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Helicoidal Minimal Surfaces in a Finsler Space of Randers Type

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Abstract. We consider the Finsler space $(\overline{M}^3, \overline{F})$ obtained by perturbing the Euclidean metric of $\mathbb{R}^3$ by a rotation. It is the open region of $\mathbb{R}^3$ bounded by a cylinder with a Randers metric. Using the Busemann–Hausdorff volume form, we obtain the differential equation that characterizes the helicoidal minimal surfaces in $\overline{M}^3$. We prove that the helicoid is a minimal surface in $\overline{M}^3$ only if the axis of the helicoid is the axis of the cylinder. Moreover, we prove that, in the Randers space $(\overline{M}^3, \overline{F})$, the only minimal surfaces in the Bonnet family with fixed axis $O\overline{x}^3$ are the catenoids and the helicoids.

1 Introduction

The development of the theory of minimal surfaces in Finsler spaces started about ten years ago, in contrast with the theory of minimal surfaces in Riemannian spaces, which has been studied for many years with contributions from many authors. In 1998, Z. Shen [6] studied submanifolds of a Finsler space and he introduced the notion of a *mean curvature form* $\mathcal{H}_\varphi$ for an immersion φ of a manifold into a Finsler space using the Busemann–Hausdorff volume form. An immersion φ is said to be *minimal* if $\mathcal{H}_\varphi \equiv 0$. In 2003, M. Souza and the second author [11] presented the first nontrivial examples of minimal surfaces in the Randers space obtained by perturbing the Euclidean metric in $\mathbb{R}^3$ by a translation. In 2004, they obtained a Bernstein type theorem for this space in collaboration with J. Spruck [10]. They showed that the partial differential equation that describes a minimal graph is elliptic when the norm of the translation b is such that $0 \leq b < \sqrt{3}/3$. However, in contrast with the Riemannian case, the equation is not elliptic for $\sqrt{3}/3 \leq b < 1$. Moreover, minimal surfaces in Finsler spaces may have isolated singularities.

Considering the same Randers space, He and Shen [5] proved a Bernstein type theorem for minimal graphs using the Holmes–Thompson volume form and Wu [12] studied surfaces that are minimal with respect to both the Busemann–Hausdorff and the Holmes–Thompson volume forms. One should mention that He and Shen also proved that the Holmes–Thompson volume form for a Randers metric $F = \alpha+\beta$ is just the volume form of the Riemannian metric α. Other interesting results on minimal surfaces in Finsler spaces were also obtained by Cui and Shen [3, 4].

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In [9], we studied minimal surfaces in the Finsler space, $(\overline{M}^3, \overline{F})$, generated by the Euclidean metric of $\mathbb{R}^3$ perturbed by a rotation. It is the open region of $\mathbb{R}^3$ bounded by a cylinder of radius 1 with a Randers metric [2] (see also [9]). One should point out that the ambient space $(\overline{M}^3, \overline{F})$ is a Randers manifold with zero flag curvature. This fact was proved by Shen in [8]. Using the Busemann–Hausdorff volume form, in [9] we proved that the only minimal surfaces of rotation in this space are the catenoids contained in $\overline{M}^3$, generated by the rotation of a catenary around the axis of the cylinder. There are no minimal surfaces of rotation whose rotational axis is different from the axis of the cylinder. We also obtained the partial differential equations that characterizes the minimal surfaces in $\overline{M}^3$ that are the graph of a function. We proved that the only planar regions that are minimal in $(\overline{M}^3, \overline{F})$ are the open disks bounded by the parallels of the cylinder and the strips of planes generated by the intersection of $\overline{M}^3$ with the planes of $\mathbb{R}^3$ that contain the cylinder axis.

In this paper, we study minimal helicoidal surfaces in $(\overline{M}^3, \overline{F})$. In Section 3, we obtain the differential equation that charaterizes the helicoidal minimal surfaces in $(\overline{M}^3, \overline{F})$, with respect to the Busemann-Hausdorff volume form (Theorem 3.3). We show that the helicoid is a minimal surface in $\overline{M}^3$ (Corollary 3.4), and that a helicoid is a minimal surface in $\overline{M}^3$, only if the axis of the helicoid is the axis of the cylinder (Theorem 3.5). In Section 4, (Theorem 4.3) we show that the only minimal surfaces in the one-parameter family of surfaces called the *Bonnet family*, with the fixed axis of the cylindrical region $(\overline{M}^3 \overline{F})$, are the catenoids and the helicoids.

2 Preliminaries

Let M be a C^∞ n-dimensional manifold. A point of the tangent bundle TM will be denoted by (x, y), where $x \in M$, $y \in T_xM$. If we consider local coordinates $x^1, \dots, x^n$ on M, then $\partial/\partial x^i$ and dx^i will be bases for T_xM and T_x^*M respectively.

A *Finsler metric* on M is a function $F\colon TM \to [0, \infty)$ that satisfies the following conditions: (Regularity) $F \in C^\infty$ in $TM \setminus \{0\}$; (Positive homogeneity) $F(x, ty) = tF(x, y), \forall t > 0, (x, y) \in TM$; (Strong convexity) $g = (g_{ij}(x, y)) = (\frac{1}{2}[F^2(x, y)]_{y^i y^j})$ is positive definite at each point of $TM \setminus \{0\}$, where $y = \sum y^i \partial/\partial x^i$. One can show that this property is independent of the local coordinates. Then (M, F) is said to be a *Finsler manifold*.

An interesting class of Finsler metrics on M are the *Randers metrics*. Such a metric is given by

$$F(x, y) = \alpha(x, y) + \beta(x, y), \tag{2.1}$$

where

$$\alpha(x, y) := \sqrt{a_{ij}(x) y^i y^j}, \qquad \beta(x, y) := b_k(x) y^k, \tag{2.2}$$

and a_{ij} are the components of a Riemannian metric, a^{ij} denotes its inverse and b_k are the components of a 1-form β, whose *norm*

$$b = \sqrt{a^{ij} b_i b_j}, \tag{2.3}$$

satisfies $0 \le b < 1$. Then (M, F) is said to be a *Randers space*.

Let (M^n, F) be an oriented Finsler manifold. For a fixed point $x \in M$, let $\{e_i|_x\}_{i=1}^n$ be an arbitrary oriented basis of T_xM and $\{\theta^i\}_{i=1}^n$, its dual basis. Let

$$\mathbb{D}_x^n := \left\{ (y^i) \in \mathbb{R}^n : F(x, y^i e_i) \leq 1 \right\}. \tag{2.4}$$

The *Busemann–Hausdorff volume form* of a Finsler metric F is defined as

$$dV_F := \sigma_F(x)\theta^1 \wedge \cdots \wedge \theta^n, \qquad \text{where } \sigma_F(x) = \frac{\mathrm{vol}(\mathbb{B}^n)}{\mathrm{vol}(\mathbb{D}_x^n)},$$

$\mathbb{B}^n$ is the unit ball of $\mathbb{R}^n$ and vol is the Euclidean volume.

Proposition 2.1 ([7]) *Let (M^n, F) be a Randers space, where the metric $F = \alpha + \beta$ is given by (2.1)–(2.2) and the norm of β, $b = \|\beta\| < 1$, is defined by (2.3). Then the volume form is given by*

$$dV_F = \left(1 - \|\beta\|^2\right)^{\frac{n+1}{2}} \sqrt{\det\left(a_{ij}(x)\right)}\, dx_1 \cdots dx_n.$$

In what follows we will use the following convention for indices: greek letters γ, τ, η, $\xi, \dots$ for indices from 1 to $n+1$, latin letters i, j, k, $l, \dots$ for indices from 1 to n. We will also use the Einstein convention for repeated indices.

We now consider an immersion $\varphi\colon M^n \to (\widetilde{M}^{n+1}, \widetilde{F})$, where $\widetilde{F}$ is a Finsler metric. Then the induced metric on M, given by $F := \varphi^*\widetilde{F}$, is also a Finsler metric (see [6]) and the *volume element induced* on (M, F) by the immersion is defined by $dV_F = \Im(x, z)dx$, where $\Im(x, z) := \frac{\mathrm{vol}(\mathbb{B}^n)}{\mathrm{vol}(\mathcal{D}_x^n)}$, $x \in M$, $z = (z_i^\eta)$, with

$$z_i^\eta := \frac{\partial \varphi^\eta}{\partial x^i}, \tag{2.5}$$

$\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, vol is the Euclidean volume,

$$\mathcal{D}_x^n = \{(y^1, \dots, y^n) \in \mathbb{R}^n | F(x, y^i z_i^\eta \widetilde{e}_\eta) \leq 1\}$$

and $\widetilde{e}_\eta$ is a local frame for $\widetilde{M}^{n+1}$. We introduce the following notation

$$\widetilde{x}^\eta := \varphi^\eta(x). \tag{2.6}$$

The concept of mean curvature form $\mathcal{H}_\varphi$ of the immersion φ was introduced by Z. Shen [6] as follows. Consider a variation of an immersion $\varphi_t\colon M \to \widetilde{M}$, $t \in (-\varepsilon, \varepsilon)$, such that $\varphi_t = \varphi_0$ away from a compact set Ω. By considering the variational vector field $\widetilde{X} = \partial\varphi_t/\partial t|_{t=0}$ one defines

$$\psi_x(\widetilde{X}) = \frac{d}{dt}\Big[\ln \Im(\varphi_t(x), \nabla\varphi_t(x))\Big]_{t=0}.$$

Let $\varphi = \varphi_0$, $F = \varphi^*\widetilde{F}$ and $V(t) = \mathrm{vol}(\Omega, \varphi_t^*\widetilde{F})$, then one can show that

$$V'(0) = \int_M \left[\psi_x(\widetilde{X}) - \operatorname{div}(\mathcal{P}(\widetilde{X})|_x)\right] dV_F,$$

where $\mathcal{P}(\widetilde{X}) = \frac{1}{\Im(z)}\frac{\partial \Im}{\partial z_i^\eta}(z)\widetilde{X}^\eta e_i$.

Now let $\varphi\colon M \to (\widetilde{M}^m, \widetilde{F})$ be an immersion. Given any vector field $\widetilde{X}$ of $\widetilde{M}$ along M, there exists a variation φ_t of φ, whose variational vector filed is $\widetilde{X}$. The mean curvature form of φ, $\mathcal{H}_\varphi$, is a 1-form on $\widetilde{M}$ restricted to $\varphi(M)$ defined by

$$\mathcal{H}_\varphi(\widetilde{X}_x) = \psi_x(\widetilde{X}) - \operatorname{div}(\mathcal{P}(\widetilde{X}))|_x.$$

In local coordinates, considering the basis $\partial/\partial x^i$ and $\partial/\partial \widetilde{x}^\eta$ one has

$$\psi_x(\widetilde{X}) = \frac{1}{\Im}\Big(\frac{\partial \Im}{\partial \widetilde{x}^\eta}\widetilde{X}^\eta + \frac{\partial \Im}{\partial z_i^\eta}\frac{\partial \widetilde{X}^\eta}{\partial x^i}\Big), \qquad \mathcal{P}(\widetilde{X}) = \frac{1}{\Im}\frac{\partial \Im}{\partial z_i^\eta}\widetilde{X}^\eta\frac{\partial}{\partial x^i}.$$

Hence, $\mathcal{H}_\varphi(\widetilde{X})$ is independent of the variation and it is given by

$$\mathcal{H}_\varphi(\widetilde{X}) = \frac{1}{\Im}\Big\{\frac{\partial \Im}{\partial \widetilde{x}^\eta} - \frac{\partial^2 \Im}{\partial \widetilde{x}^\varepsilon \partial z_i^\eta}\frac{\partial \varphi^\varepsilon}{\partial x^i} - \frac{\partial^2 \Im}{\partial z_i^\eta \partial z_j^\varepsilon}\frac{\partial^2 \varphi^\varepsilon}{\partial x^i \partial x^j}\Big\}\widetilde{X}^\eta. \tag{2.7}$$

One can show that the mean curvature form has the following properties: a) $\mathcal{H}_\varphi(\widetilde{X}_x)$ depends linearly on $\widetilde{X}_x$ at each point $x \in M$; b) $\mathcal{H}_\varphi(\widetilde{v}) = 0$, $\forall \widetilde{v} = \varphi_*(v), v \in T_xM, x \in M$.

An immersion φ is said to be *minimal* if $\mathcal{H}_\varphi(\widetilde{X}) = 0, \forall \widetilde{X} \in T_{\varphi(x)}\widetilde{M}$. It follows from the properties a) and b) that, in order to verify that an immersion is minimal, it is sufficient to show that $\mathcal{H}_\varphi(\widetilde{X}) = 0$, for a vector field $\widetilde{X}$ such that $\forall x \in M, \widetilde{X}_x \in T_{\varphi(x)}\widetilde{M} \setminus T_{\varphi(x)}\varphi(M)$.

The Zermelo navigation problem consists in choosing the paths that go from one point to another, in the least possible time, on a Riemannian manifold (M, h) under the influence of a wind or current that is represented by a vector field W on M whose length satisfies $|W| := \sqrt{h(W,W)} < 1$. The solutions of such a problem are geodesics of a Finsler metric of Randers type that is non-Riemannian except when $W = 0$. Conversely, one shows that any Randers metric appears as a solution of Zermelo's navigation problem on an appropriate Riemannian manifold, under the influence of a wind W on M, with $|W| < 1$. Therefore, a Randers space can be considered to be a perturbation of a Riemannian space, and the Randers metric is given by (see [2])

$$F(y) = \frac{1}{\lambda}\Big(\sqrt{[h(W,y)]^2 + |y|^2\lambda} - h(W,y)\Big), \qquad \lambda = 1 - |W|^2.$$

Let h be the Euclidean metric of $\mathbb{R}^3$ with coordinates $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$. We consider the Randers space obtained by perturbing h with by the rotation

$$W := (\overline{x}^2, -\overline{x}^1, 0), \text{ where } (\overline{x}^1)^2 + (\overline{x}^2)^2 < 1. \tag{2.8}$$

We then get a Finsler metric of Randers type $\overline{F} = \overline{\alpha} + \overline{\beta}$, defined on an open region of $\mathbb{R}^3$ bounded by a cylinder of radius 1 around the axis $O\overline{x}^3$, *i.e.*,

$$\overline{M}^3 = \Big\{\overline{x} \in \mathbb{R}^3; \sum_{i=1}^{2}(\overline{x}^i)^2 < 1\Big\}, \tag{2.9}$$

where

$$(2.10)\quad \overline{\alpha}(\overline{x},\overline{y}) = \frac{1}{1-\sum_{i=1}^{2}(\overline{x}^i)^2}\sqrt{(-\overline{x}^2\overline{y}^1+\overline{x}^1\overline{y}^2)^2+\Big[\sum_{\mu=1}^{3}(\overline{y}^\mu)^2\Big]\Big[1-\sum_{i=1}^{2}(\overline{x}^i)^2\Big]},$$

$$(2.11)\quad \overline{\beta}(\overline{x},\overline{y}) = \frac{-\overline{x}^2\overline{y}^1+\overline{x}^1\overline{y}^2}{1-\sum_{i=1}^{2}(\overline{x}^i)^2}.$$

This Randers manifold has zero flag curvature (see [2]) and

$$(2.12)\qquad (a_{\mu\eta}) = \begin{pmatrix} \dfrac{1-(\overline{x}^1)^2}{[1-\sum_{i=1}^{2}(\overline{x}^i)^2]^2} & \dfrac{-\overline{x}^1\overline{x}^2}{[1-\sum_{i=1}^{2}(\overline{x}^i)^2]^2} & 0 \\ \dfrac{-\overline{x}^1\overline{x}^2}{[1-\sum_{i=1}^{2}(\overline{x}^i)^2]^2} & \dfrac{1-(\overline{x}^2)^2}{[1-\sum_{i=1}^{2}(\overline{x}^i)^2]^2} & 0 \\ 0 & 0 & \dfrac{1}{1-\sum_{i=1}^{2}(\overline{x}^i)^2} \end{pmatrix}.$$

We consider an immersion $\varphi\colon M^2 \to (\overline{M}^3,\overline{F})$. The Finsler metric F induced on M^2 by φ is also of Randers type (see [1]) and is given by $F = \alpha + \beta$, where for $(x, y) \in TM^2$,

$$(2.13)\qquad \alpha(x,y) = \frac{1}{\lambda}\sqrt{[-\widetilde{x}^2 z_i^1 y^i + \widetilde{x}^1 z_i^2 y^i]^2 + \Big[\sum_{\mu=1}^{3} z_i^\mu z_j^\mu y^i y^j\Big]\lambda},$$

$$(2.14)\qquad \beta(x,y) = \frac{-\widetilde{x}^2 z_i^1 y^i + \widetilde{x}^1 z_i^2 y^i}{\lambda},$$

where we have introduced the notation

$$(2.15)\qquad \lambda := 1-|W|^2 = 1-\sum_{k=1}^{2}(\widetilde{x}^k)^2.$$

Since $\beta(x,y) = b_i(x)y^i$, it follows from (2.14) that

$$(2.16)\qquad b_i(x) = -\frac{1}{\lambda}(z_i^1\widetilde{x}^2 - z_i^2\widetilde{x}^1).$$

We now denote by $A(x)$ the 2×2 matrix given by the restriction of (2.12) to the immersion, *i.e.*, $A_{ij} := a_{\mu\eta}z_i^\mu z_j^\eta$. Then

$$(2.17)\qquad A_{ij} = \frac{1}{\lambda^2}\Big[\sum_{k=1}^{2} z_i^k z_j^k + \lambda z_i^3 z_j^3 - \sum_{k,l=1}^{2} z_i^k z_j^l \widetilde{x}^k\widetilde{x}^l\Big] = \frac{1}{\lambda}\sum_{\mu=1}^{3} z_i^\mu z_j^\mu + b_i b_j,$$

where $b_i = b_i(x)$ is given by (2.16).

We now introduce the notation

$$D^{\tau\nu} := \det \begin{pmatrix} z_1^{\tau} & z_1^{\nu} \\ z_2^{\tau} & z_2^{\nu} \end{pmatrix}, \quad \tau, \nu = 1, 2, 3, \quad \tau \neq \nu, \tag{2.18}$$

$$B := \sum_{\tau<\nu} (D^{\tau\nu})^2, \tag{2.19}$$

$$C := \sum_{k=1}^{2} \widetilde{x}^k D^{k3}. \tag{2.20}$$

Observe that $D^{\tau\nu} = -D^{\nu\tau}$.

With this notation (see [9]), we have

$$\det A = \frac{B - C^2}{\lambda^3}, \tag{2.21}$$

$$\|\beta\|^2 = \frac{(1-\lambda)B - C^2}{B - C^2}. \tag{2.22}$$

As an immediate consequence of Proposition 2.1 for $n = 2$, and equations (2.21) and (2.22), we obtain the volume element for the Randers space (M^2, F).

Lemma 2.2 *Let $\varphi\colon M^2 \to (\overline{M}^3, \overline{F})$ be an immersion and let F be the induced Randers metric on M^2 given by (2.13) and (2.14). Then the volume element on (M^2, F) is $dV_F = \Im(x, z)dx^1dx^2$, where*

$$\Im(x, z) = \frac{B^{\frac{3}{2}}}{B - C^2}, \tag{2.23}$$

with B and C given by equations (2.19) and (2.20), respectively.

We define a vector field on M^2 by $N := z_1 \times z_2$, where $z_1 = (z_1^{\eta})$ and $z_2 = (z_2^{\eta})$. Then we can write

$$N = (D^{23}, D^{31}, D^{12}), \tag{2.24}$$

where the equality follows from the notation introduced in (2.18).

The following result provides a differential equation that characterizes the minimal surfaces in the Randers space $(\overline{M}^3, \overline{F})$.

Theorem 2.3 ([9]) *Let $\varphi\colon M^2 \to (\overline{M}^3, \overline{F})$ be an immersion in a Randers space, where $\overline{F}(\overline{x}, \overline{y}) = \overline{\alpha}(\overline{x}, \overline{y}) + \overline{\beta}(\overline{x}, \overline{y})$ is given by (2.10) and (2.11). Consider local coordinates $x = (x^1, x^2)$ on M^2 and let $\widetilde{x}^{\eta} = \varphi^{\eta}(x)$ be the coordinate functions of φ. Then φ is minimal if and only if*

$$\begin{aligned} \Big\{ 2(B + 3C^2)\Big[2B\Big(\frac{\partial C}{\partial \widetilde{x}^{\varepsilon}}\frac{\partial \varphi^{\varepsilon}}{\partial x^i} + \frac{\partial C}{\partial z_j^{\varepsilon}}\frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j}\Big) - C\frac{\partial B}{\partial z_j^{\varepsilon}}\frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j}\Big]\frac{\partial C}{\partial z_i^{\eta}} \\ + (B^2 - 4BC^2 + 3C^4)\frac{\partial^2 B}{\partial z_i^{\eta}\partial z_j^{\varepsilon}}\frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j}\Big\} N^{\eta} = 0, \end{aligned} \tag{2.25}$$

where $N = z_1 \times z_2$, $z_i = (z_i^{\eta}) = (\frac{\partial \varphi^{\eta}}{\partial x^i})$, and B and C are given by equations (2.19) and (2.20), respectively.

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We conclude this section with a lemma that will simplify the computation of the last term of equation (2.25).

Lemma 2.4 ([9]) *Consider B given by* (2.19)*. Then*

$$\frac{\partial^2 B}{\partial z_i^\eta \partial z_j^\varepsilon}\frac{\partial^2 \varphi^\varepsilon}{\partial x^i \partial x^j} N^\eta = 2\sum_{\eta=1}^{3}\sum_{\mu\neq\eta}\Big(z_1^\mu \frac{\partial D^{\mu\eta}}{\partial x^2} - z_2^\mu \frac{\partial D^{\mu\eta}}{\partial x^1}\Big) N^\eta. \tag{2.26}$$

3 Helicoidal Minimal Surfaces in $(\overline{M}^3, \overline{F})$

In this section, we obtain the differential equation that characterizes a helicoidal minimal surface in the Randers space $(\overline{M}^3, \overline{F})$.

Consider the immersion $\varphi\colon M^2 \to (\overline{M}^3, \overline{F})$ defined by

$$\varphi(t,\theta) := (t\cos\theta, t\sin\theta, f(t) + a\theta), \quad a \in \mathbb{R}\setminus\{0\}, \quad 0 < t < 1. \tag{3.1}$$

This is a helicoidal surface around the axis $O\overline{x}^3$. For this immersion, we denote $\widetilde{x}^\eta = \varphi^\eta(x)$ given in (2.6) and use the notation introduced in (2.5). Then we have

$$\widetilde{x}^\eta = \delta_{\eta 1} t\cos\theta + \delta_{\eta 2} t\sin\theta + \delta_{\eta 3}\big[f(t) + a\theta\big], \tag{3.2}$$

$$\begin{aligned} z_i^\eta = {} & \delta_{i1}\big[\delta_{\eta 1} cos\theta + \delta_{\eta 2}\sin\theta + \delta_{\eta 3} f'(t)\big] \\ & + \delta_{i2}\big[-\delta_{\eta 1} t\sin\theta + \delta_{\eta 2} t\cos\theta + \delta_{\eta 3} a\big]. \end{aligned} \tag{3.3}$$

Moreover, from equation (2.18) we get

(3.4)
$$D^{\tau\nu} = \big[\delta_{\tau 1}\delta_{\nu 2} t + \delta_{\tau 1}\delta_{\nu 3}\big(a\cos\theta + t f'(t)\sin\theta\big) + \delta_{\tau 2}\delta_{\nu 3}\big(a\sin\theta - t f'(t)\cos\theta\big)\big],$$
$$\tau,\nu = 1,2,3, \quad \tau < \nu.$$

It follows from (2.19) and (2.20) that B and C are given by

$$B = \Gamma + t^2 f'^2(t) \qquad \text{and} \qquad C = at, \tag{3.5}$$

where we introduced the notation

$$\Gamma = a^2 + t^2. \tag{3.6}$$

Remark 3.1 Observe that (see [9])

$$\frac{\partial C}{\partial \widetilde{x}^\eta} = \delta_{\eta k} D^{k3}, \tag{3.7}$$

$$\frac{\partial C}{\partial z_j^\varepsilon} = \widetilde{x}^k\big[\delta_{j1}(\delta_{\varepsilon k} z_2^3 - \delta_{\varepsilon 3} z_2^k) + \delta_{j2}(\delta_{\varepsilon 3} z_1^k - \delta_{\varepsilon k} z_1^3)\big]. \tag{3.8}$$

Lemma 3.2 *For the immersion $\varphi(t,\theta)$ given by* (3.1)*, considering $x^1 = t$, $x^2 = \theta$, we have that*

(i)

$$\frac{\partial C}{\partial z_i^\eta}\frac{\partial C}{\partial \widetilde{x}^\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial x^i} N^\eta = -t^2 f'(t) B, \tag{3.9}$$

(ii)

$$\frac{\partial C}{\partial z_j^{\varepsilon}} \frac{\partial C}{\partial z_i^{\eta}} \frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j} N^{\eta} = t^4 f'(t) \big[1 + f'^2(t) \big], \tag{3.10}$$

(iii)

$$\frac{\partial B}{\partial z_j^{\varepsilon}} \frac{\partial C}{\partial z_i^{\eta}} \frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j} N^{\eta} = -2at^3 f'(t) \big[1 + f'^2(t) + t f'(t) f''(t) \big], \tag{3.11}$$

(iv)

$$\frac{\partial^2 B}{\partial z_i^{\eta} \partial z_j^{\varepsilon}} \frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j} N^{\eta} = \big[a^2 + \Gamma + t^2 f'^2(t) \big] f'(t) + \Gamma t f''(t), \tag{3.12}$$

where Γ *is given in* (3.6).

Proof (i) Since $\frac{\partial \varphi^{\varepsilon}}{\partial x^i} = z_i^{\varepsilon}$, we have that

$$\frac{\partial C}{\partial z_i^{\eta}} \frac{\partial C}{\partial \widetilde{x}^{\varepsilon}} \frac{\partial \varphi^{\varepsilon}}{\partial x^i} N^{\eta} = \Big(\frac{\partial C}{\partial \widetilde{x}^{\varepsilon}} z_1^{\varepsilon} \Big) \Big(\frac{\partial C}{\partial z_1^{\eta}} N^{\eta} \Big) + \Big(\frac{\partial C}{\partial \widetilde{x}^{\varepsilon}} z_2^{\varepsilon} \Big) \Big(\frac{\partial C}{\partial z_2^{\eta}} N^{\eta} \Big). \tag{3.13}$$

We will now compute each term on the right-hand side of this equation. Replacing η by ε in (3.7) and using (3.3) and (3.4), we have

$$\frac{\partial C}{\partial \widetilde{x}^{\varepsilon}} z_1^{\varepsilon} = \delta_{\varepsilon k} D^{k3} z_1^{\varepsilon} = D^{k3} z_1^{k} = a,$$

and

$$\frac{\partial C}{\partial \widetilde{x}^{\varepsilon}} z_2^{\varepsilon} = -t^2 f'(t).$$

Now replacing ε by η and considering $j = 1$ in (3.8), it follows from (3.3) and (3.4) that

$$\frac{\partial C}{\partial z_1^{\eta}} N^{\eta} = \widetilde{x}^1 (z_2^3 D^{23} - z_2^1 D^{12}) - \widetilde{x}^2 (z_2^3 D^{13} + z_2^2 D^{12}) = -at^2 f'(t), \tag{3.14}$$

and considering $j = 2$, we get

$$\frac{\partial C}{\partial z_2^{\eta}} N^{\eta} = t^2 [1 + f'^2(t)]. \tag{3.15}$$

Substituting each term of the right-hand side of (3.13), we conclude the proof of Lemma 3.2(i).

(ii) Observe that

$$\frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j} = \frac{\partial z_j^{\varepsilon}}{\partial x^i}, \quad x^1 = t, \quad \text{and} \quad x^2 = \theta.$$

Hence,

$$\frac{\partial C}{\partial z_j^{\varepsilon}} \frac{\partial C}{\partial z_i^{\eta}} \frac{\partial^2 \varphi^{\varepsilon}}{\partial x^i \partial x^j} N^{\eta} = \Big(\frac{\partial C}{\partial z_j^{\varepsilon}} \frac{\partial z_j^{\varepsilon}}{\partial t} \Big) \Big(\frac{\partial C}{\partial z_1^{\eta}} N^{\eta} \Big) + \Big(\frac{\partial C}{\partial z_j^{\varepsilon}} \frac{\partial z_j^{\varepsilon}}{\partial \theta} \Big) \Big(\frac{\partial C}{\partial z_2^{\eta}} N^{\eta} \Big). \tag{3.16}$$

From (2.20), and from equations (3.2) and (3.4), we have

$$\frac{\partial C}{\partial z_j^{\varepsilon}} \frac{\partial z_j^{\varepsilon}}{\partial t} = \widetilde{x}^k \frac{\partial D^{k3}}{\partial t} = 0.$$

Analogously, we get

$$\frac{\partial C}{\partial z_j^\varepsilon}\frac{\partial z_j^\varepsilon}{\partial \theta} = t^2 f'(t).$$

Substituting the last two equations into (3.16), it follows from (3.15) that (3.10) holds.

(iii) Similarly since $\frac{\partial^2\varphi^\varepsilon}{\partial x^i \partial x^j} = \frac{\partial z_j^\varepsilon}{\partial x^i}$, $x^1 = t$, and $x^2 = \theta$, we have

$$\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial C}{\partial z_i^\eta}\frac{\partial^2\varphi^\varepsilon}{\partial x^i \partial x^j}N^\eta = \Big(\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial z_j^\varepsilon}{\partial t}\Big)\Big(\frac{\partial C}{\partial z_1^\eta}N^\eta\Big) + \Big(\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial z_j^\varepsilon}{\partial \theta}\Big)\Big(\frac{\partial C}{\partial z_2^\eta}N^\eta\Big). \tag{3.17}$$

From (2.19) we get

$$\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial z_j^\varepsilon}{\partial t} = \frac{\partial B}{\partial t} = 2t\big[1 + f'^2(t) + t f'(t) f''(t)\big],$$

where the last equality follows from (3.5).

On the other hand, from (3.5) we can see that B does not depend on θ. Hence,

$$\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial z_j^\varepsilon}{\partial \theta} = \frac{\partial B}{\partial \theta} = 0. \tag{3.18}$$

Substituting the last two equations and (3.14) into (3.17), we obtain (3.11).

(iv) For this proof, we will use Lemma 2.4 with $x^1 = t$ and $x^2 = \theta$. It follows from (3.3) and (3.4) that

$$\sum_{\mu\neq 1}\Big(z_1^\mu \frac{\partial D^{\mu 1}}{\partial \theta} - z_2^\mu \frac{\partial D^{\mu 1}}{\partial t}\Big) D^{23} = t\cos\theta\big[1 - f'^2(t)\big] + a\big[2f'(t) + t f''(t)\big]\sin\theta,$$

$$\sum_{\mu\neq 2}\Big(z_1^\mu \frac{\partial D^{\mu 2}}{\partial \theta} - z_2^\mu \frac{\partial D^{\mu 2}}{\partial t}\Big) D^{31} = t\sin\theta\big[1 - f'^2(t)\big] - a\big[2f'(t) + t f''(t)\big]\cos,\theta$$

$$\sum_{\mu\neq 3}\Big(z_1^\mu \frac{\partial D^{\mu 3}}{\partial \theta} - z_2^\mu \frac{\partial D^{\mu 3}}{\partial t}\Big) D^{12} = t\big[2f'(t) + t f''(t)\big].$$

Adding the last three equalities, it follows from Lemma 2.4 and (3.5) that equation (3.12) holds. ■

The following result provides a differential equation that characterizes the helicoidal minimal surfaces in the Randers space $(\overline{M}^3, \overline{F})$.

Theorem 3.3 *Let $(\overline{M}^3, \overline{F} = \overline{\alpha} + \overline{\beta})$ be the Randers space given by the open region of $\mathbb{R}^3$ bounded by the cylinder of radius 1 around the axis $O\overline{x}^3$, where $\overline{\alpha}$ and $\overline{\beta}$ are defined by (2.10) and (2.11). Then the helicoidal surface $\varphi(t,\theta) = (t\cos\theta, t\sin\theta, f(t) + a\theta)$,*

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where $a \in \mathbb{R} \setminus \{0\}$ and $0 < t < 1$, is minimal if and only if

$$\begin{aligned}
&\Big\{ [\Gamma(\Gamma + a^2) - 4a^2t^2(\Gamma + 2a^2)]\Gamma - 3a^4t^4(2\Gamma - 3t^2)\Big\} f'(t) \\
&+ \Big[(3\Gamma + 2a^2 - 8a^2t^2)\Gamma + a^4t^2(3t^2 - 8)\Big] t^2 f'^3(t) \\
&+ \Big[(3\Gamma + a^2(1 - 4t^2)\Big] t^4 f'^5(t) + t^6 f'^7(t) \\
&+ 2[\Gamma^2 + 6a^4t^4]t^3 f'^2(t) f''(t) + (\Gamma + 4a^2t^2)t^5 f'^4(t) f''(t) \\
&+ (a^2t^2 - \Gamma)(3a^2t^2 - \Gamma)t f''(t) = 0,
\end{aligned} \tag{3.19}$$

where Γ is given in (3.6).

Proof It follows from Theorem 2.3 that φ is minimal if and only if for any t and θ,

$$\begin{aligned}
\Big\{ 2(B + 3C^2)\Big[2B\Big(\frac{\partial C}{\partial \widetilde{x}^\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial x^i} + \frac{\partial C}{\partial z_j^\varepsilon}\frac{\partial^2 \varphi^\varepsilon}{\partial x^i \partial x^j}\Big) - C\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial^2 \varphi^\varepsilon}{\partial x^i \partial x^j}\Big]\frac{\partial C}{\partial z_i^\eta} \\
+ (B^2 - 4BC^2 + 3C^4)\frac{\partial^2 B}{\partial z_i^\eta \partial z_j^\varepsilon}\frac{\partial^2 \varphi^\varepsilon}{\partial x^i \partial x^j}\Big\} N^\eta = 0.
\end{aligned}$$

From (3.5) we obtain

$$B + 3C^2 = \Gamma + \Big[3a^2 + f'^2(t)\Big] t^2, \tag{3.20}$$

$$\begin{aligned}
B^2 - 4BC^2 + 3C^4 = \Gamma^2 + \Big[(3a^2t^2 - 4\Gamma)a^2 \\
+ 2(\Gamma - 2a^2t^2) f'^2(t) + t^2 f'^4(t)\Big] t^2.
\end{aligned} \tag{3.21}$$

Therefore, substituting equations (3.5), (3.20), (3.21), and (3.9)–(3.12) given in Lemma 3.2 in the above expression, we conclude that (3.19) holds. ■

As an immediate consequence of this result we obtain the following corollary.

Corollary 3.4 *Let $(\overline{M}^3, \overline{F} = \overline{\alpha} + \overline{\beta})$ be the Randers space given by the open region of $\mathbb{R}^3$ bounded by the cylinder of radius 1 around the axis $O\overline{x}^3$, with $\overline{\alpha}$ and $\overline{\beta}$ given, respectively, by (2.10) and (2.11). Then the helicoids given by $\phi(t, \theta) = (t\cos\theta, t\sin\theta, a\theta)$, where $a \in \mathbb{R} \setminus \{0\}$ and $0 < t < 1$ are minimal surfaces in $(\overline{M}^3, \overline{F})$.*

Now, we want to show that if the axis of the helicoid is different from the axis of the cylinder, *i.e.*, axis $O\overline{x}^3$, then the helicoid is not a minimal surface in $(\overline{M}^3, \overline{F})$.

Theorem 3.5 *Let $(\overline{M}^3, \overline{F} = \overline{\alpha} + \overline{\beta})$ be the Randers space given by the open region of $\mathbb{R}^3$ bounded by the cylinder of radius 1 around the axis $O\overline{x}^3$, where $\overline{\alpha}$ and $\overline{\beta}$ are given by (2.10) and (2.11). The helicoid is a minimal surface in $(\overline{M}^3, \overline{F})$ only if the axis of the helicoid is the axis of the cylinder.*

Before we prove Theorem 3.5, we will obtain two lemmas. We start observing that if $\mathcal{M}$ is an orthogonal matrix such that $\det \mathcal{M} = 1$, then the cofactor matrix is

$$\widetilde{\mathcal{M}} = \mathcal{M}. \tag{3.22}$$

A helicoid in $(\overline{M}^3, \overline{F})$ with any axis can be locally decribed by

$$\zeta(t,\theta) := (t\cos\theta, t\sin\theta, a\theta)\mathcal{M}, \qquad a \in \mathbb{R}\setminus\{0\}, \qquad 0 < t < 1, \tag{3.23}$$

where $\mathcal{M}$ is a 3×3 orthogonal matrix. Without loss of generality, we may consider $\det\mathcal{M} = 1$.

Remark 3.6 For the immersion $\zeta(t,\theta)$, the elements of the matrix $\mathcal{M}$ will be denoted by $m_{\mu\eta}$. Then

$$\widetilde{x}^{\eta} = t\big[m_{1\eta}\cos\theta + m_{2\eta}\sin\theta\big] + m_{3\eta}a\theta, \tag{3.24}$$

$$z_i^{\eta} = \delta_{i1}\big[m_{1\eta}\cos\theta + m_{2\eta}\sin\theta\big] + \delta_{i2}\big[t(-m_{1\eta}\sin\theta + m_{2\eta}\cos\theta) + m_{3\eta}a\big]. \tag{3.25}$$

Moreover, from (2.18) and (3.25) we have that

$$\begin{aligned} D^{12} &= a[m_{13}\sin\theta - m_{23}\cos\theta] + t\,m_{33},\\ D^{13} &= a[-m_{12}\sin\theta + m_{22}\cos\theta] - t\,m_{32},\\ D^{23} &= a[m_{11}\sin\theta - m_{21}\cos\theta] + t\,m_{31}. \end{aligned} \tag{3.26}$$

From equations (2.24) and (3.26), we have that the vector field N is given by

$$N = (a\sin\theta, -a\cos\theta, t)\mathcal{M}.$$

Now using equations (2.19), (2.20), (3.22), (3.24), and (3.26), a straightforward computation shows that B and C are given as follows.

Lemma 3.7 *Considering the immersion $\zeta(t,\theta)$ given by (3.23), the functions B and C defined by (2.19) and (2.20) are given by*

$$B = a^2 + t^2, \tag{3.27}$$

$$C = at\,m_{33} + t^2(m_{23}\cos\theta - m_{13}\sin\theta) - a^2\theta(m_{23}\sin\theta + m_{13}\cos\theta). \tag{3.28}$$

By using computations entirely analogous to those in Lemma 3.2 and systematically using the relation (3.22) we can verify the following result.

Lemma 3.8 *For the immersion $\zeta(t,\theta)$ given by (3.23), we have that*

(i)

(3.29)

$$\frac{\partial C}{\partial z_i^{\eta}}\frac{\partial C}{\partial \widetilde{x}^{\varepsilon}}\frac{\partial \zeta^{\varepsilon}}{\partial x^i}N^{\eta} = B\Big\{ m_{33}\big[a^2\theta(-m_{13}\sin\theta + m_{23}\cos\theta) - t^2(m_{13}\cos\theta + m_{23}\sin\theta)\big] - at(m_{13}\cos\theta + m_{23}\sin\theta)(-m_{23}\cos\theta + m_{13}\sin\theta) + at\theta(1 - m_{33}^2)\Big\},$$

(ii)

(3.30)

$$\frac{\partial C}{\partial z_j^{\varepsilon}}\frac{\partial C}{\partial z_i^{\eta}}\frac{\partial^2 \zeta^{\varepsilon}}{\partial x^i\partial x^j}N^{\eta} = a\theta t(-m_{13}\sin\theta + m_{23}\cos\theta)\big[(B + a^2)(-m_{13}\sin\theta + m_{23}\cos\theta) - atm_{33} + a^2\theta(m_{13}\cos\theta + m_{23}\sin\theta)\big],$$

(iii)

$$\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial C}{\partial z_i^\eta}\frac{\partial^2\zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta = 2Ba\theta t(-m_{13}\sin\theta + m_{23}\cos\theta), \tag{3.31}$$

(iv)

$$\frac{\partial^2 B}{\partial z_i^\eta\partial z_j^\varepsilon}\frac{\partial^2\zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta = 0, \tag{3.32}$$

where B and C are given by (3.27) and (3.28), respectively.

Proof of Theorem 3.5 We will prove that a helicoid is not a minimal surface in $(\overline{M}^3, \overline{F})$ if the axis of the helicoid is different from the axis of the cylinder. A helicoid around an axis different from $O\overline{x}^3$ can be locally described by the immersion $\zeta(t,\theta) = \phi(t,\theta)\mathcal{M}$, where $\phi(t,\theta) = (t\cos\theta, t\sin\theta, a\theta)$ is a helicoid around $O\overline{x}^3$ and $\mathcal{M} = (m_{\mu\eta})$ is a 3×3 matrix, such that $\det\mathcal{M} = 1$.

It follows from Theorem 2.3 that ζ is minimal if and only if for any t and θ,

$$2(B+3C^2)\Big\{2B\big[\mathcal{P}_1(t,\theta)+\mathcal{P}_2(t,\theta)\big] - C\mathcal{P}_3(t,\theta)\Big\} + (B^2-4BC^2+3C^4)\mathcal{P}_4(t,\theta) = 0, \tag{3.33}$$

where B and C are given by (3.27) and (3.28), and we are denoting by $\mathcal{P}_1(t,\theta)$-$\mathcal{P}_4(t,\theta)$ the following expressions contained in (2.25):

$$\begin{aligned}
\mathcal{P}_1(t,\theta) &:= \Big(\frac{\partial C}{\partial z_i^\eta}\frac{\partial C}{\partial \widetilde{x}^\varepsilon}\frac{\partial\zeta^\varepsilon}{\partial x^i}N^\eta\Big)(t,\theta),\\
\mathcal{P}_2(t,\theta) &:= \Big(\frac{\partial C}{\partial z_j^\varepsilon}\frac{\partial C}{\partial z_i^\eta}\frac{\partial^2\zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta\Big)(t,\theta),\\
\mathcal{P}_3(t,\theta) &:= \Big(\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial C}{\partial z_i^\eta}\frac{\partial^2\zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta\Big)(t,\theta),\\
\mathcal{P}_4(t,\theta) &:= \Big(\frac{\partial^2 B}{\partial z_i^\eta\partial z_j^\varepsilon}\frac{\partial^2\zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta\Big)(t,\theta).
\end{aligned} \tag{3.34}$$

Observe that $\mathcal{P}_1(t,\theta)$-$\mathcal{P}_4(t,\theta)$ were obtained in terms of the immersion ζ in the equations (3.29)–(3.32) of Lemma 3.8. Therefore, considering $\theta = 0$, in these equations we have, for all t,

$$\mathcal{P}_1(t,0) = -Btm_{13}(tm_{33} - am_{23}),$$
$$\mathcal{P}_2(t,0) = 0,\quad \mathcal{P}_3(t,0) = 0,\quad \mathcal{P}_4(t,0) = 0.$$

Substituting these values into (3.33) and using $B \neq 0$, we obtain

$$m_{13}m_{33}t^2 - am_{13}m_{23}t = 0, \forall t.$$

Hence, since $a \neq 0$, we have

$$m_{13}m_{33} = 0 \quad\text{and}\quad m_{13}m_{23} = 0. \tag{3.35}$$

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Similarly, considering $\theta = \pi$ and (3.35), it follows that, for all t,

$$\begin{aligned}
\mathcal{P}_1(t,\pi) &= Ba\pi\left[(1-m_{23}^2)t - am_{23}m_{33}\right],\\
\mathcal{P}_2(t,\pi) &= a^2\pi m_{23}m_{33}t^2 + a\pi(B+a^2)m_{23}^2 t,\\
\mathcal{P}_3(t,\pi) &= -2Bat\pi m_{23},\\
\mathcal{P}_4(t,\pi) &= 0.
\end{aligned}$$

On the other hand, $C(t,\pi) = atm_{33} - m_{23}t^2 + a^2\pi m_{13}$.

Substituting these values into (3.33), we have

$$(1-m_{33}^2)t^3 + am_{23}m_{33}t^2 + a^2(1-m_{33}^2+2m_{23}^2)t - a^3m_{23}m_{33} = 0, \quad \forall t.$$

Therefore, $1-m_{33}^2 = 0$ and $m_{23}m_{33} = 0$. Since $m_{13}m_{33} = 0$, we conclude that

$$m_{13} = m_{23} = 0 \quad \text{and} \quad m_{33} = \pm 1. \qquad \blacksquare$$

4 Helicoidal Minimal Surfaces of the Bonnet Family in $(\overline{M}^3, \overline{F})$

Considering the immersion $\vartheta(u,v)$ given by

$$\begin{aligned}
\text{(4.1)}\quad \vartheta(u,v) = \big(&a\cos\lambda\cos u\cosh v + a\sin\lambda\sin u\sinh v,\\
&a\cos\lambda\sin u\cosh v - a\sin\lambda\cos u\sinh v, av\cos\lambda + au\sin\lambda\big).
\end{aligned}$$

This parametrization provides a one-parameter family of helicoidal minimal surfaces in Euclidean space called the *Bonnet family*. We want to verify which of these surfaces are minimal in $(\overline{M}^3, \overline{F})$.

Remark 4.1 For the immersion $\vartheta(u,v)$, given by (4.1), we have

$$\begin{aligned}
\text{(4.2)}\quad \widetilde{x}^\eta = a\big\{&\delta_{1\eta}[\cos\lambda\cos u\cosh v + \sin\lambda\sin u\sinh v]\\
&+\delta_{2\eta}[\cos\lambda\sin u\cosh v - \sin\lambda\cos u\sinh v] + \delta_{3\eta}[v\cos\lambda + u\sin\lambda]\big\},
\end{aligned}$$

and

$$\begin{aligned}
\text{(4.3)}\quad z_i^\eta = a\Big\{&\delta_{i1}\big[\delta_{\eta 1}(-\cos\lambda\sin u\cosh v + \sin\lambda\cos u\sinh v)\\
&\quad+\delta_{\eta 2}(\cos\lambda\cos u\cosh v + \sin\lambda\sin u\sinh v) + \delta_{\eta 3}\sin\lambda\big]\\
&+\delta_{i2}\big[\delta_{\eta 1}(\cos\lambda\cos u\sinh v + \sin\lambda\sin u\cosh v)\\
&\quad+\delta_{\eta 2}(\cos\lambda\sin u\sinh v - \sin\lambda\cos u\cosh v) + \delta_{\eta 3}\cos\lambda\big]\Big\}.
\end{aligned}$$

Moreover, from (2.18) and (4.3) we have that

$$\text{(4.4)}\quad \begin{aligned}
D^{12} &= -a^2\cosh v\sinh v,\\
D^{13} &= -a^2\sin u\cosh v,\\
D^{23} &= a^2\cos u\cosh v.
\end{aligned}$$

It follows from (2.19) and (2.20) that B and C are given by

$$\text{(4.5)}\quad B = a^4\cosh^4 v, \qquad C = -a^3\sin\lambda\sinh v\cosh v.$$

By using computations entirely analogous to those in Lemma 3.2 and Lemma 3.8, we can verify the following result.

Lemma 4.2 *For the immersion $\vartheta(u, v)$ given by (4.1), we have that*

(i)

$$\frac{\partial C}{\partial z_i^\eta}\frac{\partial C}{\partial \widetilde{x}^\varepsilon}\frac{\partial \vartheta^\varepsilon}{\partial x^i}N^\eta = a^7\cos\lambda\cosh^6 v, \tag{4.6}$$

(ii)

$$\frac{\partial C}{\partial z_j^\varepsilon}\frac{\partial C}{\partial z_i^\eta}\frac{\partial^2 \vartheta^\varepsilon}{\partial x^i\partial x^j}N^\eta = -a^7\cos\lambda\cosh^2 v\big[\cosh^4 v - \sin^2\lambda(\cosh^2 v + \sinh^2 v)\big], \tag{4.7}$$

(iii)

$$\frac{\partial B}{\partial z_j^\varepsilon}\frac{\partial C}{\partial z_i^\eta}\frac{\partial^2 \zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta = -4a^8\cos\lambda\sin\lambda\cosh^5 v\sinh v, \tag{4.8}$$

(iv)

$$\frac{\partial^2 B}{\partial z_i^\eta\partial z_j^\varepsilon}\frac{\partial^2 \zeta^\varepsilon}{\partial x^i\partial x^j}N^\eta = 0. \tag{4.9}$$

The following result characterizes the surfaces of the Bonnet family that are minimal in $(\overline{M}^3, \overline{F})$.

Theorem 4.3 *Let $(\overline{M}^3, \overline{F})$ be the Randers space, $\overline{F} = \overline{\alpha} + \overline{\beta}$, where $\overline{M}^3$, $\overline{\alpha}$, and $\overline{\beta}$ are given by (2.10) and (2.11). Then the only minimal surfaces in $\overline{M}^3$ of the Bonnet family given by (4.1) are the catenoids and the helicoids.*

Proof The catenoids were shown to be minimal in [9] and in Theorem 3.5 we proved that the helicoids around the $\overline{x}^3$ axis are minimal. Substituting equations (4.5)–(4.9) in the expression (2.25), a straightforward computation shows that $\cos\lambda\sin^2\lambda = 0$. It follows from (4.1) that if $\cos\lambda = 0$, then the mininal surface will be a helicoid. On the other hand, if $\sin\lambda = 0$, then the minimal surface will be a catenoid. ∎

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