ON NON-ANTICIPATIVE LINEAR TRANSFORMATIONS
OF GAUSSIAN PROCESSES WITH
EQUIVALENT DISTRIBUTIONS

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Let $\xi(t)$, $t \in T$, be a Gaussian process on a set $T$, and $H = H(\xi)$ be
the closed linear manifold generated by all values $\xi(t)$, $t \in T$, with the
inner product
$$\langle \eta_1, \eta_2 \rangle = E\eta_1\eta_2; \quad \eta_1, \eta_2 \in H.$$ We suppose that the Hilbert space $H$ is separable.

Let $\mathcal{A}$ be a linear operator on $H$; we call a random process of the
form
$$\eta(t) = \mathcal{A}\xi(t), \quad t \in T,$$
a linear transformation of the process $\xi(t)$, $t \in T$. One says that a linear
transformation $\mathcal{A}$ is non-anticipative, if
$$\mathcal{A}H_s(\xi) \subseteq H_s(\xi), \quad t \in T,$$
where $H_s(\xi)$ denotes the subspace in $H$, which is generated by all values
$\xi(s)$, $s \leq t$.

Let $P$ be a probability distribution of the Gaussian process $\xi = \xi(t)$,
$\xi \in T$, in some measurable space $(X, \mathcal{B}, P)$ of (trajectories) $x = x(t)$, $t \in T$,
where $\sigma$-algebra $\mathcal{B}$ is generated by all sets $\{x(t) \in B\}$ ($t \in T$), $B$ are Borel
sets on the real line, so $P$ is determined by finite-dimensional distributions of the random process $\xi = \xi(t)$, $t \in T$. Let $Q$ be a probability distributions of the Gaussian process $\eta = \eta(t)$, $t \in T$, represented by the
formula (1). According to well known Feldman's theorem (see, for example, [1]), $Q$ is equivalent to $P$ ($Q \sim P$) if and only if the operator
$$B = \mathcal{A}^* \mathcal{A}$$
is invertible and $I - B \in S_2$, where $S_2$ denotes the class of all Hilbert-
Schmidt operators in $H$.

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The operator $B$ connects with the correlation function $B(s, t)$ of the Gaussian distribution $Q$ as

$$B(s, t) = \langle B\xi(s), \xi(t) \rangle, \quad s, t \in T;$$

let us call $B$ the correlation operator of $Q$. Obviously, for any equivalent distribution $Q$ (i.e. $Q$ has strictly positive correlation operator $B$, such that $I - B \in S_2$), there is a linear transformation (1), which gives us a Gaussian process $\gamma(t), t \in T$, with the distribution $Q$: the general operator $\mathcal{A}$, which satisfies the condition (3), has the form

$$\mathcal{A} = V B^{1/2}$$

where $V$ is an arbitrary unitary operator in $H$. Let us consider a linear transformation (1) with $\mathcal{A} = I - \Delta$:

$$\gamma(t) = \xi(t) - \Delta \xi(t), \quad t \in T.$$

It is more convenient to reformulate Feldman's theorem in the following way: $Q \sim P$ if and only if $I - B^{1/2} \in S_2$ and $1$ does not belong to the spectrum of $I - B^{1/2}$. Indeed, $I - B \in S_2$ if and only if

$$(I - B^{1/2}) = (I - B)(I + B^{1/2})^{-1} \in S_2.$$

It is easy to see that for any operator $\Delta \in S_2$, which has no eigenvalue equal to 1, the random process $\gamma(t), t \in T$, of the form (6) has the equivalent distribution $Q$ with the correlation operator, because

$$I - B = \Delta + \Delta^*(I - \Delta) \in S_2.$$ 

But the condition $\Delta \in S_2$ is not necessary for the equivalence $Q \sim P$. Namely, by the formula (5) we have

$$\Delta = I - V B^{1/2},$$

where $V$ is some unitary operator and (for the equivalent distribution $Q$) $I - B^{1/2} \in S_2$; obviously $\Delta \in S_2$ if and only if $[\Delta - (I - B^{1/2})]B^{-1/2} = I - V \in S_2$.

Then we shall be interested in the linear transformation (6) with operators $\Delta \in S_2$. As we have obtained, it holds true if and only if

$$I - V \in S_2$$

where $V$ is an unitary operator connected with the operator $\Delta$ by the formula (7): $\Delta = I - V B^{1/2}$. According to Feldman's theorem any trans-
formation (6) such that $\Delta \in S$ and $1$ does not belong to the spectrum $\Delta$

We shall be interested also in a such property of the linear transformation (6) as to be non-anticipative that means

$$\Delta H_t(\xi) \subseteq H_t(\xi), \quad t \in T. \tag{9}$$

In the resent time it was paid attention for non-anticipative transformations in connection with Hitsuda's result [2] for the Wiener process $\xi(t), 0 \leq t \leq 1$: any Gaussian process $\eta(t), 0 \leq t \leq 1$, with an equivalent probability distribution can be represent in the form

$$\eta(t) = \xi(t) - \int_0^t \left[ \int_u^t \Delta(u, s)d\xi(u) \right] ds \tag{10}$$

where $\Delta(t, s); 0 \leq t, s \leq 1$,

$$\Delta(t, s) = 0, \quad s < t, \tag{11}$$

$$\int_0^t \Delta(t, s)^2 dtds < \infty. \tag{12}$$

Though in the paper [2] it was used some theorems on the martingales, it was clear that the representation (10) can be obtained as a result of the theory of operators in a Hilbert space: the formula (10) is given by a non-anticipative transformation (6) with $\Delta \in S$ in the case of Wiener process $\xi(t), 0 \leq t \leq 1$. The existense of such transformation in the general case follows from non-trivial Gohberg-Krein's theorems on so-called special factorization; namely, any positive operator $B$ of the type

$$B = (I - F) = (I - G)^{-1} \tag{13}$$

($F$ and $G = -FB^{-1}$ belong $S$)

can be represented in the form

$$B = (I + X)\mathcal{D}(I + X^*)$$

where $(I + X)$ is invertible, $X \in S$ and $\mathcal{D} \geq 0$; besides the operators $X$ and $\mathcal{D}$ satisfy the condition

$$XH_t \subseteq H_t, \quad \mathcal{D}H_t \subseteq H_t \quad (t \in T)$$

for a given monotone family of subspaces $H_t, t \in T$ ($H_s \subseteq H_t$ if $s \leq t$) (see the theorems 6.2 Ch. IV and 10.1 Ch. I in the book [3]). It is clear that for $H_t = H_t(\xi), t \in T$, the operator
satisfies the conditions (2) and (3), so the corresponding linear transformation (6) with $\mathcal{A} = I - \mathcal{A}$ will be non-anticipative. This proof of the existence of non-anticipative representations (6) for Gaussian processes $\gamma(t)$, $t \in T$, with equivalent distributions was suggested recently by Kallianpur and Oodaira [4] (in the case of Wiener process $\xi(t)$, $0 \leq t \leq 1$, it was done earlier by Kailath [5]). We should like to do the following essential note: for the operator $\mathcal{A}$, which was mentioned above (see (14)) it holds true that

$$\Delta = I - \mathcal{A} \in S_1,$$

so for any Gaussian process $\xi(t)$, $t \in T$, there is a non-anticipative Gaussian process $\eta(t) = \xi(t) - \Delta \xi(t)$, $t \in T$ (where $\Delta \in S_1$ satisfies the condition (9)) with a given equivalent probability distribution.

Indeed, in the representation (13) we have $(I + X)^{-1} = I + \mathcal{I}$, $\mathcal{I} = -X(I + X)^{-1} \in S$, and the operator $\mathcal{D}$ has a form

$$\mathcal{D} = (I + \mathcal{I})(I - F)(I + \mathcal{I}^*) = I + V$$

where

$$V = \mathcal{I}(I - F)(I + \mathcal{I}^*) - F(I + \mathcal{I}^*) + \mathcal{I}^* \in S_1.$$  

From relations

$$\mathcal{D}^{1/2} = (I + V)^{1/2} = I + W,$$

$$I + V = (I + W)^2 = I + W(2I + W) = I + W(I + \mathcal{D}^{1/2}),$$

we obtain that

$$W = V(I + \mathcal{D}^{1/2})^{-1} \in S_2,$$

so

$$\Delta = I - \mathcal{A} = I - (I + X)\mathcal{D}^{1/2}$$

$$= I - (I + X)(I + W) = -X(I + W) - W \in S_1.$$

It is worth to pay attention for the following fact: the linear transformation (6) with the operator $\mathcal{A} = I - \Delta$ of the form (14) is such that

$$H_t(\xi) = H_t(\eta), \quad t \in T.$$
Indeed, for the invertible positive operator $\mathcal{D}^{1/2} : \mathcal{D}^{1/2} H_t(\xi) \subseteq H_t(\xi)$, we have

$$\mathcal{D}^{1/2} H_t(\xi) = H_t(\xi)$$

because in a contrary case there is an element $h \in H_t(\xi)$, such that $h \perp \mathcal{D}^{1/2} H_t(\xi)$ and $\mathcal{D}^{1/2} h = 0$. Remind that a Volterra operator $X$ has only one point of a spectra equal to 0, so for the operator $(I + X)$ in the formula (14), $(I + X) H_t(\xi) \subseteq H_t(\xi)$, we have

$$(I + X) H_t(\xi) = H_t(\xi) .$$

Now it is obvious that the operator $\mathcal{A} = (I + X) \mathcal{D}^{1/2}$ satisfies the condition (16).

Let us consider a few examples of non-anticipative representations (6) with $\mathcal{A} \in S_x$.

**EXAMPLE 1.** Let $\xi(t)$, $0 \leq t \leq 1$, be a Gaussian process with stationary increments:

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} - \frac{1}{i\lambda} \Phi(d\lambda) ,$$

which has a spectral density $f(\lambda)$ of the type:

$$0 < \lim_{\lambda \to -\infty} f(\lambda) \leq \lim_{\lambda \to +\infty} f(\lambda) < \infty$$

(if $f(\lambda) = 1/2\pi$, we deal with Wiener process $\xi(t)$, $0 \leq t \leq 1$).

The corresponding space $H$ consists of all random variables

$$\gamma = \int_{-\infty}^{\infty} \varphi(\lambda) \Phi(d\lambda) = \int_{0}^{1} \epsilon(t) \hat{\xi}(t) dt$$

where functions $\epsilon(t)$, $0 \leq t \leq 1$, belonging to $L^2[0,1]$, and $\hat{\xi}(t)$ is the generalized derivative of process $\xi(t)$; besides\(^1\)

$$\|\gamma\|^2 = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\lambda \bigcup_{0}^{1} \epsilon(t)^2 dt$$

\(^1\) The relation $\alpha \succ \beta$ between variables $\alpha, \beta$ means that

$$0 < c_1 \leq \frac{\alpha}{\beta} \leq c_2 < \infty .$$
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(see, for example, [1] or [6]), and the formula (18) gives us the isomorphism between $H$ and $L^2[0,1]$ such that

$$H_i(\xi) \leftrightarrow L^2[0,1], \quad 0 \leq t \leq 1,$$

(20)

where $L^2[0,1]$ denotes the subspace of all functions $c(s), \ 0 \leq s \leq 1; c(s) = 0$ for $s > t$. As it follows from the conditions (19) and (20), the formula

$$\Delta \eta = \int_0^t [\hat{A} c(t)] \hat{\xi}(t) dt$$

gives us the isomorphism $\Delta \leftrightarrow \hat{\Delta}$ between Hilbert-Schmidt operators in $H$ and $L^2[0,1]$, and an operator $\Delta$ satisfies the condition (9) if and only if

$$\hat{\Delta} L^2[0,1] \subseteq L^2[0,1], \quad 0 \leq t \leq 1,$$

that is equivalent to the condition (11) for a corresponding kernel $\Delta(t,s)$:

$$\hat{\Delta}c(t) = \int_0^t \Delta(t,s)c(s)ds, \quad 0 \leq t \leq 1,$$

(remind $\hat{\Delta} \in S$ if and only if $\Delta(t,s); 0 \leq t, s \leq 1,$ satisfies the condition (12)). Thus any non-anticipative operator $\Delta \in S$ can be described by the formula

$$\Delta \eta = \int_0^t [\int_0^t \Delta(t,s)c(s)ds] \hat{\xi}(t) dt$$

(21)

with a Volterra, Hilbert-Schmidt kernel $\Delta(t,s); 0 \leq t \leq 1$. For variables $\xi(t), 0 \leq t \leq 1$, which correspond to the functions

$$c(s) = \begin{cases} 1, & 0 \leq s \leq t, \\ 0, & s > t, \end{cases}$$

we obtained from the formula (21) a general non-anticipative transformation (6) with $\Delta \in S$ as

$$\eta(t) = \xi(t) + \int_0^t \int_0^s \Delta(u,s)\dot{\xi}(u)du ds, \quad 0 \leq t \leq 1,$$

(22)

that gives us Hitsuda's representation (10) in the case of Wiener process $\xi(t), 0 \leq t \leq 1$.

**Example 2.** Let $\xi(t), 0 \leq t \leq 1$, be a Gaussian stationary process:
\[ \xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} \phi(d\lambda) \]

with a spectral density \( f(\lambda) \) of the type

\[ 0 < \lim_{\lambda \to \pm\infty} \lambda^{2n} f(\lambda) \leq \lim_{\lambda \to \pm\infty} \lambda^{2n} f(\lambda) < \infty. \] (23)

It will be convenient to introduce the process

\[ \zeta(t) = \sum_{k=0}^{n-1} c_k \xi^{(k)}(t) + \int_0^{t} \xi(s) ds \]

\[ = \int_{-\infty}^{\infty} \frac{e^{it\lambda} - 1}{i\lambda} (1 + i\lambda)^n \phi(d\lambda), \quad 0 \leq t \leq 1. \] (24)

Obviously, the spectral density of this process \( \zeta(t) \) with stationary increments satisfies the condition (17) and we can use results of our example 1 for the process \( \zeta(t), 0 \leq t \leq 1. \)

As is known (see, for example, [1] or [6]) the Hilbert space \( H = H(\xi) \) consists of all variables

\[ \eta = \int_{-\infty}^{\infty} \varphi(\lambda) \phi(d\lambda) \]

\[ \left( \varphi(\lambda) = \sum_{k=0}^{n-1} c_k (i\lambda)^k + (1 + i\lambda)^n \int_0^{t} e^{it\lambda} c(t) dt \right) \]

where \( c_1, \ldots, c_{n-1} \) are arbitrary constants and \( c(t), \in L^1[0,1] \) or

\[ \eta = \sum_{k=0}^{n-1} c_k \xi^{(k)}(0) + \int_0^{t} c(t) \xi(t) dt \] (25)

where \( \xi(t) \) denotes the generalized derivative of the process \( \zeta(t) \) determined by the transformation (24).

If we consider in the general formula (25) only functions \( c(s), \in L^1[0,t] \), we obtain the corresponding subspace \( H_t(\xi), 0 \leq t \leq 1, \) and it shows that \( H_t(\xi) \) is a \textit{direct sum} of the subspace

\[ H_0(\xi) = \bigcap_{t>0} H_t(\xi), \]

which consists of all variables \( \eta = \sum_{k=0}^{n-1} c_k \xi^{(1)}(0), \) and the subspace \( H_t(\xi) \) of all variables \( \eta = \int_0^{t} c(s) \xi(s) ds \):

\[ H_t(\xi) = H_0(\xi) + H_t(\xi), \quad 0 \leq t \leq 1; \]
in particular

\[ H(\xi) = H_{0+}(\xi) + H(\zeta). \]

Let \( P \) be a projector on the subspace \( H(\zeta) \) parallel to the subspace \( H_{0+}(\xi) \). If \( \Delta \in S_2 \) then \( P\Delta P \in S_2 \); obviously, if \( \Delta \) satisfies the condition (9) then \( P\Delta P \) satisfies to the similar condition with respect to \( H_t(\zeta) \), \( 0 \leq t \leq 1 \). As it has been shown (see (21)), the non-anticipative operator \( P\Delta P \) in \( H(\zeta) \) can be described by a Volterra, Hilbert-Schmidt kernel \( \Delta(t,s); 0 \leq t, s \leq 1 \):

\[ P\Delta P \eta = \int_0^i \int_t^1 \Delta(t,s)e(s)ds \zeta(t)dt \]  
(26)

where \( \eta \in H \) is given by the formula (25) and

\[ P\eta = \int_0^i e(t)\zeta(t)dt. \]

For any non-anticipative operator \( \Delta \) in \( H(\xi) \) we have

\[ \Delta H_{0+} = \Delta(\bigcap_{t=0} H_t) \subseteq \bigcap_{t=0} \Delta H_t \subseteq H_t = H_{0+} \]

that is equivalent to the condition

\[ (I - P)\Delta(I - P) = \Delta(I - P). \]

Then

\[ \Delta = (I - P)\Delta P + \Delta(I - P) + P\Delta P = (I - P)\Delta + P\Delta P \]

where the finite-dimensional operator \((I - P)\Delta\), mapping \( H(\xi) \) on the subspace \( H_{0+}(\xi) \), has the form

\[ (I - P)\Delta \eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) \]  
(27)

\((\eta_0, \eta_1, \cdots, \eta_{n-1} \) are some fixed elements in \( H_{0+} \)). Combining formulas (26) and (27), we obtain a general non-anticipative operator \( \Delta \in S \) as

\[ \Delta \eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) + \int_0^i \int_t^1 \Delta(t,s)e(s)ds \zeta(t)dt; \]  
(28)

in particular, for \( \eta \in H_t(\xi) \)

\[ \Delta \eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) + \int_0^i \int_u^1 \Delta(u,s)e(s)du \zeta(u)ds \]  
(29)
References


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