THE MODULUS OF WEAKLY COMPACT MULTIPLIERS ON
THE BANACH ALGEBRA $L^1(G)^{**}$ OF A LOCALLY
COMPACT GROUP

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Abstract

In this paper we give a necessary and sufficient condition under which the answer to the open problem raised by Ghahramani and Lau ('Multipliers and modulus on Banach algebras related to locally compact groups', J. Funct. Anal. 150 (1997), 478–497) is positive.


Keywords and phrases: locally compact group, weakly compact multiplier, modulus of operator.

1. Introduction

Let $G$ be a locally compact group with a left Haar measure $\lambda$ and let $L^1(G)$ be the Banach space of complex-valued integrable functions with respect to left Haar measure $\lambda$. With the norm $\|\cdot\|_1$ and with convolution

$$\vartheta * \omega(x) = \int_G \vartheta(y)\omega(y^{-1}x) \, d\lambda(y) \quad (x \in G)$$

as product, $L^1(G)$ becomes a Banach algebra [9]. Let $M(G)$ be the Banach algebra of all bounded complex-valued regular Borel measures on $G$ equipped with the convolution product $*$ and the total variation norm. The other Banach algebras we shall consider are defined as follows. Let $C(G)$ be the space of all bounded continuous complex-valued functions on $G$ with the sup-norm, and let $RUC(G)$ be the Banach space of all $f \in C(G)$ such that the mapping $x \mapsto f_x$ from $G$ into $C(G)$ is continuous, where $f_x(y) = f(yx)$ for all $x, y \in G$ [4].

Let $\mathcal{A}$ be a Banach algebra. We denote the dual of $\mathcal{A}$ by $\mathcal{A}^*$. If $\phi \in \mathcal{A}^*$, then the value of $\phi$ at an element $\vartheta \in \mathcal{A}$ will be written as $\langle \phi, \vartheta \rangle$. For every $n \in \mathcal{A}^{**}$ and $\phi \in \mathcal{A}^*$, we define the element $n\phi$ of $\mathcal{A}^*$ by the formula

$$\langle n\phi, \vartheta \rangle = \langle n, \phi\vartheta \rangle \quad (\vartheta \in \mathcal{A}),$$
where
\[ \langle \phi \vartheta, \omega \rangle = \langle \phi, \vartheta \omega \rangle \quad (\omega \in \mathcal{A}). \]

In this paper we always endow the second dual \( \mathcal{A}^{**} \) with the first Arens product ‘’, defined by
\[ \langle m \cdot n, \phi \rangle = \langle m, n\phi \rangle \]
for all \( m, n \in \mathcal{A}^{**} \) and \( \phi \in \mathcal{A}^* \). This product turns \( \mathcal{A}^{**} \) into a Banach algebra [10].

Let \( \mathcal{A} \) be a Banach algebra; a bounded operator \( T : \mathcal{A} \to \mathcal{A} \) is called a left multiplier if \( T(\vartheta \omega) = T(\vartheta)\omega \) for all \( \vartheta, \omega \in \mathcal{A} \). An element \( \vartheta \in \mathcal{A} \) is said to be a left weakly completely continuous element of \( \mathcal{A} \) if the left multiplier \( \vartheta \Gamma : \omega \mapsto \vartheta \omega \) is a weakly compact operator on \( \mathcal{A} \).

A Banach lattice \( \mathcal{A} \) is called complete if every nonempty subset of \( \mathcal{A} \) bounded from above has a supremum in \( \mathcal{A} \); see [3]. Hence for each bounded operator \( T \) from a complete Banach lattice \( \mathcal{A} \) into a complete Banach lattice \( \mathcal{B} \), we may define the modulus of \( T \) by
\[ |T|(p) = \sup\{ |T(\vartheta)| : \vartheta \in \mathcal{A}, |\vartheta| \leq p \} \]
for all positive elements \( p \in \mathcal{A} \). Then \( |T| \) is bounded; indeed,
\[ |T(\vartheta)| \leq \|T\| \|\vartheta\| \leq \|T\| \|p\| \]
for all \( \vartheta \in \mathcal{A} \) and for all positive elements \( p \in \mathcal{A} \) with \( |\vartheta| \leq p \). It is natural to ask whether the modulus of a left multiplier on a complete Banach lattice is also a left multiplier?

The modulus of left multipliers on group algebras has been studied by Ghahramani and Lau [7]. They proved that if \( T = \mu \Gamma \) on \( L^1(\mathcal{G}) \) (respectively on \( M(\mathcal{G}) \)) for some \( \mu \in \mathcal{M}(\mathcal{G}) \), then \( |T| = |\mu| \Gamma \) on \( L^1(\mathcal{G}) \) (respectively on \( M(\mathcal{G}) \)). To put it another way, they showed that the modulus of a left multiplier on \( L^1(\mathcal{G}) \) (respectively on \( M(\mathcal{G}) \)) is also a left multiplier on \( L^1(\mathcal{G}) \) (respectively on \( M(\mathcal{G}) \)). However, they proved that the modulus of a left multiplier on \( L^1(\mathcal{G})^{**} \) is not necessarily a left multiplier. They also proved that for any \( m \in L^1(\mathcal{G})^{**} \), \( |m| \Gamma \) is a left multiplier on \( L^1(\mathcal{G})^{**} \) if and only if \( |m| \Gamma = |m| \Gamma \). The question comes to mind immediately: under what conditions on a left multiplier \( T : L^1(\mathcal{G})^{**} \to L^1(\mathcal{G})^{**} \) can we assert that \( |T| \) is a left multiplier on \( L^1(\mathcal{G})^{**} \)?

Ghahramani and Lau [5–7] obtained some results on the question of the existence of nonzero weakly compact left multipliers on \( L^1(\mathcal{G})^{**} \). Losert [11], among other things,
proved that the existence of a nonzero weakly compact left multiplier on \( L^1(\mathcal{G})^{**} \) is equivalent to compactness of \( \mathcal{G} \); see also [1].

Our main purpose in this paper is to give a necessary and sufficient condition under which the answer to Problem 1.1 is positive. In fact, we show that the modulus of a weakly compact left multiplier on \( L^1(\mathcal{G})^{**} \) is a left multiplier if and only if \(|T| = \varrho \Gamma\) for some \( \varrho \in L^1(\mathcal{G}) \).

2. The results

We commence this section with the main result of the paper. First, we recall the main definitions and results that we require later. Let \( \mathcal{A} \) be a \( C^* \)-algebra. A linear functional \( p \in \mathcal{A}^* \) is called positive if \( \langle p, \phi \rangle \geq 0 \) for all positive elements \( \phi \in \mathcal{A} \). It is well known from [12] that if \( p \) is a bounded linear functional on a unital \( C^* \)-algebra, then \( p \) is positive if and only if \( \|p\| = \langle p, 1 \rangle \).

**Theorem 2.1.** Let \( \mathcal{G} \) be a locally compact group and let \( T \) be a weakly compact left multiplier on \( L^1(\mathcal{G})^{**} \). Then \( |T| \) is a left multiplier on \( L^1(\mathcal{G})^{**} \) if and only if there exists \( \varrho \in L^1(\mathcal{G}) \) such that \( |T| = \varrho \Gamma \) on \( L^1(\mathcal{G})^{**} \).

**Proof.** Let \( |T| \) be a nonzero left multiplier on \( L^1(\mathcal{G})^{**} \). Since \( T \) is weakly compact, \( |T| \) is a nonzero weakly compact left multiplier on \( L^1(\mathcal{G})^{**} \); see [3]. It follows that \( \mathcal{G} \) is compact. So \( L^1(\mathcal{G}) \) is an ideal in \( L^1(\mathcal{G})^{**} \). Hence for every \( \omega_1, \omega_2 \in L^1(\mathcal{G}) \),

\[ |T|(\omega_1 \cdot \omega_2) = |T|(\omega_1) \cdot |T|(\omega_2) \in L^1(\mathcal{G}). \]

This shows that \( |T| \) restricted to \( L^1(\mathcal{G}) \) is a weakly compact left multiplier on \( L^1(\mathcal{G}) \). By [2], there exists \( \varrho \in L^1(\mathcal{G}) \) such that \( |T| = \varrho \Gamma \) on \( L^1(\mathcal{G}) \). Define the function \( S : L^1(\mathcal{G})^{**} \to L^1(\mathcal{G})^{**} \) by

\[ S(m) = |T|(m) - \varrho \Gamma(m) \]

for all \( m \in L^1(\mathcal{G})^{**} \). Note that if \( \iota \in C(\mathcal{G}) \), then \( \iota = \omega j \) for some \( \omega \in L^1(\mathcal{G}) \) and \( j \in C(\mathcal{G}) \). Since \( m \cdot \omega \in L^1(\mathcal{G}) \), we have

\[ \langle |T|(m) - \varrho \Gamma(m), \omega j \rangle = \langle |T|(m \cdot \omega) - \varrho \Gamma(m \cdot \omega), j \rangle = 0. \]

Hence \( \langle S(m), \iota \rangle = 0 \). This implies that

\[ \langle |T|(m), \iota \rangle = \langle \varrho \Gamma(m), \iota \rangle \]

for all \( \iota \in C(\mathcal{G}) \). We show that \( S = 0 \) on \( L^1(\mathcal{G})^{**} \). Suppose that \( S(m) \neq 0 \) for some \( m \in L^1(\mathcal{G})^{**} \). Without loss of generality, we may assume that \( m \) is hermitian. Let \( m = m_+ - m_- \) be the Jordan decomposition of \( m \). Hence there exists a positive functional \( p \) on \( L^1(\mathcal{G})^* \) such that \( S(p) \neq 0 \). Choose a continuous function \( \psi \) with compact support \( K \) such that \( \|\psi\|_\infty \leq 1 \) and

\[ |\langle \varrho \Gamma(p), \psi \rangle| \geq \|\varrho \Gamma(p)\|_1 - (5/12)\|S(p)\|. \]
Let $V$ be an open set with compact closure for which $K \subseteq V$. Let $\theta$ be a continuous function on $\mathcal{G}$ such that

$$0 \leq \theta(\mathcal{G}) \leq 1, \quad \theta(K) = 1, \quad \text{and} \quad \theta(\mathcal{G} \setminus V) = 0.$$ 

There exists an element $\phi$ in the unit ball of $L^1(\mathcal{G})^*$ such that

$$|\langle S(p), \phi \rangle| \geq \frac{33}{24} \|S(p)\|.$$ 

Define the function $\kappa : \mathcal{G} \to \mathbb{C}$ by

$$\kappa(x) := \phi(x) - \theta(x)\psi(x) + \psi(x)$$

for all $x \in \mathcal{G}$. Then $\kappa$ is an element in the unit ball of $L^1(\mathcal{G})^*$ and $\langle \phi \Gamma(p), \kappa \rangle = \langle \phi \Gamma(p), \psi \rangle$.

Choose a constant $\eta \in \mathbb{C}$ such that $\|\eta(\phi - \theta \psi) + \psi\|_\infty \leq 1$ and

$$|\langle \eta \Gamma(p), \eta(\phi - \theta \psi) + \psi \rangle| = |\langle \eta \Gamma(p), \phi - \theta \psi \rangle| + |\langle \eta \Gamma(p), \psi \rangle|.$$ 

Then

$$\|\eta \Gamma(p)\|_1 \geq |\langle \eta \Gamma(p), \phi - \theta \psi \rangle| + |\langle \eta \Gamma(p), \psi \rangle|$$

$$\geq |\langle \eta \Gamma(p), \phi - \theta \psi \rangle| + \|\eta \Gamma(p)\|_1 - \frac{5}{12} \|S(p)\|.$$ 

This implies that

$$|\langle \eta \Gamma(p), \phi - \theta \psi \rangle| \leq \frac{5}{12} \|S(p)\|.$$ 

Hence

$$|\langle [T](p), \kappa \rangle| \geq |\langle S(p), \kappa \rangle| + |\langle \eta \Gamma(p), \psi \rangle| - |\langle \eta \Gamma(p), \phi - \theta \psi \rangle|$$

$$\geq \frac{1}{8} \|S(p)\| + \|\eta \Gamma(p)\|_1.$$ 

It follows that

$$\|\|T\|(p)\| \geq \frac{1}{8} \|S(p)\| + \|\eta \Gamma(p)\|_1.$$ 

On the other hand,

$$\|\|T\|(p)\| = \langle [T](p), 1 \rangle = \langle \eta \Gamma(p), 1 \rangle \leq \|\eta \Gamma(p)\|_1.$$ 

Hence that $\|S(p)\| = 0$ and so $S(p) = 0$. This contradiction shows

$$\|T\|(p) = \|\eta \Gamma(p)\|_1.$$ 

for all $m \in L^1(\mathcal{G})^{**}$. The converse is clear. \hfill \Box

As a corollary of this theorem we have the following result.

**Corollary 2.2.** Let $\mathcal{G}$ be a locally compact group and let $m$ be a left weakly completely continuous element of $L^1(\mathcal{G})^{**}$. If $[m \Gamma]$ is a left multiplier on $L^1(\mathcal{G})^{**}$, then $[m] \in L^1(\mathcal{G})^{**}$.\hfill
Therefore, there exists \( T \) such that \( |m| \Gamma = \theta \Gamma \) on \( L^1(\mathcal{G})^* \); see [7, Lemma 3.6]. Invoke Theorem 2.1 to conclude that there exists \( \vartheta \in L^1(\mathcal{G}) \) such that \( |m| \Gamma = \vartheta \Gamma \) on \( L^1(\mathcal{G})^* \). Thus
\[
|m| \Gamma(E) = \vartheta \Gamma(E).
\]
Thus \( |m| = \theta \in L^1(\mathcal{G}) \), and the proof is complete.

Another consequence of Theorem 2.1 is the following result due to Ghahramani and Lau [7].

**Corollary 2.3.** Let \( \mathcal{G} \) be a locally compact group. If \( \theta \in L^1(\mathcal{G}) \) is a weakly completely continuous element of \( L^1(\mathcal{G})^* \), then \( |\theta| \) is also a weakly completely continuous element of \( L^1(\mathcal{G})^* \). Furthermore, \( |\theta| \Gamma = |\theta| \Gamma \) on \( L^1(\mathcal{G})^* \).

**Proof.** Let \( \vartheta \in L^1(\mathcal{G}) \) be a weakly completely continuous element of \( L^1(\mathcal{G})^* \). Let \( E \) be a positive right identity of \( L^1(\mathcal{G})^* \). Then
\[
|\vartheta| \Gamma(E) = \sup \{|\vartheta| \cdot m : m \in L^1(\mathcal{G})^* \}, \ |m| \leq E
\]
\[
\leq \sup \{|\vartheta| \cdot |m| : m \in L^1(\mathcal{G})^* \}, \ |m| \leq E
\]
\[
= \sup \{|\vartheta \cdot m : m \in L^1(\mathcal{G})^* \}, \ |m| \leq E
\]
\[
|\vartheta| \Gamma(E) = \omega, \ 
\]
Since \( |E| = E \leq E \), we have \( |\vartheta| \Gamma(E) = |\theta| \). By Theorem 2.1, there exists \( \omega \in L^1(\mathcal{G}) \) such that \( |\theta| \Gamma = |\omega| \Gamma \). Thus
\[
|\vartheta| \Gamma(E) = |\omega| \Gamma(E) = \omega.
\]
So \( \omega = |\theta| \), and therefore \( |\theta| \Gamma = |\theta| \Gamma \) on \( L^1(\mathcal{G})^* \). This means that \( |\theta| \) is a weakly completely continuous element of \( L^1(\mathcal{G})^* \).

We give a corollary of this result.

**Corollary 2.4.** Let \( \mathcal{G} \) be a locally compact group and let \( T \) be a weakly compact left multiplier on \( L^1(\mathcal{G})^* \). If the range of \( T \) is contained in \( L^1(\mathcal{G}) \), then \( |T| \) is a left multiplier.

**Proof.** Let \( T \) be a weakly compact left multiplier on \( L^1(\mathcal{G})^* \) with \( T(L^1(\mathcal{G})^*) \subseteq L^1(\mathcal{G}) \). Let \( E \) be a right identity of \( L^1(\mathcal{G})^* \). If \( m \in L^1(\mathcal{G})^* \), then
\[
\langle T(m - E \cdot m), \omega \rangle = \langle T(m - E \cdot m) \cdot \omega, \eta \rangle
\]
\[
= \langle T(m \cdot \omega - E \cdot (m \cdot \omega)), \eta \rangle
\]
\[
= 0
\]
for all \( \omega \in L^1(\mathcal{G}) \) and \( \eta \in C(\mathcal{G}) \). Hence \( \langle T(m - E \cdot m), \eta \rangle = 0 \) for all \( \eta \in C(\mathcal{G}) \). Since \( T(m - E \cdot m) \in L^1(\mathcal{G}) \), it follows that \( T(m - E \cdot m) = 0 \). Hence, for any \( m \in L^1(\mathcal{G})^* \),
\[
T(m) = T(E \cdot m) = T(E) \cdot m.
\]
Therefore, there exists \( \vartheta \in L^1(\mathcal{G}) \) such that \( T = |\vartheta| \Gamma \) on \( L^1(\mathcal{G})^* \). In view of Corollary 2.3, we have \( |T| = |\vartheta| \Gamma \) on \( L^1(\mathcal{G})^* \), proving the corollary.
By Corollary 2.3, if \( \vartheta \in L^1(\mathcal{G}) \) is a nonzero weakly completely continuous element of \( L^1(\mathcal{G})^{**} \), then \( |\vartheta \Gamma| \) is a nonzero weakly compact multiplier on \( L^1(\mathcal{G})^{**} \). In the next result, we show that this is not true for all elements \( m \in L^1(\mathcal{G})^{**} \) instead of \( \vartheta \in L^1(\mathcal{G}) \).

In the following, let

\[
\text{RUC}(\mathcal{G})^\perp = \{ m \in L^1(\mathcal{G})^{**} : \langle m, \phi \rangle = 0 \text{ for all } \phi \in \text{RUC}(\mathcal{G}) \}.
\]

**Theorem 2.5.** Let \( \mathcal{G} \) be a locally compact group and let \( \ell \) be any element of \( \text{RUC}(\mathcal{G})^\perp \). Then \( |\ell \Gamma| \) is a left multiplier on \( L^1(\mathcal{G})^{**} \) if and only if \( \ell = 0 \).

**Proof.** Suppose that \( \ell \) is a nonzero element of \( \text{RUC}(\mathcal{G})^\perp \). Choose \( \phi \in L^1(\mathcal{G})^* \) with \( \langle \ell, \phi \rangle \neq 0 \). Then, for every positive function \( \rho \) in \( L^1(\mathcal{G}) \),

\[
\langle |\phi|, \rho \rangle = \sup\{ |\langle \phi, \vartheta \rangle| : \vartheta \in L^1(\mathcal{G}), |\vartheta| \leq \rho \}
= \sup\{ \int_G \phi(x)\vartheta(x) \, d\lambda(x) : \vartheta \in L^1(\mathcal{G}), |\vartheta| \leq \rho \}
\leq \sup\{ \int_G |\phi(x)|\rho(x) \, d\lambda(x) : \vartheta \in L^1(\mathcal{G}), |\vartheta| \leq \rho \}
= \|\phi\|_\infty \int_G \rho(x) \, d\lambda(x) = \|\phi\|_\infty \langle 1, \rho \rangle.
\]

Hence \( |\phi| \leq \|\phi\|_\infty 1 \) and so

\[
\rho(\mathcal{G})\langle \ell, \phi \rangle \leq \rho(\mathcal{G})\sup\{ |\langle \ell, \psi \rangle| : \psi \in L^1(\mathcal{G})^*, |\psi| \leq \|\phi\|_\infty 1 \}
\leq \rho(\mathcal{G})\sup\{ |\langle \ell, |\psi| \rangle| : \psi \in L^1(\mathcal{G})^*, |\psi| \leq \|\phi\|_\infty 1 \}
\leq \rho(\mathcal{G})\sup\{ |\langle \ell, |\phi|_1 \rangle| : \psi \in L^1(\mathcal{G})^*, |\psi| \leq \|\phi\|_\infty 1 \}
= \rho(\mathcal{G})|\phi|_\infty \langle |\ell|, 1 \rangle = |\phi|_\infty \langle |\ell|, \rho \rangle.
\]

This shows that \( |\ell| 1 \neq 0 \). Fix \( \omega \in L^1(\mathcal{G}) \) with \( \langle |\ell|1, \omega \rangle \neq 0 \). Since

\[
\langle |\ell|1, \omega \rangle = \langle |\ell|, 1\omega \rangle = \langle |\ell|, \omega, 1 \rangle,
\]

it follows that \( |\vartheta| \Gamma(\omega) \neq 0 \). On the other hand we have, for any \( \vartheta \in L^1(\mathcal{G}) \) and \( \phi \in L^1(\mathcal{G})^* \), \( \vartheta \phi \in \text{RUC}(\mathcal{G}) \). Thus

\[
\langle \ell \cdot \vartheta, \phi \rangle = \langle \ell, \vartheta \phi \rangle = 0,
\]

and hence \( \ell \cdot \vartheta = 0 \), which implies that \( \ell \cdot m = 0 \) for all \( m \in L^1(\mathcal{G})^{**} \) with \( |m| \leq \omega \); see [3, p. 234]. Therefore,

\[
|\ell \Gamma|(\omega) = \sup\{ |\ell \Gamma(m)| : m \in L^1(\mathcal{G})^{**}, |m| \leq \omega \} = 0.
\]

Now if \( |\vartheta| \) is a left multiplier, then from [7] we would have \( |\vartheta| \Gamma(\omega) = |\vartheta| \Gamma(\omega) \), a contradiction. Therefore, \( \ell = 0 \). The converse is clear. \( \Box \)

As suggested by a referee, it would be worth working on the following open problem: give an example of a weakly compact left multiplier \( T \) on \( L^1(\mathcal{G})^{**} \) such that \( |T| \) is not a left multiplier.
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References


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