that $\ell(b_1) = k(b_1) - 1 = 6 - 1 = 5$, so that $\ell(B_1) = 14 - 5 = 9$.

If $\ell_p(G)$ is the number of p-regular G-conjugacy classes, then $\ell_3(G) = 43$ and $\ell_2(G) = 25$. Thus $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \sum_{B \in \mathrm{Blk}^+(G,p)} \ell(B) + |\,\mathrm{Irr}^0(G)|.$$

This completes the proof. □

THEOREM 5.2. *Let* $G = J_4$, *and let* B *be a* p*-block of* G *with a non-cyclic defect group. Then the number of* B*-weights is the number of irreducible Brauer characters of* B.

Proof. We may suppose that $p = 2$ and $p = 3$. If $p = 2$ and $B = B_0$, then the result follows by Lemma 5.1, Table 1 and (5.1).

Suppose that $p = 3$. Since $\mathrm{Irr}(N(3^{1+2})/C(3^{1+2}))$ has seven irreducible characters, and since $\mathrm{Irr}(N(3^{1+2})/3^{1+2})$ has 14 characters, it follows that $B \in \{B_0, B_1\}$ has 7-weights of the form $(3^{1+2}, \varphi)$. If $(3^2, b)$ is a Brauer B-subgroup, then $\mathrm{Irr}^0(N(3^2)/3^2)$ has exactly two characters covering the canonical character of b, so $\mathrm{Irr}^0(N(3^2)/C(3^2))$ has two characters, B has 2-weights of the form $(3^2, \varphi)$, and B has no weight of the form $(3, \varphi)$.

$\mathrm{Irr}^0(N(3^2)/3^2)$ has 9 characters; hence B_2 has 5-weights of the form $(3^2, \varphi)$. □

6. *Radical chains*

Let $G = J_4$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. We will do some cancellations in the alternating sum of Dade's conjecture. We first list some radical p-chains $C(i)$ and their normalizers for certain integers i, and then we reduce the proof of the conjecture to the subfamily $\mathcal{R}^0(G)$ of $\mathcal{R}(G)$, where $\mathcal{R}^0(G)$ is the union of G-orbits of all $C(i)$. The subgroups of the 2-chains in Table 3 are given either by Table 1 or in the proof of Lemma 6.1.

LEMMA 6.1. *Let* $\mathcal{R}^0(G)$ *be the* G*-invariant subfamily of* $\mathcal{R}(G)$ *such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leqslant i \leqslant 4\}, & \text{with } C(i) \text{ given in Table 2 if } p = 3, \\ \{C(i) : 1 \leqslant i \leqslant 16\}, & \text{with } C(i) \text{ given in Table 3 if } p = 2. \end{cases}$$

Then

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) \tag{6.1}$$

for all integers $d \geqslant 0$.

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Table 2: Some radical 3-chains of J_4

C		$N(C)$
$C(1)$	1	J_4
$C(2)$	$1 < 3$	$6.M_{22}:2$
$C(3)$	$1 < 3 < 3^2$	$(3^2 \times 2^3).(2 \times S_3)$
$C(4)$	$1 < 3^2$	$(3^2 : 2 \times 2^3).S_4$

Proof. Let $C \in \mathcal{R}(G)$ be given by (2.1), so that we may suppose that $P_1 \in \mathcal{R}_0(G, p)$.

Case (1). Suppose that $p = 3$. Let $C' : 1 < 3 < 3^{1+2}$ and $g(C') : 1 < 3^{1+2}$. Then $N(C') = N(g(C')) = N(3^{1+2})$,

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d), \tag{6.2}$$

and we may suppose that $C \neq_G C'$ and $C \neq_G g(C')$. Similarly, let $C' : 1 < 3^2 < 3^{1+2}$ and $g(C') : 1 < 3 < 3^2 < 3^{1+2}$. Then $N(C') = N(g(C')) =_G 3^{1+2}.2^3$, and we may suppose that $C \neq_G C'$ and $C \neq_G g(C')$. Thus $C =_G C(i)$ for $1 \leqslant i \leqslant 4$.

Case (2). Suppose that $p = 2$. Let $M_1 = 2^{11} : M_{24}$, $M_2 = 2^{1+12} \cdot (3 \cdot M_{22}) : 2$, $M_3 = 2^{10} : L_5(2)$ and $M_4 = 2^{3+12} \cdot (S_5 \times L_3(2))$ be maximal subgroups of $G = J_4$. For each $R \in \mathcal{R}_0(G, 2)$, we may suppose that $R \in \mathcal{R}_0(M_i)$ such that $N_G(R) \leqslant N_{M_i}(R)$ for some i.

We first classify the radical 2-subgroups of M_i using the modified local strategy and do the fusions in G by applying Lemma 2.2. Moreover, we carry out cancellation using Lemma 2.1.

Case (2a). We may take

$$\begin{aligned}\mathcal{R}_0(M_2, 2) = \{&2^{1+12}, 2^{1+12}.2^3, 2^{1+12}.2^4, 2^{10}.2^{3+4}, 2^{3+12}.2^2, 2^{1+12}.2^{2+3},\\ &2^{1+12}.2^5, 2^{1+12}.2^3.2^3, 2^{1+12}.2^2.2^4, 2^{1+12}.2.2^{2+3}, 2^{11}.2^2.2^3.2^4,\\ &2^{1+12}.2^{2+5}, 2^{1+12}.2.2^3.2^3, S\},\end{aligned}$$

and by [17, Theorem 17], $N(R) = N_{M_2}(R)$ for all $R \in \mathcal{R}_0(M_2, 2)$, so that we may suppose that $\mathcal{R}_0(M_2, 2) \subseteq \mathcal{R}_0(G, 2)$.

Let $R \in \mathcal{R}_0(M_2, 2) \backslash \{2^{1+12}\}$ and let $\sigma(R) : 1 < Q = 2^{1+12} < R$, so that $\sigma(R)' : 1 < R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.1. Thus there is a bijection g from $\mathcal{R}^-(\sigma(R), 2^{1+12})$ onto $\mathcal{R}^0(\sigma(R), 2^{1+12})$ such that $N(C') = N(g(C'))$ and $|C'| = |g(C')| - 1$ for each $C' \in \mathcal{R}^-(\sigma(R), 2^{1+12})$. So (6.2) holds, and we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(M_2,2) \backslash \{2^{1+12}\}} (\mathcal{R}^-(\sigma(R), 2^{1+12}) \cup \mathcal{R}^0(\sigma(R), 2^{1+12})).$$

In particular, $P_1 \notin_G \mathcal{R}_0(M_2, 2) \backslash \{2^{1+12}\}$ and if $P_1 = 2^{1+12}$, then $C =_G C(6)$. We may suppose that

$$\begin{aligned}P_1 \in_G \{&2^{10}, 2^{11}, 2^{10}.2^4, 2^{11}.2^4, 2^{3+12}, 2^{3+12}.2, 2^3.2^{6+8}, 2^2.2^{5+10},\\ &2^{3+12}.D_8, 2^{3+12}.2^3, 2^{11}.2^{1+6}, 2^{3+12}.2^4, 2^{3+12}.2^3.2^2\}.\end{aligned}$$

Table 3: Some radical 2-chains of J_4

C		$N(C)$
$C(1)$	1	J_4
$C(2)$	$1 < 2^{11}$	$2^{11}: M_{24}$
$C(3)$	$1 < 2^{11} < 2^{1+12}.2^4$	$2^{1+12}.2^4.3.S_6$
$C(4)$	$1 < 2^{11} < 2^3.2^{6+8} < 2^{1+12}.2^2.2^4$	$2^{1+12}.2^2.2^4.(S_3 \times S_3)$
$C(5)$	$1 < 2^{11} < 2^3.2^{6+8}$	$2^3.2^{6+8}.(S_3 \times L_3(2))$
$C(6)$	$1 < 2^{1+12}$	$2^{1+12}.3.M_{22}: 2$
$C(7)$	$1 < 2^{10} < 2^6.2^8$	$2^6.2^8.L_4(2)$
$C(8)$	$1 < 2^{10}$	$2^{10}: L_5(2)$
$C(9)$	$1 < 2^{10} < 2^{1+12}.2^3$	$2^{1+12}.2^3(S_3 \times L_3(2))$
$C(10)$	$1 < 2^{10} < 2^{1+12}.2^3 < 2^{10}.2^{2+6}$	$2^{10}.2^{2+6}.(S_3 \times S_3)$
$C(11)$	$1 < 2^{10} < 2^{3+12}.2$	$2^{3+12}.2.(S_3 \times L_3(2))$
$C(12)$	$1 < 2^{10} < 2^{3+12}.2 < 2^6.2^8.2^3$	$2^6.2^8.2^3.L_3(2)$
$C(13)$	$1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3} < 2^{10}.2^2.2^{3+4}$	$2^{10}.2^2.2^{3+4}.S_3$
$C(14)$	$1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3}$	$2^{1+12}.2^{2+3}.(S_3 \times S_3)$
$C(15)$	$1 < 2^{3+12} < 2^{3+12}.2^2$	$2^{3+12}.2^2.(S_3 \times S_5)$
$C(16)$	$1 < 2^{3+12}$	$2^{3+12}.(S_5 \times L_3(2))$

Case (2b). Applying the local strategy [3, 4], we obtain sixteen radical subgroups of M_4. Let

$$\mathcal{X} = \{2^{3+12}.2, 2^3.2^{6+8}, 2^2.2^{5+10}, 2^{3+12}.D_8, 2^{3+12}.2^3, 2^{3+12}.2^4, 2^{3+12}.2^3.2^2\},$$

so that each subgroup of $\mathcal{X}$ is radical in G and contained in M_4, by [17]. Using the local structures, we can identify each $R \in \mathcal{X}$ with a radical subgroup of M_4.

Next we consider fusions of subgroups in $\mathcal{R}_0(M_2, 2)$ and $\mathcal{R}_0(M_4, 2)$. Let

$$R = 2^{3+12}.2^2 \in \mathcal{R}_0(M_4, 2),$$

so that $Z(R) = 2$ and $N_{M_4}(R) = 2^{3+12}.2^2.(S_3 \times S_5)$. Now G has two classes of involutions $2A$ and $2B$ such that $C(2A) = M_2$ and $C(2B) = 2^{11}: M_{22}: 2$. Since $|N_{M_4}(R)|_2 > |C(2B)|_2$, it follows that $Z(R)$ is a $2A$ involution. In particular, we may suppose that $N_{M_4}(R) \leqslant M_2$. By Lemma 2.2, there is a radical subgroup $P \in \mathcal{R}_0(M_2, 2)$ such that $R \leqslant P$ and $N_{M_4}(R) \leqslant N_{M_2}(P)$. By the local structures of subgroups of $\mathcal{R}_0(M_2, 2)$, R is a radical subgroup of M_2 such that $N_{M_4}(R) = N_{M_2}(R) = N(R)$, and so is a radical subgroup of G.

Applying the local strategy to $N_{M_4}(R)$, we find that $N_{M_4}(R)$ has exactly seven radical subgroups W, and each is radical in M_4 with $N_{N_{M_4}(R)}(W) = N_{M_4}(W)$. In particular,

$$\mathcal{R}_0(N_{M_4}(R), 2) = \mathcal{R}_0(M_4, 2)\backslash(\mathcal{X} \cup \{2^{3+12}\}). \tag{6.3}$$

If we view R as a subgroup of $\mathcal{R}_0(M_2, 2)$ (we can identify R with a radical subgroup of M_2 using local structures), then each radical subgroup W of $N_{M_2}(R) = N_{M_4}(R)$ is radical in M_2 with $N_{M_2}(W) = N_{N_{M_2}(R)}(W)$. It follows that each $W \in \mathcal{R}_0(N_{M_4}(R), 2)$ is radical in G with $N(W) = N_{M_4}(W)$, and so each $Q \in \mathcal{R}_0(M_4, 2)$ is radical in G with $N(Q) = N_{M_4}(Q)$.

We may take

$$\begin{aligned}\mathcal{R}_0(M_4, 2) = \{&2^{3+12}, 2^{3+12}.2, 2^{3+12}.2^2, 2^3.2^{6+8}, 2^2.2^{5+10}, 2^{3+12}.D_8,\\ &2^{1+12}.2^5, 2^{1+12}.2^{2+3}, 2^{3+12}.2^3, 2^{1+12}.2^2.2^4, 2^{3+12}.2^4,\\ &2^{1+12}.2^3.2^3, 2^{1+12}.2^{2+5}, 2^{11}.2^2.2^3.2^4, 2^{3+12}.2^3.2^2, S\}\end{aligned}$$

and $N(R) = N_{M_4}(R)$ for all $R \in \mathcal{R}_0(M_4, 2)$, so that we may suppose that $\mathcal{R}_0(M_4, 2) \subseteq \mathcal{R}_0(G, 2)$.

Let $R \in \mathcal{X}$ and $\sigma(R) : 1 < Q = 2^{3+12} < R$, so that $\sigma(R)' : 1 < R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.1. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R\in\mathcal{X}} (\mathcal{R}^-(\sigma(R), 2^{3+12}) \cup \mathcal{R}^0(\sigma(R), 2^{3+12})).$$

In particular, $P_1 \notin_G \mathcal{X}$, and if $P_1 = 2^{3+12}$, then $P_2 \notin_G \mathcal{X}$.

Let $K = 2^{3+12}.2^2.(S_3 \times S_5)$. We may take

$$\begin{aligned}\mathcal{R}_0(K, 2) = \{&2^{3+12}.2^2, 2^{1+12}.2^5, 2^{1+12}.2^{2+3}, 2^{1+12}.2^2.2^4,\\ &2^{1+12}.2^3.2^3, 2^{1+12}.2^{2+5}, 2^{11}.2^2.2^3.2^4, S\} \subseteq \mathcal{R}_0(M_4, 2)\end{aligned}$$

and $N_K(R) = N_{M_4}(R) = N(R)$ for all $R \in \mathcal{R}_0(K, 2)$,and

$$\mathcal{R}_0(K, 2) = \mathcal{R}_0(M_4, 2)\backslash(\mathcal{X} \cup \{2^{3+12}\}).$$

Let $R \in \mathcal{R}_0(K, 2)\backslash\{2^{3+12}.2^2\}$ and let $\sigma(R) : 1 < 2^{3+12} < Q = 2^{3+12}.2^2 < R$, so that $\sigma(R)' : 1 < 2^{3+12} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R\in\mathcal{R}_0(2^{3+12}.2^2.(S_3\times S_5),2)} (\mathcal{R}^-(\sigma(R), 2^{3+12}.2^2) \cup \mathcal{R}^0(\sigma(R), 2^{3+12}.2^2)).$$

In particular, if $P_1 =_G 2^{3+12}$, then $C \in_G \{C(15), C(16)\}$.

Case (2c). The fusions in G of subgroups of $\mathcal{R}_0(M_1, 2)$ with other subgroups in other $\mathcal{R}_0(M_i, 2)$ is similar to that of Case (2b). We may take

$$\begin{aligned}\mathcal{R}_0(M_1, 2) = \{&2^{11}, 2^{11}.2^4, 2^{1+12}.2^4, 2^3.2^{6+8}, 2^{11}.2^{1+6}, 2^{3+12}.D_8,\\ &2^{1+12}.2^2.2^4, 2^{1+12}.2.2^{2+3}, 2^{3+12}.2^4, 2^{1+12}.2^{2+5},\\ &2^{3+12}.2^3.2^2, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2.2^3.2^3, S\}\end{aligned}$$

and $N(R) = N_{M_1}(R)$ for all $R \in \mathcal{R}_0(M_1, 2)$, so we may suppose that $\mathcal{R}_0(M_1, 2) \subseteq \mathcal{R}_0(G, 2)$.

For $R \in \{2^{11}.2^4, 2^{11}.2^{1+6}\} \subseteq \mathcal{R}_0(M_1, 2)$, let $\sigma(R) : 1 < Q = 2^{11} < R$, so that $\sigma(R)' : 1 < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \{2^{11}.2^4, 2^{11}.2^{1+6}\}} (\mathcal{R}^-(\sigma(R), 2^{11}) \cup \mathcal{R}^0(\sigma(R), 2^{11})).$$

In particular, $P_1 \neq_G 2^{11}.2^4$ or $2^{11}.2^{1+6}$, and if $P_1 = 2^{11}$, then $P_2 \neq_G 2^{11}.2^4$ and $2^{11}.2^{1+6}$.

Let $L = 2^{1+12}.2^4.3.S_6$. We may take

$$\mathcal{R}_0(L, 2) = \{2^{1+12}.2^4, 2^{1+12}.2^2.2^4, 2^{1+12}.2.2^{2+3}, \\ 2^{1+12}.2^{2+5}, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2.2^3.2^3, S\} \subseteq \mathcal{R}_0(M_1, 2)$$

and $N_L(R) = N_{M_1}(R)$ for all $R \in \mathcal{R}_0(L, 2)$.

Let $R \in \mathcal{R}_0(L, 2) \backslash \{2^{1+12}.2^4\}$, and let $\sigma(R) : 1 < 2^{11} < Q = 2^{1+12}.2^4 < R$, so that $\sigma(R)' : 1 < 2^{11} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{1+12}.2^4.3.S_6, 2)} (\mathcal{R}^-(\sigma(R), 2^{1+12}.2^4) \cup \mathcal{R}^0(\sigma(R), 2^{1+12}.2^4)).$$

In particular, if $P_1 = 2^{11}$, then $P_2 \notin_G \mathcal{R}_0(L, 2) \backslash \{2^{1+12}.2^4\}$; if, moreover, $P_2 =_G 2^{1+12}.2^4$, then $C =_G C(3)$.

Let $H = 2^3.2^{6+8}.(S_3 \times L_3(2))$. We may take

$$\mathcal{R}_0(H, 2) = \{2^3.2^{6+8}, 2^{3+12}.D_8, 2^{1+12}.2^2.2^4, 2^{3+12}.2^4, \\ 2^{1+12}.2^{2+5}, 2^{3+12}.2^3.2^2, 2^{11}.2^2.2^3.2^4, S\} \subseteq \mathcal{R}_0(M_1, 2)$$

and $N_H(R) = N_{M_1}(R)$ for all $R \in \mathcal{R}_0(H, 2)$.

Let

$$\mathcal{Y} = \{2^{3+12}.D_8, 2^{3+12}.2^4, 2^{3+12}.2^3.2^2\} \subseteq \mathcal{R}_0(H, 2), \qquad R \in \mathcal{Y},$$

and let $\sigma(R) : 1 < 2^{11} < Q = 2^3.2^{6+8} < R$, so that $\sigma(R)' : 1 < 2^{11} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{Y}} (\mathcal{R}^-(\sigma(R), 2^3.2^{6+8}) \cup \mathcal{R}^0(\sigma(R), 2^3.2^{6+8})).$$

In particular, if $P_1 = 2^{11}$, then $P_2 \notin_G \mathcal{Y}$; if, moreover, $P_2 = 2^3.2^{6+8}$, then $P_3 \notin \mathcal{Y}$.

We may take

$$\mathcal{R}_0(2^{1+12}.2^2.2^4.(S_3 \times S_3), 2) = \mathcal{R}_0(H, 2) \backslash (\mathcal{Y} \cup \{2^3.2^{6+8}\};$$

moreover, $N_{2^{1+12}.2^2.2^4.(S_3 \times S_3)}(R) = N_H(R) = N_{M_1}(R)$.

For each

$$R \in \mathcal{R}_0(2^{1+12}.2^2.2^4.(S_3 \times S_3), 2) \backslash \{2^{1+12}.2^2.2^4\},$$

let $\sigma(R) : 1 < 2^{11} < 2^3.2^{6+8} < Q = 2^{1+12}.2^2.2^4 < R$, so that $\sigma(R)' : 1 < 2^{11} < 2^3.2^{6+8} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{1+12}.2^2.2^4.(S_3 \times S_3), 2) \backslash \{2^{1+12}.2^2.2^4\}} (\mathcal{R}^-(\sigma(R), 2^{1+12}.2^2.2^4) \cup \mathcal{R}^0(\sigma(R), 2^{1+12}.2^2.2^4)).$$

It follows that if $P_1 = 2^{11}$, then $C =_G C(i)$ for some $2 \leqslant i \leqslant 5$.

Case (2d). The fusions in G of subgroups of $\mathcal{R}_0(M_3, 2)$ with subgroups in other $\mathcal{R}_0(M_i, 2)$ is similar to that of Case (2b). We may take

$$\begin{aligned}\mathcal{R}_0(M_3, 2) = \{&2^{10}, 2^{10}.2^4, 2^6.2^8, 2^{3+12}.2, 2^{1+12}.2^3, 2^{10}.2^{3+4}, 2^6.2^8.2^3,\\ &2^{10}.2^{1+6}, 2^{1+12}.2^{2+3}, 2^{10}.2^{2+6}, 2^{3+12}.2^3, 2^{1+12}.2^3.2^3,\\ &2^{10}.2.2^{3+5}, 2^{10}.2^2.2^{3+4}, 2^{10}.2^6.2^3, S\}\end{aligned}$$

and $N(R) = N_{M_3}(R)$ for $R \in \mathcal{R}_0(M_3, 2)\backslash\mathcal{Z}$, where

$$\mathcal{Z} = \{2^6.2^8, 2^6.2^8.2^3, 2^{10}.2^{1+6}, 2^{10}.2^{2+6}, 2^{10}.2.2^{3+5}, 2^{10}.2^2.2^{3+4}, 2^{10}.2^6.2^3\}.$$

In addition, for $R \in \mathcal{Z}$, $C_{M_3}(2^6.2^8) = 2^6$, $C_{M_3}(2^6.2^8.2^3) \simeq 2^3 \simeq C_{M_3}(2^{10}.2^{1+6})$,

$$C_{M_3}(2^{10}.2^{2+6}) \simeq C_{M_3}(2^{10}.2.2^{3+5}) \simeq C_{M_3}(2^{10}.2^2.2^{3+4}) \simeq 2,$$

$C_{M_3}(2^{10}.2^6.2^3) \simeq 2^2$ and, moreover,

$$N_{M_3}(R) = \begin{cases} 2^6.2^8.L_4(2), & \text{if } R = 2^6.2^8,\\ 2^6.2^8.2^3.L_3(2), & \text{if } R = 2^6.2^8.2^3,\\ 2^{10}.2^{1+6}.L_3(2), & \text{if } R = 2^{10}.2^{1+6},\\ 2^{10}.2^{2+6}.(S_3 \times S_3), & \text{if } R = 2^{10}.2^{2+6},\\ 2^{10}.2.2^{3+5}.S_3, & \text{if } R = 2^{10}.2.2^{3+5},\\ 2^{10}.2^2.2^{3+4}.S_3, & \text{if } R = 2^{10}.2^2.2^{3+4},\\ 2^{10}.2^6.2^3.S_3, & \text{if } R = 2^{10}.2^6.2^3.\end{cases}$$

Let $\sigma : 1 < Q = 2^{10} < 2^{10}.2^4$, so that $\sigma' : 1 < 2^{10}.2^4$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 2^{10}) \cup \mathcal{R}^0(\sigma, 2^{10})).$$

In particular, $P_1 \neq_G 2^{10}.2^4$, and if $P_1 = 2^{10}$, then $P_2 \neq_G 2^{10}.2^4$.

Let $J = 2^6.2^8.L_4(2)$. We may take

$$\begin{aligned}\mathcal{R}_0(J, 2) = \{&2^6.2^8, 2^6.2^8.2^3, 2^{10}.2^{1+6}, 2^{10}.2^{2+6},\\ &2^{10}.2.2^{3+5}, 2^{10}.2^2.2^{3+4}, 2^{10}.2^6.2^3, S\} \subseteq \mathcal{R}_0(M_3, 2)\end{aligned}$$

and $N_J(R) = N_{M_3}(R)$ for all $R \in \mathcal{R}_0(J, 2)$.

Let $R \in \mathcal{R}_0(J, 2)\backslash\{2^6.2^8\}$ and let $\sigma(R) : 1 < 2^{10} < Q = 2^6.2^8 < R$, so that $\sigma(R)' : 1 < 2^{10} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^6.2^8.L_4(2),2)} (\mathcal{R}^-(\sigma(R), 2^6.2^8) \cup \mathcal{R}^0(\sigma(R), 2^6.2^8)).$$

In particular, if $P_1 = 2^{10}$, then $P_2 \notin_G \mathcal{R}_0(J, 2)\backslash\{2^6.2^8\}$; if, moreover, $P_2 =_G 2^6.2^8$, then $C =_G C(7)$.

Let $T = 2^{1+12}.2^3.(S_3 \times L_3(2))$. We may take

$$\begin{aligned}\mathcal{R}_0(T, 2) = \{&2^{1+12}.2^3, 2^{10}.2^{3+4}, 2^{1+12}.2^{2+3}, 2^{10}.2^{2+6},\\ &2^{1+12}.2^3.2^3, 2^{10}.2.2^{3+5}, 2^{10}.2^2.2^{3+4}, S\} \subseteq \mathcal{R}_0(M_3, 2)\end{aligned}$$

and $N_T(R) = N_{M_3}(R)$ for all $R \in \mathcal{R}_0(T, 2)$.

Let

$$\mathcal{T} = \{2^{10}.2^{3+4}, 2^{1+12}.2^{2+3}, 2^{1+12}.2^3.2^3\} \subseteq \mathcal{R}_0(T, 2),$$

$R \in \mathcal{T}$, and let $\sigma(R) : 1 < 2^{10} < Q = 2^{1+12}.2^3 < R$, so that $\sigma(R)' : 1 < 2^{10} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{T}} (\mathcal{R}^-(\sigma(R), 2^{1+12}.2^3) \cup \mathcal{R}^0(\sigma(R), 2^{1+12}.2^3)).$$

In particular, if $P_1 = 2^{10}$, then $P_2 \notin_G \mathcal{T}$; if, moreover, $P_2 = 2^{1+12}.2^3$, then $P_3 \notin \mathcal{T}$.

Let $V = 2^{3+12}.2.(S_3 \times L_3(2))$. We may take

$$\begin{aligned}\mathcal{R}_0(V, 2) = \{&2^{3+12}.2, 2^6.2^8.2^3, 2^{1+12}.2^{2+3}, 2^{3+12}.2^3,\\ &2^{1+12}.2^3.2^3, 2^{10}.2^2.2^{3+4}, 2^{10}.2^6.2^3, S\} \subseteq \mathcal{R}_0(M_3, 2)\end{aligned}$$

and $N_V(R) = N_{M_3}(R)$ for all $R \in \mathcal{R}_0(V, 2)$.

Let $\sigma : 1 < 2^{10} < Q = 2^{3+12}.2 < 2^{3+12}.2^3$, so that $\sigma(R)' : 1 < 2^{10} < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 2^{3+12}.2) \cup \mathcal{R}^0(\sigma, 2^{3+12}.2)).$$

In particular, if $P_1 = 2^{10}$, then $P_2 \notin_G 2^{3+12}.2^3$ and if moreover, $P_2 = 2^{3+12}.2$, then $P_3 \neq 2^{3+12}.2^3$.

We may take

$$\mathcal{R}_0(2^{10}.2^{2+6}.(S_3 \times S_3), 2) = \{2^{10}.2^{2+6}, 2^{10}.2.2^{3+5}, 2^{10}.2^2.2^{3+4}, S\} \subseteq \mathcal{R}_0(M_3, 2);$$

moreover, $N_{2^{10}.2^{2+6}.(S_3\times S_3)}(R) = N_{M_3}(R)$.

For each

$$R \in \mathcal{R}_0(2^{10}.2^{2+6}.(S_3 \times S_3), 2)\backslash\{2^{10}.2^{2+6}\},$$

let

$$\sigma(R) : 1 < 2^{10} < 2^{1+12}.2^3 < Q = 2^{10}.2^{2+6} < R,$$

so that

$$\sigma(R)' : 1 < 2^{10} < 2^{1+12}.2^3 < R.$$

A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{10}.2^{2+6}.(S_3\times S_3),2)\backslash\{2^{10}.2^{2+6}\}} (\mathcal{R}^-(\sigma(R), 2^{10}.2^{2+6}) \cup \mathcal{R}^0(\sigma(R), 2^{10}.2^{2+6})).$$

It follows that if $P_1 = 2^{10}$ and $P_2 = 2^{1+12}.2^3$, then $C \in \{C(9), C(10)\}$.

We may take

$$\mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2) = \{2^6.2^8.2^3, 2^{10}.2^2.2^{3+4}, 2^{10}.2^6.2^3, S\} \subseteq \mathcal{R}_0(M_3, 2);$$

moreover, $N_{2^6.2^8.2^3.L_3(2)}(R) = N_{M_3}(R)$.

For each

$$R \in \mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2)\backslash\{2^6.2^8.2^3\},$$

let

$$\sigma(R) : 1 < 2^{10} < 2^{3+12}.2 < Q = 2^6.2^8.2^3 < R,$$

so that $\sigma(R)' : 1 < 2^{10} < 2^{3+12}.2 < R$. A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^6.2^8.2^3.L_3(2),2)\backslash\{2^6.2^8.2^3\}} (\mathcal{R}^-(\sigma(R), 2^6.2^8.2^3) \cup \mathcal{R}^0(\sigma(R), 2^6.2^8.2^3)).$$

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So if $P_1 = 2^{10}$ and $P_2 = 2^{3+12}.2$, then $P_3 \notin_G \mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2)\backslash\{2^6.2^8.2^3\}$, and if moreover, $P_3 = 2^6.2^8.2^3$, then $C =_G C(11)$.

We may take

$$\mathcal{R}_0(2^{1+12}.2^{2+3}.(S_3 \times S_3), 2) = \{2^{1+12}.2^{2+3}, 2^{1+12}.2^3.2^3, 2^{10}.2^2.2^{3+4}, S\} \subseteq \mathcal{R}_0(M_3, 2)$$

and, moreover, $N_{2^{1+12}.2^{2+3}.(S_3\times S_3)}(R) = N_{M_3}(R)$.

Let

$$\sigma : 1 < 2^{10} < 2^{3+12}.2 < Q = 2^{1+12}.2^{2+3} < 2^{1+12}.2^3.2^3,$$

so that

$$\sigma' : 1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^3.2^3.$$

A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 2^{1+12}.2^{2+3}) \cup \mathcal{R}^0(\sigma, 2^{1+12}.2^{2+3})).$$

It follows that if $P_1 = 2^{10}$ and $P_2 = 2^{3+12}.2$, then $P_3 \neq_G 2^{1+12}.2^3.2^3$ and if, moreover, $P_3 = 2^{1+12}.2^{2+3}$, then $P_4 \neq_G 2^{1+12}.2^3.2^3$.

Let

$$\sigma : 1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3} < Q = 2^{10}.2^2.2^{3+4} < S,$$

so that

$$\sigma' : 1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3} < S.$$

A similar proof to that of Case (2a) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 2^{10}.2^2.2^{3+4}) \cup \mathcal{R}^0(\sigma, 2^{10}.2^2.2^{3+4})).$$

It follows that if $P_1 = 2^{10}$, then $C \in \{C(i) : 7 \leqslant i \leqslant 14\}$. □

7. *The proof of Dade's ordinary conjecture*

Let $L = N(C)$ be the normalizer of a radical p-chain. If L is a maximal subgroup of J_4, then the character table of L can be found in the library of character tables distributed with GAP. If L is not a maximal subgroup, its character table can be calculated easily using MAGMA.

The tables listing the degrees of irreducible characters referenced in the proof of Theorem 7.1 are given in Appendix A.

THEOREM 7.1. *Let B be a p-block of $G = J_4$ with a positive defect. Then B satisfies Dade's ordinary conjecture.*

Proof. By [11] and [9], we may suppose that $p = 2$ or $p = 3$, and a defect group of B is non-cyclic. By Lemma 5.1, $B \in \{B_0, B_1, B_2\}$ when $p = 3$ and $B = B_0$ when $p = 2$.

Case (1). Suppose that $p = 3$. If $B = B_2$, then

$$\mathrm{k}(N(C(1)), B_2, d) = \mathrm{k}(N(C(4)), B_2, d) = \begin{cases} 9, & \text{if } d = 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{k}(N(C(3)), B_2, d) = \mathrm{k}(N(C(2)), B_2, d) = \begin{cases} 18, & \text{if } d = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Table 4: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 20$

Defect d	20	19	18	17	16	15	14	13	12	11	10	otherwise
k(7, d, u)	32	16	36	28	2	2	10	2	0	0	1	0
k(8, d, u)	32	16	20	8	2	0	2	0	0	0	1	0
k(9, d, u)	32	32	52	24	18	8	18	12	2	2	1	0
k(10, d, u)	32	32	84	100	26	26	26	14	6	4	1	0
k(11, d, u)	32	56	28	16	38	32	6	0	0	0	0	0
k(12, d, u)	32	56	44	36	38	34	14	2	0	0	0	0
k(13, d, u)	32	72	92	124	62	58	30	22	10	2	0	0
k(14, d, u)	32	72	60	48	54	40	22	20	6	0	0	0

This proves the theorem when $B = B_2$.

If $B = B_0$ or B_1, then

$$\mathrm{k}(N(C(1)), B, d) = \mathrm{k}(N(C(2)), B, d) = \begin{cases} 9, & \text{if } d = 3, \\ 5, & \text{if } d = 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{k}(N(C(3)), B, d) = \mathrm{k}(N(C(4)), B, d) = \begin{cases} 9, & \text{if } d = 3, \\ 2, & \text{if } d = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This proves the theorem when $p = 3$.

Case (2). Suppose that $p = 2$, so that $B = B_0$. We set $\mathrm{k}(i, d, u) = \mathrm{k}(N(C(i)), B, d, u)$ for integers i, d and u.

We first consider the chains C with $\mathrm{d}(N(C)) = 20$, so that $C =_G C(i)$ for $7 \leqslant i \leqslant 14$. The values $\mathrm{k}(i, d, u)$ are given in Table 4.

It follows that

$$\sum_{i=7}^{14} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = 0.$$

Finally, suppose that $C = C(i)$ is a chain with $\mathrm{d}(N(C)) = 21$. Then $C =_G C(i)$ for $1 \leqslant i \leqslant 6$ or $15 \leqslant i \leqslant 16$. The values $\mathrm{k}(i, d, u)$ are given in Table 5.

It follows that

$$\sum_{\mathrm{d}(N(C))=21} (-1)^{|C|} \mathrm{k}(N(C), B_0, d, u) = 0,$$

and Theorem 7.1 follows. □

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Table 5: Values of $k(i, d, u)$ when $p = 2$ and $d(N(C(i))) = 21$

Defect d	21	20	19	18	17	16	15	14	13	12	11	7	otherwise
k(1, d, u)	32	8	4	4	1	0	4	4	1	0	0	1	0
k(2, d, u)	32	8	4	12	7	0	4	4	1	0	0	0	0
k(3, d, u)	32	24	12	28	11	12	11	12	11	6	1	0	0
k(4, d, u)	32	40	28	40	21	30	27	14	15	4	1	0	0
k(5, d, u)	32	24	20	24	9	18	20	6	1	0	0	0	0
k(6, d, u)	32	24	12	4	1	4	5	12	11	6	1	1	0
k(15, d, u)	32	40	28	16	11	22	21	14	15	4	1	0	0
k(16, d, u)	32	24	20	16	3	18	20	6	1	0	0	0	0

Appendix A. *Degrees of character tables for chain normalisers of* J_4

Table A.1: The degrees of characters in $\mathrm{Irr}(J_4)$

Degree:	1	1333	299367	887778	889111
Number:	1	2	2	2	1
Degree:	1187145	1776888	3403149	4290927	32307363
Number:	2	1	2	1	2
Degree:	32897107	35411145	95288172	230279749	259775040
Number:	2	2	1	1	2
Degree:	300364890	366159104	393877506	394765284	460559498
Number:	1	1	1	1	1
Degree:	493456605	690839247	786127419	789530568	885257856
Number:	1	1	3	1	2
Degree:	1016407168	1085604531	1089007680	1182518964	1183406741
Number:	2	1	1	1	2
Degree:	1184295852	1445942610	1509863773	1579061136	1842237992
Number:	1	3	1	1	1
Degree:	1903741279	1981808640	2001151845	2267824128	2692972480
Number:	1	3	3	1	1
Degree:	2727495848	3054840657			
Number:	1	1			

Table A.2: The degrees of characters in $\mathrm{Irr}(2^{11} : M_{24})$

Degree:	1	23	45	231	252	253	483
Number:	1	1	2	2	1	1	1
Degree:	759	770	990	1035	1265	1288	1771
Number:	1	2	2	3	1	2	1
Degree:	2024	2277	3312	3520	5313	5544	5796
Number:	1	1	1	1	2	1	1
Degree:	10395	10626	11385	15180	15939	21252	26565
Number:	1	1	1	1	3	1	1
Degree:	28336	34155	41216	42504	48576	53130	57960
Number:	1	4	1	1	1	1	2
Degree:	68310	69552	70840	79695	85008	91080	127512
Number:	1	2	2	3	2	1	2
Degree:	141680	154560	159390	185472	226688	239085	
Number:	1	2	1	2	2	2	

Table A.3: The degrees of characters in $\mathrm{Irr}(2^{1+12}.2^4.3.S_6)$

Degree:	1	5	6	9	10	12	15	16	18	30
Number:	2	4	2	2	2	1	4	1	3	5
Degree:	45	60	72	90	108	135	180	270	288	360
Number:	8	1	2	8	1	12	5	4	4	10
Degree:	384	540	576	640	720	768	1080	1152	1280	1440
Number:	2	4	1	4	4	1	16	4	1	8
Degree:	1536	1728	1920	2160	2304	2560	2880	3072	3840	
Number:	2	4	2	6	4	4	6	1	5	

Table A.4: The degrees of characters in $\mathrm{Irr}(2^{1+12}.2^2.2^4.(S_3 \times S_3))$

Degree	1	2	3	4	6	9	12	18	24	36	48
Number	4	4	16	1	10	12	5	26	4	22	1
Degree	72	96	144	192	256	288	384	512	576	768	1024
Number	36	4	20	13	4	26	14	4	14	11	1

Table A.5: The degrees of characters in $\mathrm{Irr}(2^3.2^{6+8}.(L_3(2) \times S_3))$

Degree:	1	2	3	6	7	8	9	12	14	16
Number:	2	1	6	4	2	2	4	1	1	1
Degree:	18	21	24	42	63	84	126	168	252	336
Number:	2	10	2	8	8	6	8	6	13	5
Degree:	448	504	672	896	1008	1344	1792	2016	2688	4032
Number:	4	14	8	4	3	14	1	10	2	2

Table A.6: The degrees of characters in $\mathrm{Irr}(2^{1+12}.3.M_{22}:2)$

Degree:	1	21	42	45	55	90	99	154
Number:	2	2	1	4	2	2	2	2
Degree:	198	210	231	385	420	462	560	640
Number:	1	4	2	2	1	2	1	4
Degree:	660	693	768	1386	2016	2772	3465	3584
Number:	1	4	1	2	2	4	4	2
Degree:	4158	5544	6930	7680	8316	8448	10395	13440
Number:	2	2	4	2	1	2	8	2
Degree:	13860	15360	16128	19712	20160	20790	21120	22176
Number:	5	1	3	1	5	4	2	2
Degree:	24192	26880	27720	28160	42240	49152		
Number:	4	3	2	2	1	1		

Table A.7: The degrees of characters in $\mathrm{Irr}(2^6.2^8.L_4(2))$

Degree:	1	7	14	15	20	21	28	35	45	56	64
Number:	1	1	1	3	1	3	1	1	8	1	1
Degree:	70	90	105	120	140	210	280	315	420	448	560
Number:	1	3	9	3	4	5	4	6	13	2	1
Degree:	630	840	1260	1680	2240	2520	3360	4032	4480	6720	7168
Number:	6	8	17	1	4	12	2	2	2	1	1

Table A.8: The degrees of characters in $\mathrm{Irr}(2^{10}:L_5(2))$

Degree:	1	30	124	155	217	280	310	315
Number:	1	1	1	3	1	1	1	6
Degree:	465	496	651	868	930	960	1024	1085
Number:	8	1	3	3	7	1	1	2
Degree:	1240	1860	2170	2480	3255	4340	6510	7812
Number:	3	1	1	1	4	4	4	2
Degree:	8680	9765	13020	13888	19530	26040	39060	
Number:	2	4	5	1	2	2	4	

Table A.9: The degrees of characters in $\mathrm{Irr}(2^{1+12}.2^3.(S_3 \times L_3(2)))$

Degree:	1	2	3	6	7	8	12	14	16
Number:	2	1	4	4	6	2	1	5	1
Degree:	21	28	42	63	64	84	126	128	168
Number:	12	1	8	8	2	17	14	1	10
Degree:	192	252	336	384	448	504	512	672	768
Number:	4	33	5	4	6	12	2	6	1
Degree:	896	1008	1024	1344	1792	2016	2688		
Number:	5	12	1	6	1	2	2		

Table A.10: The degrees of characters in $\mathrm{Irr}(2^{10}.2^{2+6}.(S_3 \times S_3))$

Degree:	1	2	3	4	6	8	9	12	16	18	24	36	48
Number:	4	4	12	5	10	4	16	26	1	18	14	53	1
Degree:	64	72	96	128	144	192	256	288	384	512	576	768	1024
Number:	4	82	4	4	24	18	5	22	10	4	4	1	1

Table A.11: The degrees of characters in $\mathrm{Irr}(2^{3+12}.2.(S_3 \times L_3(2))$

Degree:	1	2	3	6	7	8	12	14	16
Number:	4	2	8	8	4	4	2	2	2
Degree:	21	42	63	84	112	126	168	224	252
Number:	8	16	8	10	8	28	2	8	16
Degree:	336	448	504	672	1008	1344	2016		
Number:	16	2	10	16	12	4	8		

Table A.12: The degrees of characters in $\mathrm{Irr}(2^6.2^8.2^3.L_3(2))$

Degree:	1	3	6	7	8	14	21	28	42	56
Number:	4	8	4	12	4	16	8	10	36	2
Degree:	84	112	168	224	336	448	672	896	1344	
Number:	34	16	30	20	22	10	14	2	4	

Table A.13: The degrees of characters in $\mathrm{Irr}(2^{10}.2^2.2^{3+4}.S_3)$

Degree:	1	2	3	4	6	8	12	24	48	64	96	128	192	256	384	512
Number:	16	20	16	10	52	2	82	122	62	16	58	20	14	10	2	2

Table A.14: The degrees of characters in $\mathrm{Irr}(2^{1+12}.2^{2+3}.(S_3 \times S_3))$

Degree:	1	2	3	4	6	9	12	18	24	36	48
Number:	8	8	16	2	16	8	4	48	4	54	22
Degree:	64	72	96	128	144	192	256	288	384	576	768
Number:	8	44	14	8	32	10	2	26	12	4	4

Table A.15: The degrees of characters in $\mathrm{Irr}(2^{3+12}.2^2.(S_3 \times S_5))$

Degree:	1	2	3	4	5	6	8	10	12	15	18
Number:	4	2	4	4	4	2	2	2	5	12	2
Degree:	30	45	60	90	120	180	240	360	480	576	640
Number:	6	8	1	26	2	18	5	12	20	8	8
Degree:	720	768	960	1152	1280	1440	1536	1920	2304	2560	3072
Number:	6	6	13	4	8	2	2	2	1	2	1

Table A.16: The degrees of characters in $\mathrm{Irr}(2^{3+12}.(S_5 \times L_3(2)))$

Degree:	1	3	4	5	6	7	8	12	15	18
Number:	2	4	2	2	3	2	2	4	4	2
Degree:	24	28	30	32	35	36	40	42	48	105
Number:	2	2	2	2	2	1	2	1	1	8
Degree:	210	315	420	630	840	1120	1344	1260	1680	2240
Number:	8	8	6	8	4	8	6	5	1	8
Degree:	2520	2688	3360	4032	4480	5040	5376	6720	5040	8064
Number:	6	2	8	2	2	1	1	4	2	2

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Table A.17: The degrees of characters in $\mathrm{Irr}(6.M_{22}.2)$

Degree:	1	10	21	42	45	55	56	90	99	120	132	154	198
Number:	2	4	2	1	4	2	2	2	2	2	2	2	1
Degree:	210	231	240	252	308	330	385	420	440	462	560	660	768
Number:	6	2	1	3	1	2	2	4	2	2	1	2	2

Table A.18: The degrees of characters in $\mathrm{Irr}((3^2 \times 2^3).(2 \times S_3))$

Degree:	1	2	3	4	6	12
Number:	8	8	8	2	12	2

Table A.19: The degrees of characters in $\mathrm{Irr}((3^2 : 2 \times 2^3).S_4)$

Degree:	1	2	3	4	6	8	12	16	24
Number:	4	6	4	2	4	4	1	2	4

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