ON CLASSICAL KRULL DIMENSION OF GROUP-GRADED RINGS

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For any ring $R$ graded by a finite group, we give a bound on the classical Krull dimension of $R$ in terms of the dimension of the initial component $R_e$. It follows that if $R_e$ has finite classical Krull dimension, then the same is true of the whole ring $R$, too.

Let $G$ be a finite group with identity $e$. A ring $R$ is said to be $G$-graded if $R = \bigoplus_{g \in G} R_g$ is a direct sum of additive subgroups $R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

There are many results relating properties of a group-graded ring $R = \bigoplus_{g \in G} R_g$ and its initial component $R_e$, where $e$ is the identity of the group (see [5, 7, 8] and [9]). Ring-theoretic dimensions of group-graded rings have been considered by several authors (see, for example, Bell [1], Chin and Quinn [2], Cohen and Montgomery [3], Năstăsescu [6]).

Rings with Krull dimension form an important class and have many nice properties (see [5]). Suppose that the set $S = \text{Spec}(R)$ of prime ideals of $R$ satisfies a.c.c. Define the sets $S_\alpha$ inductively. Let $S_0$ be the set of all maximal elements in $S$; and for each ordinal $\alpha$ denote by $S_\alpha$ the set of all $s \in S$ such that $t \in S$, $t > s$ implies $t \in S_\beta$ for some $\beta < \alpha$. Then there exists the least ordinal $\alpha$ such that $S_\alpha = S$. This ordinal is called the classical Krull dimension of $R$. If it is finite, then it is also equal to the right Krull dimension of $R$ defined on the lattice of right ideals of $R$ (see [5, Chapter 6]).

Denote by cl-K-dim($R$) the classical Krull dimension of $R$. For any ordinal $\alpha$ and positive integer $n$, we introduce ordinals $\alpha_n$, setting $\alpha_1 = \alpha + 1$, $\alpha_{n+1} = (\alpha + 1)(\alpha_n + 1)$. We shall use the results on prime ideals due to Cohen and Montgomery [3] and prove the following theorem.

**Theorem 1.** Let $G$ be a finite group with identity $e$ and $|G| = n$, and let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring. If $R_e$ has classical Krull dimension $\alpha$, then $R$ has classical Krull dimension, too, and cl-K-dim($R$) $< \alpha_n$.

This theorem is related to an open question [4, Problem 5].
It is interesting to note that the analogous assertion is not valid for Krull dimension defined on the lattice of right ideals. Indeed, if we take any group $G$ with identity $e$ and an element $g \neq e$ in $G$, and take a ring $R$ with zero multiplication which has no Krull dimension, then we can view $R$ as a group-graded ring with $R_e = 0$, $R_g = R$, and $R_h = 0$ for all $h \in G \setminus \{e, g\}$. [2, Example 2.4] shows that our theorem does not transfer to rings graded by infinite groups, even in the case of the infinite cyclic group.

We need the following lemma (see [3, Theorems 7.1 and 7.3], or [9, Theorem 17.9]).

**Lemma 2.** [3] Let $G$ be a finite group with identity $e$, and let $R$ be a $G$-graded ring.

1. If $P$ is a prime ideal of $R$, then there exist $n \leq |G|$ primes $Q_1, Q_2, \ldots, Q_n$ of $R_e$ minimal over $P \cap R_e$, and we have $P \cap R_e = Q_1 \cap Q_2 \cap \ldots \cap Q_n$.
2. If $P \subseteq Q$ are prime ideals of $R$ and $P \neq Q$, then $P \cap R_e \neq Q \cap R_e$.

**Proof of Theorem 1:** Suppose to the contrary that $R$ contains a strictly increasing chain of prime ideals $P_1 \subset P_2 \subset \ldots \subset P_{\alpha_n}$. Lemma 2(i) tells us that, for each $\gamma \leq \alpha_n$, there exists a finite set $S_\gamma$ of prime ideals of $R_e$ minimal over $R_e \cap P_\gamma$ and such that

$$
\bigcap_{P \in S_\gamma} P = R_e \cap P_\gamma
$$

and $|S_\gamma| \leq |G| = n$.

Put $S = \bigcup_{\gamma \leq \alpha_n} S_\gamma$. If a prime ideal contains an intersection of a finite number of ideals, then it contains at least one of them. Therefore, for any $\delta < \varepsilon < \alpha_n$ and $P \in S_\varepsilon$, there exists $Q \in S_\delta$, such that $Q \subseteq P$.

For $\delta < \varepsilon \leq \alpha_n$, $Q \in S_\delta$, and $P \in S_\varepsilon$, we shall write

$$Q \ll P$$

if and only if, for all $\mu$, $\delta < \mu < \varepsilon$, we can fix $I_\xi \in S_\mu$ so that $I_\mu \subseteq I_\nu$ whenever $\delta \leq \mu \leq \nu \leq \varepsilon$, where $I_\delta = Q$ and $I_\varepsilon = P$.

We shall show by induction on $\gamma \leq \alpha_n$ that, for each $P \in S_\gamma$, there exists $Q \in S_1$ such that $Q \ll P$. The case of $\gamma = 1$ is trivial. Suppose that this has been proved for all $\delta < \gamma$. Take any ideal $P \in S_\gamma$.

If $\gamma$ is not a limit ordinal, then there exists $\gamma - 1$ and we can take $P' \in S_{\gamma - 1}$ such that $P' \subseteq P$. By the induction assumption $Q \ll P'$ for some $Q \in S_1$. It follows that $Q \ll P$.

Consider the case where $\gamma$ is a limit ordinal. Denote by $L$ the set of all $Q \in \bigcup_{\delta < \gamma} S_\delta$ such that $Q \subseteq P$. By induction on $\delta$ we shall define ideals $Q_\delta \in S_\delta$, for all $\delta < \gamma$. 

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Given that $S_1$ is finite and every ideal $Q$, where $Q \in S_\nu \cap L \neq \emptyset$, $1 < \nu \leq \gamma$, contains at least one ideal of $S_1 \cap L$, it follows that there exists $Q_1 \in S_1 \cap L$ such that for any $\mu < \gamma$ we can find $\mu < \nu < \gamma$ and $Q \in S_\nu$ satisfying $Q_1 \ll Q$. By the definition of $\ll$, for any $\mu < \gamma$, we can find $Q \in S_\mu$ satisfying $Q_1 \ll Q$. Put

$$L_1 = L \cap \left\{ Q \in \bigcup_{1 < \nu < \gamma} S_\nu \mid Q_1 \ll Q \right\}.$$  

We have ensured that $L_1$ intersects all $S_\nu$, for $1 < \nu < \gamma$.

Suppose that for some $\delta < \gamma$ ideals $Q_\varepsilon$ have been defined for all $\varepsilon < \delta$, and suppose that these ideals form an ascending chain. In addition, assume that the sets

$$L_\varepsilon = L \cap \left\{ Q \in \bigcup_{\varepsilon < \nu < \gamma} S_\nu \mid Q_\varepsilon \ll Q \right\}$$

intersect all $S_\nu$ for $\varepsilon < \nu < \gamma$. Obviously, $M \subseteq L$ and $M = \bigcap_{\varepsilon < \delta} L_\varepsilon \cap S_\delta$ is not empty, because all sets $L_1 \supseteq L_2 \supseteq \ldots \supseteq L_\delta \supseteq \ldots$ are nonempty. As in the paragraph above, given that $M$ is finite, there exists $Q_\delta \in M$ such that for any $\delta < \mu < \gamma$ we can find $\mu < \nu < \gamma$ and $Q \in S_\nu$ satisfying $Q_\delta \subseteq Q$. Thus the ascending chain of ideals $Q_\delta$, $\delta < \gamma$, has been defined.

Since $Q_\delta \subseteq P$ for all $\delta < \gamma$, we see that $Q_1 \ll P$, as required.

Next, we are going to reduce the set $S$. Take any $P^{(1)}_\varepsilon \in S_{\alpha_n}$ and fix a chain of ideals $P^{(1)}_\gamma \in S_\gamma$ such that $P^{(1)}_\mu \subseteq P^{(1)}_\nu$ for all $\mu \leq \nu \leq \alpha_n$. Given that $\text{cl-K-dim}(R_\varepsilon) = \alpha$ and $\alpha_n = (\alpha + 1)(\alpha_n - 1) + 1$, there exists $0 \leq \delta < \alpha_n$ such that

$$P^{(1)}_{\delta + 1} = P^{(1)}_{\delta + 2} = \ldots = P^{(1)}_{\delta + \alpha_n - 1 + 1} \subseteq R_\varepsilon.$$  

Put $S^{(1)}_\mu = S_\mu \setminus \{P^{(1)}_\mu\}$ for $\delta \leq \mu \leq \delta + \alpha_n - 1 + 1$, and

$$S^{(1)}_\gamma = \bigcup_{\delta \leq \mu \leq \delta + \alpha_n - 1} S^{(1)}_\mu.$$  

For any $\delta < \mu < \gamma \leq \delta + \alpha_n - 1 + 1$, and any ideal $I \in S^{(1)}_\gamma$ there exists $Q \in S_\mu$ such that $Q \subseteq I$. If $Q \not\subseteq S^{(1)}_\mu$, then $Q = P^{(1)}_\mu = P^{(1)}_\nu$; whence $I \supseteq P^{(1)}_\nu$, a contradiction. Therefore $Q \in S^{(1)}_\mu$.

Thus $S^{(1)}_\gamma$ satisfies the same property we used for $S$, but now $\left| S^{(1)}_\gamma \right| \leq n - 1$ for all $\gamma$.

Suppose that for some $\gamma$ such that $\delta < \gamma \leq \delta + \alpha_n - 1$ the set $S^{(1)}_\gamma$ is empty. Then, for any $\gamma < \mu \leq \delta + \alpha_n - 1 + 1$ and $Q \in S_\mu$, we have $P^{(1)}_\mu = P^{(1)}_\nu = P_\gamma \subseteq Q$. Hence
Let us apply the same argument as above to $S^{(1)}$. Take an ideal $P^{(2)}$ in $S^{(1)}_{\delta+\alpha_{n-1}}$. Find $P^{(2)}_{\delta+1} \subseteq S^{(1)}_{\delta+1}$ with $P^{(2)}_{\delta+1} \ll P^{(2)}$. Take a chain

$$P^{(2)}_{\delta+1} \subseteq P^{(2)}_{\delta+2} \subseteq \ldots \subseteq P^{(2)}_{\delta+\alpha_{n-1}} = P^{(2)} \subseteq R_e,$$

where $P^{(2)}_{\gamma} \in S^{(1)}_{\gamma}$ for all $\delta < \gamma \leq \delta + \alpha_{n-1}$. Find a new ordinal $\delta_2$ such that $P^{(2)}_{\delta_2+1} = P^{(2)}_{\delta_2+2} = \ldots = P^{(2)}_{\delta_2+\alpha_{n-2}+1}$. Put $S^{(2)}_{\gamma} = S^{(1)}_{\gamma} \setminus \{P^{(2)}_{\gamma}\}$,

$$S^{(2)}_{\gamma} = \bigcup_{\delta_2 < \gamma \leq \delta_2 + \alpha_{n-2}} S^{(2)}_{\gamma}.$$

Then the set $S^{(2)}_{\gamma}$ satisfies the same property we used for $S$, but now $|S^{(2)}_{\gamma}| \leq n - 2$ for all $\gamma$. As above, all sets $S^{(2)}_{\gamma}$ will be nonempty for $\delta_2 < \gamma \leq \delta_2 + \alpha_{n-2}$.

If we repeat this reduction $n - 1$ times, we get a set

$$S^{(n-1)}_{\gamma} = \bigcup_{\delta_{n-1} < \gamma \leq \delta_{n-1} + \alpha_1} S^{(n-1)}_{\gamma},$$

satisfying the same conditions and such that $|S^{(n-1)}_{\gamma}| \leq 1$ for all $\delta_{n-1} < \gamma \leq \delta_{n-1} + \alpha_1$. As earlier we can show that all sets $S^{(n-1)}_{\gamma}$ are nonempty for $\delta_{n-1} < \gamma \leq \delta_{n-1} + \alpha_1$. Thus $|S^{(n-1)}_{\gamma}| = 1$ for all $\gamma$.

Given that $\alpha_1 = \alpha + 1$, we get $S^{(n-1)}_{\gamma} = S^{n-1}_{\gamma+1}$ for some $\delta_2 \leq \gamma < \delta_2 + \alpha_1$. It follows from Lemma 2(ii) that $P_\gamma = P_{\gamma+1}$. This contradiction completes the proof.

**Remark.** For a finite cl-$K$-$\dim(R_e)$ our proof simplifies since several steps become redundant.

**References**


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