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NONLINEAR OSCILLATION OF FOURTH ORDER DIFFERENTIAL EQUATIONS

TAKAŜI KUSANO AND MANABU NAITO

1. Introduction. In this paper we are concerned with the fourth order nonlinear differential equation

$$(A)\quad [r(t)y'']'' + yF(y^2, t) = 0,$$

where the following conditions are always assumed to hold:

(a) $r(t)$ is continuous and positive for $t \geqq 0$, and

$$\int_0^\infty \frac{t}{r(t)}\,dt = \infty;$$

(b) $yF(y^2, t)$ is continuous for $|y| < \infty$, $t \geqq 0$, and $F(z, t)$ is positive for $z > 0$, $t \geqq 0$.

Following Nehari [**11**] and Coffman and Wong [**3**], we classify equations of the form (A) according to the nonlinearity of $yF(y^2, t)$ with respect to y. Equation (A) is called *superlinear* or *sublinear* according as $F(z, t)$ is, respectively, nondecreasing or nonincreasing in z, i.e.,

$$F(z_1, t) \leqq F(z_2, t), \quad 0 < z_1 < z_2, \quad t \geqq 0,$$

or

$$F(z_1, t) \geqq F(z_2, t), \quad 0 < z_1 < z_2, \quad t \geqq 0.$$

Furthermore, (A) is called *strongly superlinear* if, for some $\epsilon > 0$, $z^{-\epsilon}F(z, t)$ is nondecreasing in z, i.e.,

$$z_1^{-\epsilon}F(z_1, t) \leqq z_2^{-\epsilon}F(z_2, t), \quad 0 < z_1 < z_2, \quad t \geqq 0,$$

and *strongly sublinear* if, for some $\epsilon > 0$, $z^{\epsilon}F(z, t)$ is nonincreasing in z, i.e.,

$$z_1^{\epsilon}F(z_1, t) \geqq z_2^{\epsilon}F(z_2, t), \quad 0 < z_1 < z_2, \quad t \geqq 0.$$

We confine our discussion to those solutions $y(t)$ of (A) which exist on some ray $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)| : t_0 \leqq t < \infty\} > 0$$

for every $t_0 \in [T_y, \infty)$. Such a solution $y(t)$ is said to be *oscillatory* if it has arbitrarily large zeros. If this condition does not hold, that is, if $y(t)$ is eventually positive or negative, then $y(t)$ is said to be *nonoscillatory*. Equation (A) itself is called *oscillatory* if all of its solutions are oscillatory.

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In the oscillation theory of nonlinear differential equations one of the important problems is to find necessary and sufficient conditions for the equations under consideration to be oscillatory. Beginning with the pioneering work of Atkinson [**1**] there have been a number of papers devoted to the investigation of this problem; see, e.g., Belohorec [**2**], Coffman and Wong [**3**], Kiguradze [**4**], Kusano and Naito [**5**], Ličko and Švec [**8**], Onose [**12**], and Ryder and Wend [**13**]. The purpose of this paper is to proceed further in this direction to present some new oscillation criteria for Equation (A). We aim at determining the effect which $r(t)$ exercises upon the oscillatory character of (A) in conjunction with its nonlinearity. In Section 2 we give necessary and sufficient conditions for (A) which is either superlinear or sublinear to have non-oscillatory solutions with special asymptotic properties. In Section 3 we provide necessary and sufficient conditions for (A) which is strongly superlinear or strongly sublinear to be oscillatory. The present paper has points of contact with the earlier work by Leighton and Nehari [**6**], Lovelady [**9**], Terry and Wong [**14**], and Wong [**15**].

2. Nonoscillation theorems. We begin with a lemma which gives information on the behavior of possible nonoscillatory solutions of (A).

LEMMA 1. *Suppose* (a) *and* (b) *hold. If* $y(t)$ *is an eventually positive solution of* (A), *then one of the following cases holds:*

(I) $y'(t) > 0$, $y''(t) > 0$ *and* $[r(t)y''(t)]' > 0$ *for all large* t;

(II) $y'(t) > 0$, $y''(t) < 0$ *and* $[r(t)y''(t)]' > 0$ *for all large* t.

In either case there are positive numbers T *and* a *such that*

$$(1) \qquad y(t) \leqq a \int_0^t \frac{(t-s)s}{r(s)}\, ds \quad \text{for } t \geqq T.$$

Proof. Let $y(t) > 0$ for $t \geqq t_0$. From (A), $[r(t)y''(t)]'' < 0$, $t \geqq t_0$, which implies that $[r(t)y''(t)]'$ is decreasing for $t \geqq t_0$. Therefore, $[r(t)y''(t)]'$ is eventually of constant sign. Suppose $[r(t)y''(t)]' < 0$ at some $t_1 > t_0$. Then, $[r(t)y''(t)]' \leqq -c_1$ for $t \geqq t_1$, where $c_1 = -[r(t)y''(t)]'_{t=t_1} > 0$. Upon integrating this inequality we see that there are numbers $t_2 > t_1$ and $c_2 > 0$ such that $r(t)y''(t) \leqq -c_2 t$ for $t \geqq t_2$. We divide the last inequality by $r(t)$, integrate it from t_2 to t, and then let $t \to \infty$. Using (a) we have $\lim_{t\to\infty} y'(t) = -\infty$, which yields $\lim_{t\to\infty} y(t) = -\infty$. But this contradicts the positivity of $y(t)$. Hence we must have $[r(t)y''(t)]' > 0$ for $t \geqq t_0$. It follows that $r(t)y''(t)$ is increasing, so that it is eventually of one sign. If $r(t)y''(t) < 0$ for $t \geqq t_0$, then $y'(t)$ is eventually positive. In fact, if $y'(t)$ is negative at some point, then using the decreasing property of $y'(t)$, we obtain $\lim_{t\to\infty} y(t) = -\infty$, a contradiction. If there exists $t_3 > t_0$ such that $r(t)y''(t) > 0$ for $t \geqq t_3$, then $r(t)y''(t) \geqq c_3$ for $t \geqq t_3$, where $c_3 = r(t_3)y''(t_3)$. Multiplying this inequality by $t/r(t)$ and integrating from t_3 to t, we obtain

$$(2) \qquad ty'(t) - y(t) - t_3 y'(t_3) + y(t_3) \geqq c_3 \int_{t_3}^t \frac{s}{r(s)}\, ds, \quad t \geqq t_3.$$

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From (2) and in view of (a) we see that $\lim_{t\to\infty} ty'(t) = \infty$, which implies that $y'(t) > 0$ for all large t.

If we integrate the inequality $[r(t)y''(t)]'' < 0$ four times from t_0 to t, we have

$$(3) \qquad y(t) \leqq a_0 + a_1 t + a_2 \int_{t_0}^{t} \frac{t-s}{r(s)} ds + a_3 \int_{t_0}^{t} \frac{(t-s)s}{r(s)} ds,$$

where $a_0, \ldots, a_3$ are positive constants. The inequality (1) follows immediately from (3). This completes the proof.

Evidently, similar inequalities hold for an eventually negative solution of (A). In what follows we use the notation

$$R(t) = \int_0^t \frac{(t-s)s}{r(s)} ds.$$

According to Lemma 1, if $y(t)$ is a nonoscillatory solution of (A), then it is eventually monotonic and there are positive constants a_1 and a_2 such that $a_1 \leqq |y(t)| \leqq a_2 R(t)$ for all large t. Therefore, among all nonoscillatory solutions of (A), those which are asymptotic to functions of the form $aR(t)$, $a \neq 0$, as $t \to \infty$ may be regarded as the "maximal" solutions, and those which are bounded and asymptotic to nonzero constants as $t \to \infty$ may be regarded as the "minimal" solutions. In case (A) is superlinear or sublinear necessary and sufficient conditions for the existence of these special types of nonoscillatory solutions can be established without difficulty.

THEOREM 1. *Let* (A) *be either superlinear or sublinear. A necessary and sufficient condition for* (A) *to have a nonoscillatory solution which is asymptotic to* $aR(t)$, $a \neq 0$, *as* $t \to \infty$ *is that*

$$(4) \qquad \int^{\infty} R(t)F(c^2 R(t)^2, t)dt < \infty \quad \textit{for some } c > 0.$$

Proof. (Necessity) Let $y(t)$ be a nonoscillatory solution of (A) such that $\lim_{t\to\infty} y(t)/R(t) = a \neq 0$. We may suppose that $a > 0$. There are positive numbers t_1, a_1, a_2 such that

$$(5) \qquad a_1 R(t) \leqq y(t) \leqq a_2 R(t) \quad \text{for } t \geqq t_1.$$

Integrating (A) from t_1 to t, we obtain

$$(6) \qquad [r(t)y''(t)]' - [r(t)y''(t)]'_{t=t_1} + \int_{t_1}^{t} y(s)F(y(s)^2, s)ds = 0.$$

Since $[r(t)y''(t)]' > 0$ by Lemma 1, we see from (6) that

$$(7) \qquad \int_{t_1}^{\infty} y(t)F(y(t)^2, t)dt < \infty.$$

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From (5) and (7) we conclude that

$$a_1 \int_{t_1}^{\infty} R(t) F(a_1^2 R(t)^2, t) dt < \infty \quad \text{if } (A) \text{ is superlinear,}$$

$$a_1 \int_{t_1}^{\infty} R(t) F(a_2^2 R(t)^2, t) dt < \infty \quad \text{if } (A) \text{ is sublinear.}$$

(Sufficiency) Assume (4) holds. Put $a = c/2$ if (A) is superlinear and $a = c$ if (A) is sublinear. Take $T > 0$ so large that

$$(8) \qquad \int_T^{\infty} R(t) F(c^2 R(t)^2, t) dt < \frac{1}{4}$$

and consider the integral equation

$$(9) \qquad y(t) = (\Phi y)(t),$$

where

$$(10) \quad (\Phi y)(t) = aR(t) + R(t) \int_t^{\infty} y(s) F(y(s)^2, s) ds$$
$$+ \int_T^t R(s) y(s) F(y(s)^2, s) ds$$
$$+ \int_T^t \left(\int_0^s \frac{\sigma}{r(\sigma)} d\sigma \right) (t - s) y(s) F(y(s)^2, s) ds$$
$$+ \int_T^t \left(\int_s^t \frac{t - \sigma}{r(\sigma)} d\sigma \right) s y(s) F(y(s)^2, s) ds.$$

As easily verified by differentiation a solution of (9) is a solution of Equation (A). To solve (9) with the help of Schauder's fixed point theorem we introduce the linear space $C_R[T, \infty)$ of all continuous functions $y : [T, \infty) \to R$ such that

$$||y||_R = \sup\{R(t)^{-2} |y(t)| : t \geqq T\} < \infty.$$

Obviously $C_R[T, \infty)$ is a Banach space with norm $|| \cdot ||_R$. We seek a fixed point of the operator Φ in the set

$$Y = \{y \in C_R[T, \infty) : aR(t) \leqq y(t) \leqq 2aR(t) \quad \text{for } t \geqq T\},$$

which is a bounded, convex and closed subset of $C_R[T, \infty)$. We shall show that Φ is continuous and maps Y into a compact subset of Y.

i) *Φ maps Y into Y.* If $y \in Y$, then by (10) $(\Phi y)(t) \geqq aR(t)$, $t \geqq T$, and in view of (8)

$$(\Phi y)(t) \leqq aR(t) + R(t) \int_t^{\infty} y(s) F(y(s)^2, s) ds$$
$$+ 2 \int_T^t R(s) y(s) F(y(s)^2, s) ds$$
$$\leqq aR(t) + 2R(t) \int_T^{\infty} y(s) F(y(s)^2, s) ds$$
$$\leqq aR(t) + 4aR(t) \int_T^{\infty} R(s) F(c^2 R(s)^2, s) ds \leqq 2aR(t), \qquad t \geqq T.$$

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ii) Φ *is continuous*. Let $\{y_n\}$ be a sequence of elements of Y such that $\lim_{n\to\infty} ||y_n - y||_R = 0$. Since Y is closed, $y \in Y$ and

$$|(\Phi y_n)(t) - (\Phi y)(t)| \leqq 2R(t) \int_T^\infty F_n(s)ds, \quad t \geqq T,$$

where

$$F_n(s) = |y_n(s)F(y_n(s)^2, s) - y(s)F(y(s)^2, s)|.$$

It follows that

$$(11) \quad ||\Phi y_n - \Phi y||_R \leqq 2R(T)^{-1} \int_T^\infty F_n(s)ds.$$

Since $\lim_{n\to\infty} F_n(s) = 0$ and $F_n(s) \leqq 4aR(s)F(c^2R(s)^2, s)$ for $s \geqq T$, applying the Lebesgue dominated convergence theorem, we conclude from (11) that $\lim_{n\to\infty} ||\Phi y_n - \Phi y||_R = 0$. This proves the continuity of Φ.

iii) $\overline{\Phi Y}$ *is compact*. It suffices to show that the family of functions $\{R^{-2}\Phi y : y \in Y\}$ is uniformly bounded and equicontinuous on $[T, \infty)$. Since the uniform boundedness is evident, we need only to demonstrate the equicontinuity. This will be accomplished if we show that, for any given $\epsilon > 0$, the interval $[T, \infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have oscillations less than ϵ; see Levitan [7, § 3].

If $y \in Y$, then we have for $t_2 > t_1 \geqq T$

$$(12) \quad \begin{aligned} |(R^{-2}\Phi y)(t_2) - (R^{-2}\Phi y)(t_1)| &\leqq 2aR(t_1)^{-1} \\ &\quad + 4R(t_1)^{-1} \int_T^\infty y(s)F(y(s)^2, s)ds \\ &\leqq 2aR(t_1)^{-1} \\ &\quad + 8aR(t_1)^{-1} \int_T^\infty R(s)F(c^2R(s)^2, s)ds \\ &\leqq 4aR(t_1)^{-1}. \end{aligned}$$

Since $R(t_1)^{-1} \to 0$ as $t_1 \to \infty$, it follows from (12) that, for a given $\epsilon > 0$, there exists $T^* > T$ such that

$$(13) \quad |(R^{-2}\Phi y)(t_2) - (R^{-2}\Phi y)(t_1)| < \epsilon \quad \text{if } t_2 > t_1 \geqq T^*.$$

Let $y \in Y$ and $T \leqq t_1 < t_2 \leqq T^*$. A simple computation yields the following

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inequality:

$$
\begin{aligned}
&|(R^{-2}\Phi y)(t_2) - (R^{-2}\Phi y)(t_1)| \leqq a|R(t_2)^{-1} - R(t_1)^{-1}| \\
&\quad + |R(t_2)^{-1} - R(t_1)^{-1}| \int_{t_1}^{\infty} y(s)F(y(s)^2, s)ds \\
&\quad + R(t_2)^{-1} \int_{t_1}^{t_2} y(s)F(y(s)^2, s)ds \\
&\quad + |R(t_2)^{-2} - R(t_1)^{-2}| \int_{T}^{t_2} R(s)y(s)F(y(s)^2, s)ds \\
&\quad + R(t_1)^{-2} \int_{t_1}^{t_2} R(s)y(s)F(y(s)^2, s)ds \\
&\quad + |t_2R(t_2)^{-2} - t_1R(t_1)^{-2}| \int_{T}^{t_2} \left(\int_0^s \frac{\sigma}{r(\sigma)} d\sigma \right) y(s)F(y(s)^2, s)ds \\
&\quad + t_1R(t_1)^{-2} \int_{t_1}^{t_2} \left(\int_0^s \frac{\sigma}{r(\sigma)} d\sigma \right) y(s)F(y(s)^2, s)ds \\
&\quad + |R(t_2)^{-2} - R(t_1)^{-2}| \int_{T}^{t_2} \left(\int_0^s \frac{\sigma}{r(\sigma)} d\sigma \right) sy(s)F(y(s)^2, s)ds \\
&\quad + R(t_1)^{-2} \int_{t_1}^{t_2} \left(\int_0^s \frac{\sigma}{r(\sigma)} d\sigma \right) sy(s)F(y(s)^2, s)ds \\
&\quad + |t_2R(t_2)^{-2} - t_1R(t_1)^{-2}| \int_{T}^{t_2} \left(\int_s^{t_2} \frac{d\sigma}{r(\sigma)} \right) sy(s)F(y(s)^2, s)ds \\
&\quad + t_1R(t_1)^{-2} \int_{t_1}^{t_2} \frac{d\sigma}{r(\sigma)} \cdot \int_{T}^{t_2} sy(s)F(y(s)^2, s)ds \\
&\quad + t_1R(t_1)^{-2} \int_{t_1}^{t_2} \left(\int_{t_1}^s \frac{d\sigma}{r(\sigma)} \right) sy(s)F(y(s)^2, s)ds \\
&\quad + |R(t_2)^{-2} - R(t_1)^{-2}| \int_{T}^{t_2} \left(\int_s^{t_2} \frac{\sigma}{r(\sigma)} d\sigma \right) sy(s)F(y(s)^2, s)ds \\
&\quad + R(t_1)^{-2} \int_{t_1}^{t_2} \frac{\sigma}{r(\sigma)} d\sigma \cdot \int_{T}^{t_2} sy(s)F(y(s)^2, s)ds \\
&\quad + R(t_1)^{-2} \int_{t_1}^{t_2} \left(\int_{t_1}^s \frac{\sigma}{r(\sigma)} d\sigma \right) sy(s)F(y(s)^2, s)ds.
\end{aligned}
$$

Using the inequality $y(s)F(y(s)^2, s) \leqq 2aR(s)F(c^2R(s)^2, s)$, $s \geqq T$, in the above inequality, we conclude that there exists a $\delta > 0$ such that for all $y \in Y$

(14) $\quad |(R^{-2}\Phi y)(t_2) - (R^{-2}\Phi y)(t_1)| < \epsilon \quad \text{if } t_2 - t_1 < \delta.$

The inequalities (13) and (14) enable us to divide the interval $[T, \infty)$ into a finite number of subintervals on each of which the oscillation of every $R^{-2}\Phi y$, $y \in Y$, is less than ϵ. It follows that $\overline{\Phi Y}$ is compact.

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From the above considerations we see that the Schauder fixed point theorem can be applied to the operator Φ. Let $y \in Y$ be a fixed point of Φ. Then, $y = y(t)$ is clearly a solution of the integral equation (9) on $[T, \infty)$. Since, by l'Hospital's rule,

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = \lim_{t\to\infty} \frac{y'(t)}{R'(t)} = \lim_{t\to\infty} \frac{r(t)y''(t)}{r(t)R''(t)}$$

$$= \lim_{t\to\infty} \frac{r(t)y''(t)}{t} = \lim_{t\to\infty} [r(t)y''(t)]' = a,$$

$y(t)$ is a solution of (A) with the required asymptotic property. This completes the proof of Theorem 1.

THEOREM 2. *Let (A) be either superlinear or sublinear. A necessary and sufficient condition for (A) to have a bounded nonoscillatory solution is that*

$$(15) \qquad \int^{\infty} R(t)F(c^2, t)dt < \infty \quad \textit{for some } c > 0.$$

Proof. (Necessity) Let $y(t)$ be a bounded nonoscillatory solution of (A). Without loss of generality we may suppose that $y(t) > 0$ eventually. Observe that Case (II) of Lemma 1 holds. There are positive numbers t_1, a_1, a_2 such that

$$(16) \qquad a_1 \leqq y(t) \leqq a_2 \quad \text{for } t \geqq t_1.$$

Multiplying (A) by $R(t)$ and integrating from t_1 to t, we obtain

$$(17) \qquad \begin{aligned} \int_{t_1}^{t} R(s)y(s)F(y(s)^2, s)ds &= -\int_{t_1}^{t} R(s)[r(s)y''(s)]''ds \\ &= -R(t)[r(t)y''(t)]' + R'(t)r(t)y''(t) - ty'(t) + y(t) + c, \end{aligned}$$

where c is a constant. Using the inequalities of Case (II) of Lemma 1 and the boundedness of $y(t)$, we see from (17) that

$$(18) \qquad \int_{t_1}^{\infty} R(t)y(t)F(y(t)^2, t)dt < \infty.$$

In view of (16) and (18) we have

$$a_1 \int_{t_1}^{\infty} R(t)F(a_1^2, t)dt < \infty \quad \text{if } (A) \text{ is superlinear,}$$

$$a_1 \int_{t_1}^{\infty} R(t)F(a_2^2, t)dt < \infty \quad \text{if } (A) \text{ is sublinear.}$$

(Sufficiency) Let $a = c/2$ if (A) is superlinear and $a = c$ if (A) is sublinear. Choose $T > 0$ so large that

$$\int_{T}^{\infty} R(t)F(c^2, t)dt < \frac{1}{4}.$$

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The required solution of (A) is obtained as a solution of the integral equation

(19) $y(t) = (\Psi y)(t)$,

where

$$(\Psi y)(t) = a + R(t)\int_t^\infty y(s)F(y(s)^2, s)ds + \int_T^t R(s)y(s)F(y(s)^2, s)ds$$
$$+ t\int_t^\infty \left(\int_t^s \frac{s-\sigma}{r(\sigma)}\,d\sigma\right)y(s)F(y(s)^2, s)ds$$
$$+ \int_0^t \frac{\sigma}{r(\sigma)}\,d\sigma \,.\, \int_t^\infty (s-t)y(s)F(y(s)^2, s)ds.$$

A solution of (19) in turn is determined as a fixed point of the operator Ψ. The underlying Banach space is $C[T, \infty)$ of all bounded and continuous functions $y : [T, \infty) \to R$ with norm $||y|| = \sup\{|y(t)| : t \geqq T\}$ and the set Y on which Ψ acts is $Y = \{y \in C[T, \infty) : a \leqq y(t) \leqq 2a \text{ for } t \geqq T\}$, which is a bounded, convex and closed subset of $C[T, \infty)$.

As in the proof of the sufficiency part of Theorem 1 it can be shown that Ψ is a continuous operator which maps Y into a compact subset of Y. Therefore, by the Schauder fixed point theorem, Ψ has a fixed point $y \in Y$, which provides a solution $y = y(t)$ of (A). Since

$$y'(t) = \int_t^\infty \left(\int_t^s \frac{s-\sigma}{r(\sigma)}\,d\sigma\right)y(s)F(y(s)^2, s)ds > 0,$$

this solution tends to a limit in $[a, 2a]$ as $t \to \infty$. This sketches the proof of the sufficiency part of Theorem 2. The details are left to the reader.

We now consider the differential equation

(B) $\quad [r(t)y'']'' + p(t)f(y) = 0.$

The conditions we always assume for r, p, f are (a) and the following:

(c) $p(t)$ is continuous and positive for $t \geqq 0$;

(d) $f(y)$ is continuous and nondecreasing for $y \in R$, and $yf(y) > 0$ for $y \neq 0$.

It is easily seen that the arguments and techniques developed in the proof of Theorems 1 and 2 apply equally well to equation (B). Thus we have the following

THEOREM 3. (i) *A necessary and sufficient condition for (B) to have a nonoscillatory solution which is asymptotic to $aR(t)$, $a \neq 0$, as $t \to \infty$ is that*

$$\int^\infty f(cR(t))p(t)dt < \infty \quad \textit{for some } c > 0.$$

(ii) *A necessary and sufficient condition for (B) to have a bounded nonoscillatory solution is that*

$$\int^\infty R(t)p(t)dt < \infty.$$

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An important particular case of (A) and (B) is

$$(C) \qquad [r(t)y'']'' + p(t)|y|^\gamma \operatorname{sgn} y = 0, \quad \gamma > 0,$$

where $r(t)$ and $p(t)$ satisfy (a) and (c), respectively. When specialized to (C) the above theorems give the following corollary.

COROLLARY. (i) *Equation* (C) *has a solution which is asymptotic to* $aR(t)$, $a \neq 0$, *as* $t \to \infty$ *if and only if*

$$\int^\infty [R(t)]^\gamma p(t)dt < \infty.$$

(ii) *Equation* (C) *has a bounded nonoscillatory solution if and only if*

$$\int^\infty R(t)p(t)dt < \infty.$$

3. Oscillation theorems. In this section we establish necessary and sufficient conditions in order that Equation (A) which is either strongly superlinear or strongly sublinear be oscillatory.

LEMMA 2. *Suppose* (a) *and* (b) *hold. If* $y(t)$ *is an eventually positive solution of* (A), *then for all sufficiently large* T

$$(20) \quad y(t) \geqq R_T(t)[r(t)y''(t)]' + \int_T^t R_T(s)y(s)F(y(s)^2, s)ds, \quad t \geqq T,$$

where

$$R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)} ds.$$

Proof. Let T be so large that $y(t)$ satisfies Case (I) or Case (II) of Lemma 1 for $t \geqq T$. Suppose first that Case (I) holds. Since $y''(t) > 0$ and $[r(t)y''(t)]'$ is decreasing, we have

$$(21) \quad r(t)y''(t) \geqq \int_T^t [r(s)y''(s)]'ds \geqq (t-T)[r(t)y''(t)]', \quad t \geqq T.$$

Using (21) and the positivity of $y'(t)$, we find

$$y'(t) \geqq \int_T^t y''(s)ds \geqq \int_T^t \frac{s-T}{r(s)} [r(s)y''(s)]'ds, \quad t \geqq T,$$

with the help of which we obtain

$$(22) \quad y(t) \geqq \int_T^t y'(s)ds \geqq \int_T^t \left(\int_T^s \frac{\sigma - T}{r(\sigma)} [r(\sigma)y''(\sigma)]'d\sigma \right) ds$$
$$\geqq \int_T^t \left(\int_T^s \frac{\sigma - T}{r(\sigma)} d\sigma \right) [r(s)y''(s)]'ds, \quad t \geqq T.$$

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Integrating the last integral by parts, we obtain from (22)

$$(23)\quad y(t) \geqq R_T(t)[r(t)y''(t)]' - \int_T^t R_T(s)[r(s)y''(s)]''ds,$$

which implies (20).

Next suppose that Case (II) of Lemma 1 holds. We multiply (A) by $R_T(t)$ and integrate it over $[T, t]$. Repeated application of integration by parts then yields

$$(24)\quad \begin{aligned} &R_T(t)[r(t)y''(t)]' - R_T'(t)r(t)y''(t) + (t-T)y'(t) \\ &\qquad - y(t) + y(T) + \int_T^t R_T(s)y(s)F(y(s)^2, s)ds = 0. \end{aligned}$$

Since $y''(t) < 0$, $y'(t) > 0$ and $y(T) > 0$, (20) follows immediately from (24). Thus the proof is complete.

A characterization of the oscillation situation for the strongly sublinear equation (A) is given in the following theorem.

THEOREM 4. *Let (A) be strongly sublinear. A necessary and sufficient condition for (A) to be oscillatory is that*

$$(25)\quad \int^\infty R(t)F(c^2R(t)^2, t)dt = \infty \quad \textit{for all } c > 0.$$

Proof. The necessity part is an immediate consequence of Theorem 1. To prove the sufficiency part let $y(t)$ be a nonoscillatory solution of (A). We may suppose that $y(t) > 0$ for $t \geqq T$, since a similar argument holds if $y(t) < 0$ for $t \geqq T$. From Lemma 2 we have

$$(26)\quad y(t) \geqq R_T(t)[r(t)y''(t)]' \quad \text{for } t \geqq T.$$

According to Lemma 1 there are positive constants k and $T_1 > T$ such that $y(t) \leqq kR_T(t)$ for $t \geqq T_1$. From this inequality and the strong sublinearity we see that

$$(27)\quad [y(t)]^{2\epsilon}F(y(t)^2, \mathrm{t}) \geqq [kR_T(t)]^{2\epsilon}F(k^2R_T(t)^2, t).$$

Using (26), (27) and the fact that the value of ϵ in the definition of strong sublinearity can be chosen arbitrarily small, we obtain

$$(28)\quad \begin{aligned} &\{-\{[r(t)y''(t)]'\}^{2\epsilon}\}' \\ &= 2\epsilon\{[r(t)y''(t)]'\}^{2\epsilon-1}\cdot[y(t)]^{1-2\epsilon}\cdot[y(t)]^{2\epsilon}F(y(t)^2, t) \\ &\geqq 2\epsilon k^{2\epsilon}\{[r(t)y''(t)]'\}^{2\epsilon-1}\cdot\{R_T(t)[r(t)y''(t)]'\}^{1-2\epsilon} \\ &\qquad\qquad \cdot[R_T(t)]^{2\epsilon}F(k^2R_T(t)^2, t) \\ &= 2\epsilon k^{2\epsilon}R_T(t)F(k^2R_T(t)^2, t), \quad t \geqq T_1. \end{aligned}$$

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An integration of (28) yields

$$\int_{T_1}^{\infty} R_T(t)F(k^2R_T(t)^2, t)dt < \infty,$$

which contradicts (25). This finishes the proof.

We now turn to the strongly superlinear equation (A).

THEOREM 5. *Let (A) be strongly superlinear. A necessary and sufficient condition for (A) to be oscillatory is that*

$$\int^{\infty} R(t)F(c^2, t)dt = \infty \quad \textit{for all } c > 0. \tag{29}$$

Proof. That (29) is necessary follows readily from Theorem 2. To show that (29) is sufficient let there exist a nonoscillatory solution $y(t)$ of (A). Without loss of generality we may assume that $y(t) > 0$ for $t \geqq T$. By Lemma 2 we get

$$y(t) \geqq \int_T^t R_T(s)y(s)F(y(s)^2, s)ds, \quad t \geqq T. \tag{30}$$

There is a constant $k > 0$ such that $y(t) \geqq k$ for $t \geqq T$, since $y'(t) > 0$ by Lemma 1. Using the strong superlinearity, we have

$$\begin{aligned} y(s)F(y(s)^2, s) &= [y(s)]^{1+2\epsilon} \cdot [y(s)]^{-2\epsilon}F(y(s)^2, s) \\ &\geqq k^{-2\epsilon}[y(s)]^{1+2\epsilon}F(k^2, s). \end{aligned} \tag{31}$$

From (30) and (31) we obtain

$$[y(t)]^{-1-2\epsilon} \leqq \left(k^{-2\epsilon}\int_T^t [y(s)]^{1+2\epsilon}R_T(s)F(k^2, s)ds\right)^{-1-2\epsilon} \tag{32}$$

for $t \geqq T$. Multiplying both sides of (32) by $[y(t)]^{1+2\epsilon}R_T(t)F(k^2, t)$ and integrating over $[T_1, t]$, $T_1 > T$, we find

$$\int_{T_1}^t R_T(s)F(k^2, s)ds \leqq \frac{k^{2\epsilon(1+2\epsilon)}}{2\epsilon}\left(\int_T^u [y(s)]^{1+2\epsilon}R_T(s)F(k^2, s)ds\right)^{-2\epsilon}\Bigg|_{u=t}^{u=T_1},$$

which implies

$$\int_{T_1}^{\infty} R_T(t)F(k^2, t)dt < \infty.$$

This clearly contradicts (29) and the proof is complete.

COROLLARY. (i) *Equation (C) with $0 < \gamma < 1$ is oscillatory if and only if*

$$\int^{\infty} [R(t)]^{\gamma}p(t)dt = \infty.$$

(ii) *Equation (C) with $\gamma > 1$ is oscillatory if and only if*

$$\int^{\infty} R(t)p(t)dt = \infty.$$

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We conclude by providing an oscillation criterion for Equation (B) which improves a recent result of Lovelady [**9**].

THEOREM 6. *In addition to* (a), (c), (d) *assume that if* $\alpha > 0$, *then*

$$\int_\alpha^\infty \frac{dy}{f(y)} < \infty \quad \text{and} \quad \int_{-\infty}^{-\alpha} \frac{dy}{f(y)} > -\infty. \tag{33}$$

A necessary and sufficient condition for (B) *to be oscillatory is that*

$$\int^\infty R(t)p(t)dt = \infty. \tag{34}$$

Proof. From Theorem 3 (ii) it follows that (34) holds if (B) is oscillatory. To prove the converse we use the method adapted from Macki and Wong [**10**]. Let $y(t)$ be a nonoscillatory solution of (B). We may suppose that $y(t) > 0$ for $t \geqq T$. Lemma 2 implies that

$$y(t) \geqq \int_T^t R_T(s)p(s)f(y(s))ds, \quad t \geqq T. \tag{35}$$

Using (35) and the fact that $f(y)$ is nondecreasing, we see that

$$\frac{f(y(t))}{f\left(\int_T^t R_T(s)p(s)f(y(s))ds\right)} \geqq 1, \quad t > T. \tag{36}$$

Multiplying (36) by $R_T(t)p(t)$ and integrating over $[T_1, T_2]$, $T_1 > T$, we obtain

$$\int_{T_1}^{T_2} R_T(s)p(s)ds \leqq \int_{U_1}^{U_2} \frac{dy}{f(y)}, \tag{37}$$

where

$$U_i = \int_T^{T_i} R_T(s)p(s)f(y(s))ds, \quad i = 1, 2.$$

Since the right hand side of (37) remains bounded on account of (33), we arrive at

$$\int_{T_1}^\infty R_T(t)p(t)dt < \infty$$

in the limit as $T_2 \to \infty$. This, however, is a contradiction to (34). It follows that Equation (B) is oscillatory if (34) holds. This completes the proof.

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Hiroshima University,
Hiroshima, Japan

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