J. Aust. Math. Soc. **87** (2009), 263–273
doi:10.1017/S1446788708080993

WHEN LATTICE HOMOMORPHISMS OF ARCHIMEDEAN VECTOR LATTICES ARE RIESZ HOMOMORPHISMS

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(Received 4 March 2008; accepted 11 September 2008)

Communicated by A. J. Pryde

Abstract

Let A, B be Archimedean vector lattices and let $(u_i)_{i\in I}$, $(v_i)_{i\in I}$ be maximal orthogonal systems of A and B, respectively. In this paper, we prove that if T is a lattice homomorphism from A into B such that $T(\lambda u_i) = \lambda v_i$ for each $\lambda \in \mathbb{R}_+$ and $i \in I$, then T is linear. This generalizes earlier results of Ercan and Wickstead (*Math. Nachr* **279** (9–10) (2006), 1024–1027), Lochan and Strauss (*J. London Math. Soc.* (2) **25** (1982), 379–384), Mena and Roth (*Proc. Amer. Math. Soc.* **71** (1978), 11–12) and Thanh (*Ann. Univ. Sci. Budapest. Eotvos Sect. Math.* **34** (1992), 167–171).

2000 *Mathematics subject classification*: primary 46A40; secondary 47B65.
Keywords and phrases: weak order unit, lattice homomorphism, Riesz homomorphism.

1. Introduction

In this paper, we give some conditions under which a lattice homomorphism is linear. Our starting point is a theorem of Mena and Roth [6] (generalized by Thanh [8], by Lochan and Strauss [4] and recently by Ercan and Wickstead [3]) where T is a lattice homomorphism of $C(X)$-spaces and hence via the Kakutani representation theorem, T acts between two uniformly complete Archimedean vector lattices A and B with (strong or weak) order units. Perhaps the most general result in this direction is the work of Ercan and Wickstead [3]. More precisely, they deduced from the theorem of Mena and Roth and by using the Kakutani representation theorem, that if A and B are uniformly complete Archimedean vector lattices A and B with weak order units $e_1 \in A$ and $e_2 \in B$ and if T is a lattice homomorphism from A to B, such that $T(\lambda e_1) = \lambda e_2$ for each $\lambda \in \mathbb{R}$, then T is linear. Finally, using the same argument, they gave a corresponding result for the case where T acts between two uniformly complete Archimedean vector lattices with disjoint complete systems of projection, for example, between two σ-Dedekind complete vector lattices. To the best of our knowledge, there is no proof, without use of representation theory, of these versions of the Mena–Roth theorem. The first aim of this paper is to give not only a proof of the theorem of Mena and Roth, which relies on a new, constructive and intrinsic approach but also it does not make use of the uniformly completeness of A and B. Finally, we are concerned with lattice homomorphisms that act between Archimedean vector lattices. More precisely, we prove that if A, B are Archimedean vector lattices, if $(u_i)_{i\in I}$, $(v_i)_{i\in I}$ are maximal orthogonal systems of A and B, respectively, and if T is a lattice homomorphism from A into B such that $T(\lambda u_i) = \lambda v_i$ for each $\lambda \in \mathbb{R}_+$ and $i \in I$, then T is linear. This generalizes earlier results of [3, 4, 6] and [8].

We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notation and concepts that are not explained in this paper we refer to the standard monographs [1, 5, 7].

2. Definitions and notations

We assume throughout this paper that all vector lattices (or Riesz spaces) under consideration are Archimedean.

A map T between vector lattices A and B is called *a lattice homomorphism* if

$$T(a \wedge b) = T(a) \wedge T(b) \quad \text{and} \quad T(a \vee b) = T(a) \vee T(b) \quad \text{for each } a, b \in A.$$

A linear lattice homomorphism is called *Riesz homomorphism*.

Let A be a (real) vector lattice. A vector subspace I of A is called an *order ideal* (or *o-ideal*) whenever $|a| \le |b|$ and $b \in I$ imply $a \in I$. Every o-ideal is a vector sublattice of A. The principal o-ideal generated by $0 \le e \in A$ is denoted by A_e and it is a sublattice of A.

Let A be a vector lattice, let $0 \le v \in A$, the sequence $\{a_n, n = 1, 2, \ldots\}$ in A is called (v) *relatively uniformly convergent* to $a \in A$ if, for every real number $\varepsilon > 0$, there exists a natural number n_ε such that $|a_n - a| \le \varepsilon v$ for all $n \ge n_\varepsilon$. This will be denoted by $a_n \to a$ (v). If $a_n \to a$ (v) for some $0 \le v \in A$, then the sequence $\{a_n, n = 1, 2, \ldots\}$ is called (*relatively*) *uniformly convergent* to a, which is denoted by $a_n \to a$ (*r.u.*). The notion of (v) *relatively uniformly Cauchy sequence* is defined in the obvious way. Relatively uniformly limits are unique in Archimedean vector lattices, see [5, Theorem 63.2]. A vector lattice A is called *relatively uniformly complete* whenever every relatively uniformly Cauchy sequence in A has a unique limit. Every relatively uniformly complete vector lattice is Archimedean. Let A be a vector lattice (or Riesz space). A subset S of the positive cone A^+ is called an *orthogonal system of* A if $0 \notin S$ and if $u \wedge v = 0$ for each pair (u, v) of distinct elements of S. It is clear from Zorn's lemma that every orthogonal system of A is contained in a maximal orthogonal system. An element e of a vector lattice A is called *weak order unit* (*strong order unit*) of A whenever $\{e\}$ is a maximal orthogonal system of A (respectively, $A_e = A$). An ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \ge 0$ imply $ac \wedge b = ca \wedge b = 0$. Any f-algebra is automatically commutative and has positive squares.

3. The main results

Our main goal is to establish the result corresponding to the Mena–Roth theorem for lattice homomorphisms on vector lattice with (strong and weak) order units. The following proposition is an essential ingredient for our main results.

Before continuing with the next result, we recall the following notion.

Let A be a vector lattice and let $0 \leq a \in A$. An element $0 \leq e \in A$ is called *a component* of a if $e \wedge (a - e) = 0$.

PROPOSITION 1. *Let A be a Dedekind complete vector lattice with strong order unit e, let $e_1, e_2, \ldots, e_n$ be components of e and let B be a vector lattice. If T is a lattice homomorphism from A into B such that $T(\lambda e) = \lambda T(e)$ for each $\lambda \in \mathbb{R}_+$, then T satisfies the following property:*

$$T\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i T(e_i) \quad \forall \lambda_i \in \mathbb{R}_+ \ (1 \leq i \leq n).$$

PROOF. *Step 1.* We show first that $T(\lambda e_1) = \lambda T(e_1)$ for each $\lambda \in \mathbb{R}_+$. Let us denote $e_i^c = e - e_i$, for all $1 \leq i \leq n$. Since $e_1^c + e_1 = e_1^c \vee e_1 = e$, $e_1^c \wedge e_1 = 0$ and since T is a lattice homomorphism, then $T(e_1^c) \wedge T(e_1) = 0$. It follows that

$$\begin{aligned} T(\lambda e) &= T(\lambda(e_1^c \vee e_1)) \\ &= T(\lambda e_1^c \vee \lambda e_1) \\ &= T(\lambda e_1^c) \vee T(\lambda e_1) \\ &= T(\lambda e_1^c) + T(\lambda e_1) \quad \forall \lambda \in \mathbb{R}_+. \end{aligned}$$

By the fact that

$$\begin{aligned} T(\lambda e) &= \lambda T(e) \\ &= \lambda T(e_1^c + e_1) \\ &= \lambda T(e_1^c \vee e_1) \\ &= \lambda T(e_1^c) \vee \lambda T(e_1) \\ &= \lambda T(e_1^c) + \lambda T(e_1) \quad \forall \lambda \in \mathbb{R}_+, \end{aligned}$$

we deduce that

$$T(\lambda e_1^c) - \lambda T(e_1^c) = \lambda T(e_1) - T(\lambda e_1) \quad \forall \lambda \in \mathbb{R}_+.$$

Moreover, since $\lambda e_1^c \wedge e_1 = e_1^c \wedge \lambda e_1 = 0$, for each $\lambda \in \mathbb{R}_+$ and since T is a lattice homomorphism, we have

$$T(\lambda e_1^c) \wedge T(e_1) = \lambda T(e_1^c) \wedge T(e_1) = 0 \quad \forall \lambda \in \mathbb{R}_+.$$

Hence,

$$|T(\lambda e_1^c) - \lambda T(e_1^c)| \wedge \lambda T(e_1) = 0 \quad \forall \lambda \in \mathbb{R}_+.$$

Using the same argument, we have

$$|T(\lambda e_1^c) - \lambda T(e_1^c)| \wedge T(\lambda e_1) = 0 \quad \forall \lambda \in \mathbb{R}_+.$$

Therefore

$$|T(\lambda e_1^c) - \lambda T(e_1^c)| \wedge |\lambda T(e_1) - T(\lambda e_1)| = 0 \quad \forall \lambda \in \mathbb{R}_+.$$

Hence,

$$T(\lambda e_1^c) - \lambda T(e_1^c) = \lambda T(e_1) - T(\lambda e_1) = 0 \quad \forall \lambda \in \mathbb{R}_+.$$

Step 2. We show that $T(\alpha e_1 + \beta e_2) = \alpha T(e_1) + \beta T(e_2)$ for each $\alpha, \beta \in \mathbb{R}_+$. To this end, we remark that $A_e = A$ can be seen as an f-algebra with e as unit (where its f-algebra multiplication is denoted by juxtaposition; see [2, Remark 19.5]). In order to reach our aim, we first show that $T(\lambda e_1 e_2) = \lambda T(e_1 e_2)$ and $T(\lambda e_1^c e_2^c) = \lambda T(e_1^c e_2^c)$ for each $\lambda \in \mathbb{R}_+$. To this end, let $\lambda \in \mathbb{R}_+$. Since

$$\begin{aligned} \lambda e &= \lambda e^2 \\ &= \lambda(e_1^c + e_1)(e_2^c + e_2) \\ &= \lambda e_1 e_2 + \lambda e_1 e_2^c + \lambda e_1^c e_2 + \lambda e_1^c e_2^c. \end{aligned}$$

We point out that $\lambda e_1 e_2$, $\lambda e_1 e_2^c$, $\lambda e_1^c e_2$ and $\lambda e_1^c e_2^c$ are mutually disjoint. Then by using the fact that T is a lattice homomorphism, we obtain

$$\begin{aligned} T(\lambda e) &= T(\lambda e_1 e_2 + \lambda e_1 e_2^c + \lambda e_1^c e_2 + \lambda e_1^c e_2^c) \\ &= T(\lambda e_1 e_2 \vee \lambda e_1 e_2^c \vee \lambda e_1^c e_2 \vee \lambda e_1^c e_2^c) \\ &= T(\lambda e_1 e_2) \vee T(\lambda e_1 e_2^c) \vee T(\lambda e_1^c e_2) \vee T(\lambda e_1^c e_2^c) \\ &= T(\lambda e_1 e_2) + T(\lambda e_1 e_2^c) + T(\lambda e_1^c e_2) + T(\lambda e_1^c e_2^c). \end{aligned}$$

As

$$\begin{aligned} T(\lambda e) &= \lambda T(e) \\ &= \lambda T(e_1 e_2 + e_1 e_2^c + e_1^c e_2 + e_1^c e_2^c) \\ &= \lambda T(e_1 e_2 \vee e_1 e_2^c \vee e_1^c e_2 \vee e_1^c e_2^c) \\ &= \lambda(T(e_1 e_2) \vee T(e_1 e_2^c) \vee T(e_1^c e_2) \vee T(e_1^c e_2^c)) \\ &= \lambda(T(e_1 e_2) + T(e_1 e_2^c) + T(e_1^c e_2) + T(e_1^c e_2^c)) \\ &= \lambda T(e_1 e_2) + \lambda T(e_1 e_2^c) + \lambda T(e_1^c e_2) + \lambda T(e_1^c e_2^c). \end{aligned}$$

Thus,

$$\begin{aligned} T(\lambda e_1 e_2) - \lambda T(e_1 e_2) = \lambda T(e_1 e_2^c) + \lambda T(e_1^c e_2) + \lambda T(e_1^c e_2^c) \\ - (T(\lambda e_1 e_2^c) + T(\lambda e_1^c e_2) + T(\lambda e_1^c e_2^c)). \end{aligned}$$

Let

$$X = |T(\lambda e_1 e_2) - \lambda T(e_1 e_2)|$$

and let

$$Y = |\lambda T(e_1 e_2^c) + \lambda T(e_1^c e_2) + \lambda T(e_1^c e_2^c) - (T(\lambda e_1 e_2^c) + T(\lambda e_1^c e_2) + T(\lambda e_1^c e_2^c))|.$$

Since $\lambda e_1 e_2 \wedge \lambda e_1 e_2^c = \lambda e_1 e_2 \wedge e_1 e_2^c = 0$ and since T is a lattice homomorphism, it follows that

$$T(\lambda e_1 e_2) \wedge \lambda T(e_1 e_2^c) = T(\lambda e_1 e_2) \wedge T(\lambda e_1 e_2^c) = 0$$

and

$$\lambda T(e_1 e_2) \wedge \lambda T(e_1 e_2^c) = \lambda T(e_1 e_2) \wedge T(\lambda e_1 e_2^c) = 0.$$

Hence,

$$|\lambda T(e_1 e_2^c) - T(\lambda e_1 e_2^c)| \wedge T(\lambda e_1 e_2) = 0$$

and

$$|\lambda T(e_1 e_2^c) - T(\lambda e_1 e_2^c)| \wedge \lambda T(e_1 e_2) = 0.$$

Therefore,

$$|\lambda T(e_1 e_2^c) - T(\lambda e_1 e_2^c)| \wedge X = 0.$$

Using the same argument

$$|\lambda T(e_1^c e_2) - T(\lambda e_1^c e_2)| \wedge X = 0$$

and

$$|\lambda T(e_1^c e_2^c) - T(\lambda e_1^c e_2^c)| \wedge X = 0.$$

As a conclusion $X \wedge Y = 0$. It follows that

$$\begin{aligned} T(\lambda e_1 e_2) - \lambda T(e_1 e_2) &= \lambda T(e_1 e_2^c) + \lambda T(e_1^c e_2) + \lambda T(e_1^c e_2^c) \\ &\quad - (T(\lambda e_1 e_2^c) + T(\lambda e_1^c e_2) + T(\lambda e_1^c e_2^c)) \\ &= 0. \end{aligned} \tag{3.1}$$

By using the same argument, we obtain

$$\lambda T(e_1 e_2^c) = T(\lambda e_1 e_2^c) \tag{3.2}$$

$$\lambda T(e_1^c e_2) = T(\lambda e_1^c e_2) \tag{3.3}$$

$$\lambda T(e_1^c e_2^c) = T(\lambda e_1^c e_2^c). \tag{3.4}$$

Now let $\alpha, \beta \in \mathbb{R}_+$. Since

$$\begin{aligned} \alpha e_1 + \beta e_2 &= \alpha e_1 e + \beta e_2 e \\ &= \alpha e_1 (e_2^c + e_2) + \beta e_2 (e_1^c + e_1) \\ &= \alpha e_1 e_2^c + \alpha e_1 e_2 + \beta e_2 e_1^c + \beta e_2 e_1 \\ &= (\alpha + \beta) e_1 e_2 + \alpha e_1 e_2^c + \beta e_2 e_1^c. \end{aligned}$$

A simple combination between the fact that $(\alpha+\beta)e_1e_2$, $\alpha e_1e_2^c$ and $\beta e_2e_1^c$ are mutually disjoint and the fact that T is a lattice homomorphism, we get by using the equalities (3.1), (3.2), (3.3) and (3.4)

$$\begin{aligned}
T(\alpha e_1+\beta e_2) &= T((\alpha+\beta)e_1e_2+\alpha e_1e_2^c+\beta e_2e_1^c)\\
&= T((\alpha+\beta)e_1e_2\vee \alpha e_1e_2^c\vee \beta e_2e_1^c)\\
&= T((\alpha+\beta)e_1e_2)\vee T(\alpha e_1e_2^c)\vee T(\beta e_2e_1^c)\\
&= T((\alpha+\beta)e_1e_2)+T(\alpha e_1e_2^c)+T(\beta e_2e_1^c)\\
&= (\alpha+\beta)T(e_1e_2)+\alpha T(e_1e_2^c)+\beta T(e_2e_1^c)\\
&= \alpha T(e_1e_2)+\alpha T(e_1e_2^c)+\beta T(e_1e_2)+\beta T(e_2e_1^c)\\
&= \alpha T(e_1e_2)\vee \alpha T(e_1e_2^c)+\beta T(e_1e_2)\vee \beta T(e_2e_1^c)\\
&= \alpha T((e_1e_2)\vee(e_1e_2^c))+\beta T((e_1e_2)\vee(e_2e_1^c))\\
&= \alpha T(e_1)+\beta T(e_2).
\end{aligned}$$

A simple combination between the two previous steps gives the desired result. □

The following proposition is a surprising and interesting consequence of the above result.

Proposition 2. *Let A be a Dedekind complete vector lattice with a strong order unit e and let B be a vector lattice. If T is a lattice homomorphism from A into B such that $T(\lambda e)=\lambda T(e)$ for each $\lambda\in\mathbb{R}_+$, then T satisfies the following properties:*

(1) $T(a+b)=T(a)+T(b)$ *for each* $a,b\in A_+$;
(2) $T(\lambda a)=\lambda T(a)$ *for each* $\lambda\in\mathbb{R}_+$.

Proof. Let a, $b\in A_+$, let $\lambda\in\mathbb{R}_+$ and let

$$L=\left\{k_n\in A; k_n=\sum_{i=1}^{i=n}\alpha_ie_i,\ \alpha_i\in\mathbb{R}_+,\ e_i \text{ is a component of } e,\ n=1,2,\ldots\right\}.$$

By using the Freudenthal spectral theorem [5, Theorem 40.2], there exist two sequences $(k_n)_n$, $(l_n)_n$ such that $k_n, l_n\in L$, $k_n\nearrow a$ $(r.u.)$ and $l_n\nearrow b$ $(r.u.)$. It follows that there exists $n_0\in\mathbb{N}^*$ such that, for all $n\geq n_0$, we have

$$0\leq a-k_n\leq e/n \quad\text{and}\quad 0\leq b-l_n\leq e/n.$$

It follows that

$$0\leq a\leq (e/n)+k_n \quad\text{and}\quad 0\leq b\leq (e/n)+l_n$$

for all $n\geq n_0$. Then

$$0\leq a+b\leq 2(e/n)+k_n+l_n. \tag{3.5}$$

By applying the lattice homomorphism T, we deduce that

$$0\leq T(a)\leq T((e/n)+k_n)\quad 0\leq T(\lambda a)\leq T(\lambda(e/n)+\lambda k_n)$$

and

$$0 \leq T(b) \leq T((e/n) + l_n).$$

By using the previous proposition,

$$T((e/n) + k_n) = T(e/n) + T(k_n) = (1/n)T(e) + T(k_n)$$
$$T(\lambda(e/n) + \lambda k_n) = T(\lambda(e/n)) + T(\lambda k_n) = \lambda(1/n)T(e) + \lambda T(k_n)$$

and

$$T((e/n) + l_n) = T(e/n) + T(l_n) = (1/n)T(e) + T(l_n).$$

Therefore

$$T(k_n) \nearrow T(a)\ (r.u.),\ T(\lambda k_n) \nearrow T(\lambda a)\ (r.u.) \quad \text{and} \quad T(l_n) \nearrow T(b)\ (r.u.).$$

Again, by the previous proposition, we have $T(\lambda k_n) = \lambda T(k_n)$. Then

$$T(\lambda a) = \lambda T(a).$$

Moreover, by the previous proposition, we obtain $T(k_n) + T(l_n) = T(k_n + l_n)$. Since $T(k_n) + T(l_n) \nearrow T(a) + T(b)\ (r.u.)$ and since $k_n + l_n \nearrow a + b\ (r.u.)$ and by applying the map T to the inequality (3.5), we find that $T(k_n + l_n) \nearrow T(a + b)\ (r.u.)$. Then $T(a) + T(b) = T(a + b)$ and we are done. □

The following is essential to prove the first main result.

PROPOSITION 3. *Let A be a vector lattice, let A^δ be its Dedekind completion and let B be a Dedekind complete vector lattice. If T is a lattice homomorphism from A into B, then $T_{/A_+}$ has an extension to a lattice homomorphism of $(A^\delta)_+$ into B.*

PROOF. Let $x \in (A^\delta)_+ \backslash A_+$ and let $M = \{(x \vee a) \wedge b,\ a, b \in A_+\}$. It is clear that M is the sublattice of $(A^\delta)_+$ generated by x and A_+. Let us define

$$T_1(x) = \sup\{T(z),\ z \leq x,\ z \in A_+\}.$$

Thus, we can define $T' : M \to B$ by

$$T'((x \vee a) \wedge b) = (T_1(x) \vee T(a)) \wedge T(b) \quad (a, b \in A_+).$$

Clearly T' is a lattice homomorphism of M into B. By Zorn's lemma, T' has a maximal extension to a lattice homomorphism T^* of a sublattice N of $(A^\delta)_+$ into B. We prove that $N = (A^\delta)_+$. Suppose that there exists $y \in (A^\delta)_+ \backslash N$. Let $P = \{(y \vee a) \wedge b,\ a, b \in N\}$. Then P is the sublattice of $(A^\delta)_+$ generated by y and N. Let us define

$$T_1^*(y) = \sup\{T(z),\ z \leq y,\ z \in N\}.$$

Then we can define $T^\sharp : P \to B$ by

$$T^\sharp((y \vee a) \wedge b) = (T_1^*(y) \vee T^*(a)) \wedge T^*(b) \quad (a, b \in N).$$

Therefore, $T^\sharp$ is a lattice homomorphism of P into B. This contradicts maximality of T^* and so $N = (A^\delta)_+$, as required. □

We are now in position to give a generalized version of the Mena–Roth theorem for lattice homomorphisms on vector lattices with strong order units. The proof is identical in concept to [3, Lemma 1].

THEOREM 4. *Let A be a vector lattice with a strong order unit e and let B be a vector lattice. If T is a lattice homomorphism from A into B such that $T(\lambda e) = \lambda T(e)$ for each $\lambda \in \mathbb{R}_+$, then T is linear.*

PROOF. It is shown in the previous proposition that $T_{/A_+}$ has an extension to a lattice homomorphism of $(A^\delta)_+$ into the vector lattice Dedekind completion B^δ of B. According to Proposition 2, T is additive from $(A^\delta)_+$ into the Dedekind completion of B and so on A_+. It is well known, by [1, Theorem 1.7] that T extends uniquely to a positive operator T' from A to B. Hence,

$$T'(x) = T(x^+) - T(x^-).$$

Since T is a lattice homomorphism,

$$(T(x))^- = (-T(x)) \vee 0 = -(T(x) \wedge 0) = -T(x \wedge 0) = -T(-x^-) = T(x^-).$$

Hence,

$$T'(x) = T(x^+) - T(x^-) = (T(x))^+ - (T(x))^- = T(x).$$

Therefore, $T = T'$. □

The next corollary improves earlier results of [3, 4, 6] and [8] which assumed uniform completeness.

COROLLARY 5. *Let A be a vector lattice with a strong order unit e and let B be a vector lattice. If T is a lattice homomorphism from A into B such that $T(\lambda e) = \lambda T(e)$ for each $\lambda \in \mathbb{R}$, then T is linear.*

The following results are a consequence of the above theorem.

COROLLARY 6. *Let A be a vector lattice with a weak order unit e and let B be a vector lattice with a weak order unit f. If T is a lattice homomorphism from A into B such that $T(\lambda e) = \lambda f$ for each $\lambda \in \mathbb{R}_+$, then T is linear.*

PROOF. Let $a, b \in A$ and let $g = |a| + |b|$ and let $\beta \in \mathbb{R}_+$. According to the previous theorem, T is linear on A_e and since $g \wedge ne, e \in A_e$,

$$\begin{aligned} T(\beta g) &= \sup(T(\beta g) \wedge nf) \\ &= \sup(T(\beta g) \wedge \beta nf) \\ &= \sup(T(\beta g) \wedge \beta T(ne)) \\ &= \sup(T(\beta g \wedge \beta ne)) \\ &= \beta \sup(T(g \wedge ne)) \end{aligned}$$

for each $\beta \in \mathbb{R}_+$. Moreover,

$$\begin{aligned}\beta T(g) &= \sup(\beta T(g) \wedge nf) \\ &= \sup(\beta T(g) \wedge \beta nf) \\ &= \sup(\beta T(g) \wedge \beta T(ne)) \\ &= \beta \sup(T(g) \wedge T(ne)) \\ &= \beta \sup(T(g \wedge ne))\end{aligned}$$

for each $\beta \in \mathbb{R}_+$. Therefore, $T(\beta g) = \beta T(g)$, for each $\beta \in \mathbb{R}_+$. Again by the fact that T is linear on A_e, we have

$$\begin{aligned}\beta T(g) + \beta T(e) &= T(\beta g) + T(\beta e) \\ &= \sup(T(\beta g \wedge ne)) + T(\beta e) \\ &= \sup(T(\beta g \wedge ne) + T(\beta e)) \\ &= \sup(T(\beta g \wedge ne + \beta e)) \\ &= \sup(T((\beta g + \beta e) \wedge (n + \beta)e)) \\ &= \sup(T(\beta g + \beta e) \wedge (n + \beta) f) \\ &= \sup(T(\beta g + \beta e) \wedge nf) \\ &= T(\beta g + \beta e)\end{aligned}$$

for each $\beta \in \mathbb{R}_+$. Therefore, $T_{/A_{g+e}} : A_{g+e} \to B$ is a lattice homomorphism which satisfies $T(\beta g + \beta e) = T(\beta g) + T(\beta e) = \beta T(g) + T(\beta e)$ for each $\beta \in \mathbb{R}_+$. According to the previous results, $T_{/A_{g+e}}$ is linear. Since $a + \lambda b \in A_{g+e}$, for each $\lambda \in \mathbb{R}$, it follows that $T(a + \lambda b) = T(a) + \lambda T(b)$, for each $\lambda \in \mathbb{R}$, which gives the desired result. □

As a consequence, we deduce a result of Ercan and Wickstead [3, Lemma 1].

COROLLARY 7. *Let A be a uniformly complete vector lattice with a weak order unit e and let B be a vector lattice with a weak order unit f. If T is a lattice homomorphism from A into B such that $T(\lambda e) = \lambda f$ for each $\lambda \in \mathbb{R}$, then T is linear.*

Next, we broach the problem of finding a sufficient condition for a lattice homomorphism of vector lattices, which do not have a weak order unit, to be linear. This will culminate in a second version of the Mena–Roth theorem and the Ercan–Wickstead theorem.

THEOREM 8. *Let A, B be vector lattices and let $(u_i)_{i\in I}$, $(v_i)_{i\in I}$ be maximal orthogonal systems of A and B, respectively. If T is a lattice homomorphism from A into B such that $T(\lambda u_i) = \lambda v_i$ for each $\lambda \in \mathbb{R}_+$ and $i \in I$, then T is linear.*

PROOF. It is shown in Proposition 3, that $T_{/A_+}$ has an extension to a lattice homomorphism of $(A^\delta)_+$ into the vector lattice Dedekind completion B^δ of B. According to Proposition 2, T is additive on $(A^\delta)_+$. Let $x, y \in A_+$ and $\lambda \in \mathbb{R}_+$. Then

$$\begin{aligned} T(x) &= \sup_{H,n}\Big(\sum_{i\in H}(T(x)\wedge nT(u_i))\Big) \\ &= \sup_{H,n}\Big(\sum_{i\in H}T(x\wedge nu_i)\Big) \\ &= \sup_{H}\Big(\sum_{i\in H}T(x_i)\Big) \end{aligned}$$

where x_i is the projection component of x in the order band generated u_i in A^δ (denoted by B_{u_i}) and H is a finite subset of I. Hence,

$$T(\lambda x) = \sup_{H}\Big(\sum_{i\in H}T(\lambda x_i)\Big).$$

By Corollary 6, $T_{/B_{u_i}}$ is linear, then $T(\lambda x_i) = \lambda T(x_i)$. Hence, $T(\lambda x) = \lambda T(x)$. Moreover,

$$\begin{aligned} T(x+y) &= \sup_{H}\Big(\sum_{i\in H}T((x+y)_i)\Big) \\ &= \sup_{H}\Big(\sum_{i\in H}T(x_i+y_i)\Big) \\ &= \sup_{H}\Big(\sum_{i\in H}(T(x_i)+T(y_i))\Big) \\ &= \sup_{H}\Big(\sum_{i\in H}T(x_i)\Big) + \sup_{H}\Big(\sum_{i\in H}T(y_i)\Big) \\ &= T(x)+T(y). \end{aligned}$$

To complete the proof, it is sufficient to use [1, Theorem 1.7], as in the previous theorem. □

We finish this paper with the following remark.

REMARK 9. We note that, in the results of Mena and Roth [6], Thanh [8], Lochan and Strauss [4] and Ercan and Wickstead [3], the assumption that A is a uniformly complete vector lattice is superfluous, as shown in the previous results.

Acknowledgement

I am grateful to the anonymous referee for pointing out an error in my previous version and for providing suggestions that greatly improved the paper.

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