TWO REMARKS ON $PQ^\epsilon$-PROJECTIVITY OF
RIEMANNIAN METRICS

VLADIMIR S. MATVEEV AND STEFAN ROSEMANN
Institute of Mathematics, FSU Jena, Jena 07737, Germany
e-mails: vladimir.matveev@uni-jena.de, stefan.rosemann@uni-jena.de

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Abstract. We show that $PQ^\epsilon$-projectivity of two Riemannian metrics introduced
in [15] (P. J. Topalov, Geodesic compatibility and integrability of geodesic flows, J.
Math. Phys. 44(2) (2003), 913–929.) implies affine equivalence of the metrics unless
$\epsilon \in \{0, -1, -3, -5, -7, \ldots\}$. Moreover, we show that for $\epsilon = 0$, $PQ^\epsilon$-projectivity implies
projective equivalence.

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1. Introduction.

1.1. $PQ^\epsilon$-projectivity of Riemannian metrics. Let $g$, $\bar{g}$ be two Riemannian metrics
on an $m$-dimensional manifold $M$. Consider $(1, 1)$-tensors $P, Q$ that satisfy

\begin{align}
    g(P., .) & = -g(., P.), \quad g(Q., .) = -g(., Q.) \\
    \bar{g}(P., .) & = -\bar{g}(., P.), \quad \bar{g}(Q., .) = -\bar{g}(., Q.) \\
    PQ & = \epsilon Id,
\end{align}

where $Id$ is the identity on $TM$ and $\epsilon$ is a real number, $\epsilon \neq 1, m + 1$. The following
definition was introduced in [15].

Definition 1. The metrics $g$, $\bar{g}$ are called $PQ^\epsilon$-projective if for a certain $1$-form $\Phi$
the Levi-Civita connections $\nabla$ and $\bar{\nabla}$ of $g$ and $\bar{g}$ satisfy

\begin{equation}
    \bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX
\end{equation}

for all vector fields $X, Y$.

Example 1. If the two metrics $g$, $\bar{g}$ are affinely equivalent, i.e. $\nabla = \bar{\nabla}$, then these
are $PQ^\epsilon$-projective with $P, Q, \epsilon$ arbitrary and $\Phi \equiv 0$.

Example 2. Suppose that $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. It follows that equation
(2) becomes

\begin{equation}
    \bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X.
\end{equation}
By Levi-Civita [4], equation (3) is equivalent to the condition that \( g \) and \( \bar{g} \) have the same geodesics considered as unparametrised curves, i.e. \( g \) and \( \bar{g} \) are \textit{projectively equivalent}. The theory of projectively equivalent metrics has a very long tradition in differential geometry, see for example [5, 6, 8, 10, 13] and the references therein.

**Example 3.** Suppose that \( P = Q = J \) and \( \epsilon = -1 \). It follows that \( J \) is an almost complex structure, i.e. \( J^2 = -Id \), and by equation (1) the metrics \( g \) and \( \bar{g} \) are required to be Hermitian with respect to \( J \). Equation (2) now reads

\[
\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX.
\] (4)

This equation defines the \textit{h-projective equivalence} of the Hermitian metrics \( g \) and \( \bar{g} \), and was introduced for the first time by Otsuki and Tashiro in [12, 14] for the Kaehlerian metrics. The theory of \( h \)-projectively equivalent metrics was introduced as an analog of projective geometry in the \( \text{K\"{e}hlerian} \) situation and has been studied actively over the years, see for example [1–3, 7, 11] and the references therein.

**Remark 1.** \( PQ' \)-projectivity of the Riemannian metrics is a special case of the so-called \( F \)-planar mappings introduced and investigated in [9], whose defining equation, i.e. equation (1) in [9] clearly generalises equation (2) above.

**1.2. Results.** The aim of our paper is to give a proof of the following two theorems.

**Theorem 1.** Let Riemannian metrics \( g \) and \( \bar{g} \) be \( PQ' \)-projective. If \( g \) and \( \bar{g} \) are not affinely equivalent, the number \( \epsilon \) is either zero or an odd negative integer, i.e. \( \epsilon \in \{0, -1, -3, -5, -7, ...\} \).

**Theorem 2.** Let Riemannian metrics \( g \) and \( \bar{g} \) be \( PQ' \)-projective. If \( \epsilon = 0 \) then \( g \) and \( \bar{g} \) are projectively equivalent.

**1.3. Motivation and open questions.** As was shown in [15], \( PQ' \)-projectivity of the metrics \( g, \bar{g} \) allows us to construct a family of commuting integrals for the geodesic flow of \( g \) (see Fact 2 and equation (9)). The existence of these integrals is an interesting phenomenon on its own. Besides, it appeared to be a powerful tool in the study of projectively equivalent and \( h \)-projectively equivalent metrics (Examples 2 and 3), see [3, 5–8]. Moreover, it was shown in [15] that given one pair of \( PQ' \)-projective metrics, one can construct an infinite family of \( PQ' \)-projective metrics. Under some non-degeneracy condition, this gives rise to an infinite family of integrable flows.

From the other side, the theories of projectively equivalent and \( h \)-projectively equivalent metrics appeared to be very useful mathematical theories of deep interest. The results in our paper suggest to look for other examples in the case when \( \epsilon = -1, -3, -5, ... \). If \( \epsilon = -1 \) but \( P^2 \neq -Id \), a lot of examples can be constructed using the ‘hierarchy construction’ from [15]. It is interesting to ask whether every pair of \( PQ^{-1} \)-projective metrics is in the hierarchy of some \( h \)-projectively equivalent metrics?

Another attractive problem is to find interesting examples for \( \epsilon = -3, -5, ... \). Besides the relation to integrable systems provided by [15], one could find other branches of differential geometry of similar interest as projective or \( h \)-projective geometry.
1.4. PDE for \(PQ^\varepsilon\)-projectivity. Given a pair of Riemannian metrics \(g, \bar{g}\) and tensors \(P, Q\) satisfying equation (1), we introduce the \((1, 1)\)-tensor \(A = A(g, \bar{g})\) defined by

\[
A = \left( \frac{\det \bar{g}}{\det g} \right)^{\frac{1}{m+1}} \bar{g}^{-1}g.
\] (5)

Here we view the metrics as vector bundle isomorphisms \(g : TM \to T^*M\) and \(\bar{g}^{-1} : T^*M \to TM\). We see that \(A\) is non-degenerate and self-adjoint with respect to \(g\) and \(\bar{g}\). Moreover, \(A\) commutes with \(P\) and \(Q\).

**Fact 1.** (Lemma 2 in [15], see also Theorems 5 and 6 in [9]). Two metrics \(g\) and \(\bar{g}\) are \(PQ^\varepsilon\)-projective if for a certain vector field \(\Lambda_1\), the \((1, 1)\)-tensor \(A\) defined in (5) is a solution of

\[
(\nabla_X A) Y = g(Y, X)\Lambda + g(Y, \Lambda)X + g(Y, QX)P\Lambda
\]

\[+ g(Y, P\Lambda)QX \text{ for all } X, Y \in TM.\] (6)

Conversely, if \(A\) is a \(g\)-self-adjoint positive solution of (6), which commutes with \(P\) and \(Q\), the Riemannian metric

\[
\bar{g} = (\det A)^{-\frac{1}{m+1}} g A^{-1}
\]

is \(PQ^\varepsilon\)-projective to \(g\).

**Remark 2.** Taking the trace of the \((1, 1)\)-tensors in equation (6) acting on the vector field \(Y\), we obtain

\[
\Lambda = \frac{1}{2(1 - \varepsilon)} \text{grad trace } A.\] (7)

Hence, (6) is a linear first-order PDE on the \((1, 1)\)-tensor \(A\).

**Remark 3.** From Fact 1 it follows that the metrics \(g, \bar{g}\) are affinely equivalent if and only if \(\Lambda \equiv 0\) on the whole \(M\).

**Remark 4.** Relation between the 1-form \(\Phi\) in equation (2) and the vector field \(\Lambda\) in equation (6) is given by \(\Lambda = -Ag^{-1}\Phi\) (again \(g^{-1} : T^*M \to TM\) is considered as a bundle isomorphism), see [15]. Recall from Example 2 that projective equivalence is a special case of \(PQ^\varepsilon\)-projectivity with \(\Phi(P) = 0\) or \(Q = 0\) and \(\varepsilon = 0\). In view of Fact 1, we now have that \(g\) and \(\bar{g}\) are projectively equivalent if and only if \(A = A(g, \bar{g})\) given by equation (5) (with \(\varepsilon = 0\), satisfies equation (6) with \(P\Lambda = 0\) or \(Q = 0\), i.e.

\[
(\nabla_X A) Y = g(Y, X)\Lambda + g(Y, \Lambda)X \text{ for all } X, Y \in TM.\] (8)

2. Proof of the results.

2.1. Topalov's integrals. We first recall the following.

**Fact 2.** (Proposition 3 in [15]). Let \(g\) and \(\bar{g}\) be \(PQ^\varepsilon\)-projective metrics and let \(A\) be defined by (5). We identify \(TM\) with \(T^*M\) by \(g\), and consider the canonical symplectic
structure on $TM \cong T^*M$. Then the functions $F_t : TM \to \mathbb{R}$,

$$F_t(X) = |\det (A - tId)|^{-\frac{1}{2}} g((A - tId)^{-1}X, X), \quad X \in TM$$  \hspace{1cm} (9)

are commuting quadratic integrals for the geodesic flow of $g$.

**Remark 5.** Note that the function $F_t$ in equation (9) is not defined in the points $x \in M$ such that $t \in \text{spec } A_x$. It will be clear from the proof of Theorem 1 that in the nontrivial case one can extend the functions $F_t$ to these points as well.

**2.2. Proof of Theorem 1.** Suppose that $g$ and $\bar{g}$ are $PQ^\epsilon$-projective Riemannian metrics, and let $A = A(g, \bar{g})$ be the corresponding solution of equation (6) defined by equation (5). Since $A$ is self-adjoint with respect to the positively definite metric $g$, the eigenvalues of $A$ in every point $x \in M$ are real numbers. We denote these by $\mu_1(x) \leq \cdots \leq \mu_m(x)$; depending on the multiplicity, some of the eigenvalues might coincide. The functions $\mu_i$ are continuous on $M$. Denote by $M^0 \subseteq M$ the set of points where the number of different eigenvalues of $A$ is maximal on $M$. Since the functions $\mu_i$ are continuous, $M^0$ is open in $M$. Moreover, it was shown in [15] that $M^0$ is dense in $M$ as well. The implicit function theorem now implies that $\mu_i$ are differentiable functions on $M^0$.

From Remark 3 and equation (7) we immediately obtain that $g$ and $\bar{g}$ are affinely equivalent if and only if all eigenvalues of $A$ are constant. Suppose that $g$ and $\bar{g}$ are not affinely equivalent, that is there is a non-constant eigenvalue $\rho$ of $A$ with multiplicity $k \geq 1$. Let us choose a point $x_0 \in M^0$ such that $d\rho_{|x_0} \neq 0$, define $c := \rho(x_0)$ and consider the hypersurface $H = \{x \in U : \rho(x) = c\}$, where $U \subseteq M^0$ is a geodesically convex neighbourhood of $x_0$. We think that $U$ is sufficiently small such that $\mu(x) \neq c$ for all eigenvalues $\mu$ of $A$ different from $\rho$ and all $x \in U$.

**Lemma 1.** There is a smooth nowhere vanishing $(0, 2)$-tensor $T$ on $U$ such that on $U \setminus H$, $T$ coincides with

$$\text{sgn}(\rho - c)|\det (A - cId)|^{-\frac{1}{2}} g((A - cId)^{-1}X, X).$$  \hspace{1cm} (10)

**Proof.** Let us denote by $\rho = \rho_1, \rho_2, \ldots, \rho_r$ different eigenvalues of $A$ on $M^0$ with multiplicities $k = k_1, k_2, \ldots, k_r$, respectively. Since the eigenspace distributions of $A$ are differentiable on $M^0$, we can choose a local frame $\{U_1, \ldots, U_m\}$ on $U$ such that $g$ and $A$ are given by matrices

$$g = \text{diag}(1, \ldots, 1) \quad \text{and} \quad A = \text{diag}(\rho_1, \ldots, \rho_1, \rho_2, \ldots, \rho_2, \ldots, \rho_r, \ldots, \rho_r)$$

$k$ times $k_r$ times
with respect to this frame. The tensor (10) can now be written as

\[
\text{sgn}(\rho - c) \left| \det (A - cI) \right|^{\frac{1}{r}} g(A - cI)^{-1} = \\
(\rho - c) \prod_{i=2}^{r} |\rho_i - c|^{\frac{1}{r}} \text{diag} \left( \frac{1}{\rho - c}, \ldots, \frac{1}{\rho - c}, \ldots, \frac{1}{\rho_r - c}, \ldots, \frac{1}{\rho_r - c} \right)
\]

\[
= \prod_{i=2}^{r} |\rho_i - c|^{\frac{1}{r}} \text{diag} \left( 1, \ldots, 1, \ldots, \frac{\rho - c}{\rho_i - c}, \ldots, \frac{\rho - c}{\rho_i - c} \right).
\]

(11)

Since \(\rho_i \neq c\) on \(U \subseteq M^0\) for \(i = 2, \ldots, r\), we see that (11) is a smooth nowhere vanishing \((0, 2)\)-tensor on \(U\).

**Lemma 2.** The multiplicity of the non-constant eigenvalues of \(A\) is equal to \(1 - \epsilon\).

**Proof.** Let us consider the integral \(F_c : TM \to \mathbb{R}\) defined in equation (9). Using the tensor \(T\) from Lemma 1, we can write \(F_c\) as

\[
F_c(X) = \text{sgn}(\rho - c) \left| \det (A - cI) \right|^{\frac{1}{r} - \frac{1}{k}} T(X, X), \quad X \in TM.
\]

(12)

Our goal is to show that \(\frac{1}{1-\epsilon} - \frac{1}{k} = 0\).

First suppose that \(\frac{1}{1-\epsilon} - \frac{1}{k} > 0\) and let \(y \in U \setminus H\). We choose a geodesic \(\gamma : [0, 1] \to U\) such that \(y = \gamma(0)\) and \(\gamma(1) \in H\), see Figure 1. Since \(\rho(\gamma(t)) \xrightarrow{t \to 1} c\), we see from equation (12) that \(f_c(\gamma(t)) \xrightarrow{t \to 1} 0\). It follows that \(f_c(\dot{\gamma}(0)) \xrightarrow{t \to 1} 0\). On the other hand, since \(F_c\) is an integral for the geodesic flow of \(g\) (see Fact 2), the value \(F_c(\dot{\gamma}(t))\) is independent of \(t\), and hence \(F_c(\dot{\gamma}(0)) = 0\). We have shown that \(F_c(\dot{\gamma}(0)) = 0\) for all initial velocities \(\dot{\gamma}(0) \in T_\gamma M\) of geodesics connecting \(y\) with points of \(H\). Since \(H\) is a hypersurface, it follows that the quadric \(\{X \in T_\gamma M : F_c(X) = 0\}\) contains an open subset that implies \(F_c \equiv 0\) on \(T_\gamma M\). This is a contradiction to Lemma 1, since \(T\) is non-vanishing in \(y\). We obtain that \(\frac{1}{1-\epsilon} - \frac{1}{k} \leq 0\).

Let us now treat the case when \(\frac{1}{1-\epsilon} - \frac{1}{k} < 0\). We choose a vector \(X \in T_{x_0} M\) which is not tangent to \(H\) and satisfies \(T(X, X) \neq 0\). Such a vector exists, since \(T_{x_0} M \setminus T_{x_0} H\) is open in \(T_{x_0} M\) and \(T\) is not identically zero on \(T_{x_0} M\) by Lemma 1. Let us consider the geodesic \(\gamma\) with \(\gamma(0) = x_0\) and \(\dot{\gamma}(0) = X\), see Figure 2. Since \(X \notin T_{x_0} H\), the geodesic
\( \gamma \) has to leave \( H \) for \( t > 0 \). In a point \( \gamma(t) \in U \setminus H \) the value \( F_c(\dot{\gamma}(t)) \) will be finite. On the other hand, since \( f_\epsilon(\gamma(t)) \xrightarrow{t \to 0} \infty \) and \( T(\dot{\gamma}(0), \dot{\gamma}(0)) \neq 0 \), we have \( F_c(\dot{\gamma}(t)) \xrightarrow{t \to 0} \infty \). Again this contradicts the fact that the value of \( F_c \) must remain constant along \( \dot{\gamma} \) by Fact 2. We have shown that \( 1 \frac{1}{1 - \epsilon} - \frac{1}{k} = 0 \), and finally Lemma 2 is proven. \( \square \)

As a consequence of Lemma 2, if the metrics \( g, \tilde{g} \) are not affinely equivalent (i.e. at least one eigenvalue of \( A \) is non-constant), \( \epsilon \) is an integer less or equal to zero. If \( \epsilon \neq 0 \), the condition \( PQ = \epsilon Id \) in equation (1) implies that \( P \) is non-degenerate and by the first condition in equation (1), \( g(P, . . .) \) is a non-degenerate 2-form on each eigenspace of \( A \) (note that \( A \) and \( P \) commute). This implies that for \( \epsilon \neq 0 \) the eigenspaces of \( A \) have even dimension, in particular, \( 1 - \epsilon \in \{2, 4, 6, 8, \ldots \} \). Theorem 1 is proven.

2.3. Proof of Theorem 2. Let \( g, \tilde{g} \) be two \( PQ^\epsilon \)-projective metrics and let \( A \) be the corresponding solution of equation (6) defined by equation (5). As it was already stated in the proof of Theorem 1, the eigenspace distributions of \( A \) are differentiable in a neighbourhood of almost every point of \( M \). First let us prove the following.

Lemma 3. Let \( X \) be an eigenvector of \( A \) corresponding to the eigenvalue \( \rho \). If \( \mu \) is another eigenvalue of \( A \) and \( \rho \neq \mu \), then \( X(\mu) = 0 \). In particular, \( \text{grad} \mu \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \mu \).

Remark 6. Lemma 3 is known for projectively equivalent (Example 2) and \( h \)-projectively equivalent (Example 3) metrics. For projectively equivalent metrics, it is a classical result that was already known to Levi-Civita [4]. For \( h \)-projectively equivalent metrics, it follows from [1, 7].

Proof. Let \( Y \) be an eigenvector field of \( A \) corresponding to the eigenvalue \( \mu \). For arbitrary \( X \in TM \), we obtain \( \nabla_X(AY) = \nabla_X(\mu Y) = X(\mu)Y + \mu \nabla_X Y \) and \( \nabla_X(AY) = (\nabla_X A)Y + A \nabla_X Y \). Combining these equations and replacing the expression \( (\nabla_X A)Y \) by equation (6) we obtain

\[
(A - \mu Id)\nabla_X Y = X(\mu)Y - g(Y, X)\Lambda - g(Y, \Lambda)X - g(Y, QX)\Lambda - g(Y, PA)QX.
\]

(13)

Now let \( X \) be an eigenvector of \( A \) corresponding to the eigenvalue \( \rho \) and suppose that \( \rho \neq \mu \). Since \( A \) is \( g \)-self-adjoint, the eigenspaces of \( A \) corresponding to different eigenvalues are orthogonal to each other. Moreover, since \( A \) and \( Q \) commute, \( Q \) leaves...
the eigenspaces of $A$ invariant. Using equation (13) we obtain
\[(A - \mu \text{Id}) \nabla_X Y + g(Y, \Lambda)X + g(Y, P \Lambda)QX = X(\mu)Y.\]

Since the left-hand side is orthogonal to the $\mu$-eigenspace of $A$, we necessarily have $X(\mu) = 0$. We have shown that $g(\text{grad } \mu, X) = X(\mu) = 0$ for any eigenvalue $\mu$ and any eigenvector field $X$ corresponding to an eigenvalue which is different from $\mu$. This forces $\text{grad } \mu$ to be contained in the eigenspace of $A$ corresponding to $\mu$.

Now suppose $\epsilon = 0$. Let us denote the non-constant eigenvalues of $A$ by $\rho_1, \ldots, \rho_l$. Using Lemma 2, the corresponding eigenspaces are 1-dimensional and Lemma 3 implies that these are spanned by the gradients $\text{grad } \rho_1, \ldots, \text{grad } \rho_l$ respectively. Since $P$ and $A$ commute, $P$ leaves the eigenspaces of $A$ invariant, hence $P \text{grad } \rho_i = p_i \text{grad } \rho_i$ for some real number $p_i$. Now $P$ is skew with respect to $g$ and we obtain $0 = g(\text{grad } \rho_i, P \text{grad } \rho_i) = p_i g(\text{grad } \rho_i, \text{grad } \rho_i)$, which implies that

\[P \text{grad } \rho_i = 0.\]

On the other hand, by equation (7)
\[\Lambda = \frac{1}{2} \text{grad } \text{trace } A = \frac{1}{2} (\text{grad } \rho_1 + \ldots + \text{grad } \rho_l).\]

Combining the last two equations, we obtain $P \Lambda = 0$. It follows from Remark 4 that $g$ and $\bar{g}$ are projectively equivalent and hence Theorem 2 is proved.

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