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STRUCTURAL IMPLICATIONS OF NORMS WITH HÖLDER RIGHT-HAND DERIVATIVES

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We study a nonsmooth extension of Gateaux differentiability satisfying a directional Hölder condition. In particular, we show that a Banach space is an Asplund space if it has an equivalent norm with a directionally Hölder right-hand derivative at each point of its sphere.

1. INTRODUCTION

An *Asplund space* is a Banach space where every continuous convex function on an open convex subset of the space is Fréchet differentiable on a dense G_δ subset of its domain. It is well known that a Banach space is Asplund if it has an equivalent norm Fréchet differentiable on its unit sphere, but Haydon [**14**] has given an example of an Asplund space which has no equivalent norm Gateaux differentiable on its unit sphere. Several variant differentiability properties have been studied which, when applied to an equivalent norm or a continuous bump function imply that a Banach space is Asplund.

A real-valued function ϕ on an open subset A of a Banach space X has a *right-hand derivative at* $x \in A$ *in the direction* $h \in X$ if

$$\phi'_+(x)(h) = \lim_{t \to 0^+} \frac{\phi(x+th) - \phi(x)}{t} \tag{1.1}$$

exists and has a *Fréchet right-hand derivative at* x if given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{\phi(x+th) - \phi(x)}{t} - \phi'_+(x)(h)\right| < \varepsilon \quad \text{for all } 0 < t < \delta \text{ and all } h \in S_X. \tag{1.2}$$

If $\phi'_+(x)(h)$ exists for all $h \in X$ and is linear in h then ϕ is *Gateaux differentiable* at x and if ϕ is also Fréchet right-hand differentiable at x then ϕ is *Fréchet differentiable* at x. Godefroy [**12**] showed that a Banach space is Asplund if it admits an equivalent norm which has a Fréchet right-hand derivative at each point on its unit sphere.

We say that ϕ is *directionally Hölder differentiable at* $x \in A$ if there exists a linear functional $\phi'(x) \in X^*$ and $\delta > 0$ such that given $h \in S_X$ there are $\alpha_h > 0$ and $C_h > 0$ where

$$\left|\phi(x+th) - \phi(x) - t\phi'(x)(h)\right| \leqslant C_h |t|^{1+\alpha_h} \quad \text{for all } |t| < \delta. \tag{1.3}$$

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It has been shown in [**18**] that a Banach space is Asplund if it has a continuous bump function which is directionally Hölder differentiable on X.

These two results have motivated us to study continuous convex functions whose right-hand derivative satisfies a directional Hölder condition.

We say that ϕ has a *directionally Hölder right-hand derivative at* $x \in A$ if there exists $\delta > 0$ such that given $h \in S_X$ there are $\alpha_h > 0$ and $C_h > 0$ where

$$(1.4) \qquad \left|\phi(x+th) - \phi(x) - t\phi'_+(x)(h)\right| \leqslant C_h t^{1+\alpha_h} \text{ for all } 0 < t < \delta.$$

If there exists an $\alpha > 0$ such that this inequality is satisfied with $\alpha_h = \alpha$ for all $h \in S_X$ we say that ϕ has a *directionally* α*-Hölder right-hand derivative at* x. Notice that for a continuous convex function ϕ on an open convex subset A of X, $\phi'_+(x)(h)$ exists for each $h \in X$ and is a continuous sublinear function in h, [**17**, p.5].

In Section 2 we study the particular properties of Hölder right-hand derivatives of continuous convex functions on open convex subsets of a Banach space. Although for such functions directional Hölder differentiablity implies Fréchet differentiablity, we give an example to show that directional Hölder right-hand differentiabilty does not necessarily imply Fréchet right-hand differentiability. Notwithstanding, we show that if such a function has a directional Hölder right-hand derivative on a residual subset of its domain then it is Fréchet differentiable on a residual subset of its domain.

In Section 3 we develop the notion of Hölder exposed faces of the dual ball and the relationship to the directionally Hölder right-hand differentiability of the norm. Using the result of Godefroy [**12**] based on Simons' inequality we show that a Banach space is Asplund if it has an equivalent norm directionally Hölder right-hand differentiable on the unit sphere. Finally, we see that a Banach space with an equivalent norm directionally Hölder right-hand differentiable on a residual subset of the unit sphere is superreflexive if the strongly exposed points of the unit ball are dense in the unit sphere. We also observe that Kunen's Asplund space does not admit an equivalent norm with a directionally Hölder right-hand derivative on its sphere.

2. Hölder right-hand derivatives of convex functions

Given a continuous convex function ϕ on an open convex subset A of a Banach space X, we say that ϕ is *Hölder differentiable* at $x \in A$ if there exist $\alpha > 0$, $C > 0$ and $\delta > 0$ such that

$$(2.1) \qquad \phi(x+th) - \phi(x) - t\phi'(x)(h) \leqslant C\,|t|^{1+\alpha} \quad \text{for all } |t| < \delta \text{ and } h \in S_X.$$

Clearly, if ϕ is Hölder differentiable at $x \in A$ then it is Fréchet differentiable at x. We begin with a variant of a surprising result of Borwein and Noll [**2**, Proposition 2.2].

Theorem 2.1. *If a continuous convex function ϕ on an open convex subset A of a Banach space is directionally Hölder differentiable at $x \in A$, then ϕ is Hölder differentiable at x.*

PROOF: We include the proof sketched in [**16**, p.615] for completeness. Consider $0 < \delta \leqslant 1$ such that given $h \in S_X$ there exists $\alpha_h > 0$ and $C_h > 0$ for which

$$\phi(x+th) - \phi(x) - t\phi'(x)(h) \leqslant C_h |t|^{1+\alpha_h} \quad \text{for all } |t| < \delta.$$

For each $n \in \mathbb{N}$, write

$$F_n \equiv \left\{h \in B_X : \phi(x+th) - \phi(x) - t\phi'(x)(h) \leqslant n \|th\|^{1+(1/n)} \quad \text{for all } |t| < \delta\right\}.$$

Now F_n is closed and $\bigcup_{n \in \mathbf{N}} F_n = B_X$. Since B_X is second category and F_n is symmetric there is an $n_0 \in \mathbb{N}$ such that $\pm h_0 + rB_X \subseteq F_{n_0}$ for some $r > 0$ and $h_0 \in B_X$. Given $h \in S_X$, then $\pm h_0 + rh \in F_{n_0}$, and for $|t| < r\delta$ we have

$$\begin{aligned} \phi(x+th) - \phi(x) - t\phi'(x)(th) &\leqslant \frac{1}{2}\left[\phi\Big(x + \frac{t}{r}(h_0 + rh)\Big) - \phi(x) - \phi'(x)\Big(\frac{t}{r}(h_0+rh)\Big)\right] \\ &+ \frac{1}{2}\left[\phi\Big(x + \frac{t}{r}(-h_0 + rh)\Big) - \phi(x) - \phi'(x)\Big(\frac{t}{r}(-h_0+rh)\Big)\right] \\ &\leqslant \frac{n_0}{2}\left\|\frac{t}{r}(-h_0+th)\right\|^{1+(1/n_0)} + \frac{n_0}{2}\left\|\frac{t}{r}(h_0+rh)\right\|^{1+(1/n_0)} \\ &\leqslant n_0\left(\frac{|t|}{r}\right)^{1+(1/n_0)} = C|t|^{1+(1/n_0)} \end{aligned}$$

where $C \equiv n_0/r^{1+(1/n_0)}$. ∎

However, the analogue of Theorem 2.1 does not hold for Hölder right-hand derivatives.

EXAMPLE 2.2. On any infinite dimensional Banach space there is an equivalent norm and a point on its unit sphere where for any given $\alpha > 0$ the norm has a directionally α-Hölder right-hand derivative but does not have a Fréchet right-hand derivative.

PROOF: Our construction is similar to that in [**1**, p.1126]. Write $X = Y \times \mathbb{R}$. Now Y is infinite dimensional, so by the Josefson-Nissenzweig theorem [**6**, p.219] there is a sequence $f_k \in S_{Y^*}$ such that $f_k \to_{w^*} 0$. Define an equivalent norm on X by

$$\|(y,t)\| = \sup_k\left\{\|y\| + |t|, |4f_k(y) + (1-1/k)t|\right\}.$$

We consider the right-hand derivative of $\|\cdot\|$ at $(0,1)$ in the arbitrary fixed direction (y,r). To compute this, we fix $N > 0$ such that $|f_k(y)| \leqslant \|y\|/4$ for all $k \geqslant N$ and we choose $\delta > 0$ such that

$$\text{(2.2)} \qquad (a)\ 1 + tr \geqslant \frac{1}{2} \text{ for } |t| < \delta; \qquad \text{and} \qquad (b)\ \|ty\| \leqslant \frac{1}{8N} \text{ for } |t| < \delta.$$

According to (2.2a), we have

$$\|ty\| + |1+tr| = t\|y\| + 1 + tr \quad \text{for } 0 \leqslant t < \delta. \tag{2.3}$$

On the other hand, for $k \leqslant N$ and $0 \leqslant t < \delta$, using (2.2a) and then (2.2b) we obtain

$$\begin{aligned} \Big|4f_k(ty) + (1-1/k)(1+tr)\Big| &\leqslant 4\|ty\| + 1 + tr - 1/(2k) \\ &\leqslant 4\|ty\| + 1 + tr - 1/(2N) \leqslant 1 + tr. \end{aligned} \tag{2.4}$$

Now, $\big|f_k(y)\big| \leqslant \|y\|/4$ for $k > N$, and so for $k > N$, (2.2a) implies that

$$\Big|4f_k(ty) + (1-1/k)(1+tr)\Big| \leqslant \|ty\| + |1+tr| = t\|y\| + 1 + tr \quad \text{for } 0 \leqslant t < \delta. \tag{2.5}$$

The definition of $\|\cdot\|$ along with (2.3), (2.4) and (2.5) imply that

$$\Big\|(0+ty, 1+tr)\Big\| = t\|y\| + 1 + tr \quad \text{for } 0 \leqslant t < \delta.$$

Using this and the fact that $\Big\|(0,1)\Big\| = 1$, we obtain

$$\Big\|(0,1)\Big\|'_+(y,r) = \lim_{t\to 0^+} \frac{\Big\|(0+ty,1+tr)\Big\| - \Big\|(0,1)\Big\|}{t} = \lim_{t\to 0^+} \frac{\|ty\| + 1 + tr - 1}{t} = \|y\| + r.$$

Consequently,

$$\Big\|(0,1) + t(y,r)\Big\| - \Big\|(0,1)\Big\| - \Big\|(0,1)\Big\|'_+(ty,tr) = 0 \quad \text{for } 0 \leqslant t < \delta \tag{2.6}$$

which implies that for any $\alpha > 0$, $\|\cdot\|$ has a directionally α-Hölder right-hand derivative at $(0,1)$.

To complete the proof, we show that $\|\cdot\|$ does not have a Fréchet right-hand derivative at $(0,1)$. For this we consider the directions $\{(y_k,0)\}_{k=1}^{\infty}$ where $y_k \in S_Y$ is chosen so that $f_k(y_k) \geqslant 3/4$. Then

$$\frac{\Big\|\Big(0+\frac{1}{k}y_k, 1+\frac{1}{k}0\Big)\Big\| - 1}{\frac{1}{k}} \geqslant k\Big[4f_k(y_k)/k + (1-1/k) - 1\Big] \geqslant 2.$$

Because $\Big\|(0,1)\Big\|'_+(y_k,0) = \|y_k\| = 1$, this shows that the right-hand derivative is not approached uniformly over the set of directions $\Big\{(y_k,0)\Big\}_{k=1}^{\infty} \subset B_X$. ∎

Using the fact that $\sup\Big\{|x_\gamma| : |x_\gamma| < \|x\|_\infty\Big\} < \|x\|_\infty$ for $x = \{x_\gamma\} \in c_0(\Gamma)$, we can easily establish the following example.

EXAMPLE 2.3. The usual norm on $c_0(\Gamma)$ has a directionally α-Hölder right-hand derivative at each point on its sphere.

The separable reduction argument known as Gregory's Theorem will enable us to deduce information about the differentiability of convex functions that have directionally

Hölder right-hand derivatives on residual subsets of their domain. For our purposes we need the following technical variant of the form of the theorem given in [**10**, p.163]. We recall that a set-valued mapping Φ from an open subset A of a Banach space X into subsets of the dual X^* is said to be *norm upper semicontinuous* at $x \in A$ if for every open subset W of X^* where $\Phi(x) \subseteq W$ there is an open neighbourhood U of x such that $\Phi(U) \subseteq W$. A continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \longmapsto \partial\phi(x)$ is single-valued and norm upper semicontinuous at $x \in A$, [**17**, p.19].

PROPOSITION 2.4. *Consider a set-valued mapping Φ from an open subset A of a Banach space X into subsets of the dual X^* such that the set of points where Φ is single-valued and norm upper semicontinuous is not residual in A. Then given any residual subset A_w of A there exists a separable subspace Y of X and an open subset G of A such that $G \cap A_w \cap Y$ is a residual subset of $G \cap Y$ and $\Phi|_Y$ is nowhere single-valued and norm upper semicontinuous on $G \cap Y$.*

PROOF: As in [**10**, p. 163], let G be an open subset of A where for some $m_0 \in \mathbb{N}$ the set

$$\Big\{x \in A : \text{ there exists an } f \in \Phi(x) \text{ and sequences } \{y_k\} \to x \text{ as } k \to \infty \text{ and } \{g_k\} \text{ where } g_k \in \Phi(y_k) \text{ satisfies } \|g_k - f\| > 1/m_0 \text{ for all } k \in \mathbb{N}\Big\}$$

is dense in G. The separable subspace Y is constructed as the closure of the union of a nested sequence of separable subspaces $\{Y_s\}$ as in [**10**, p.163], but we put an extra constraint on their construction. Let Y_0 be any separable subspace such that $G \cap Y_0 \neq \emptyset$, but having constructed Y_s we choose a countable set D_s in $G \cap A_w$ such that $G \cap Y_s \subseteq \overline{D_s}$ and insist that Y_{s+1} include D_s in its span. So as in [**10**, p.163] we obtain that $\Phi|_Y$ is nowhere single-valued and norm upper semi-continuous on $G \cap Y$. As $G \cap A_w \cap Y$ is residual in $G \cap Y$ we need only check on density. Given $x \in G \cap Y$ and $\varepsilon > 0$ there is an $s \in \mathbb{N}$ such that $d(x, Y_s) < \varepsilon$. But by the constraints of our construction we have $d(x, D_s) < \varepsilon$ and $D_s \subseteq Y_{s+1} \cap G \cap A_w \subseteq Y \cap G \cap A_w$. ∎

While Examples 2.2 and 2.3 show that a directionally Hölder right-hand derivative is not an overly restrictive concept, the following result gives us insight into its structural implications which we shall explore more fully in the next section.

Given a continuous convex function ϕ on an open convex subset A of a Banach space X we say that ϕ is *locally uniformly Hölder differentiable* on A if given $x_0 \in A$ there exists an open neighbourhood U of x in A and $\alpha > 0$, $C > 0$ and $\delta > 0$ such that

$$(2.7) \quad \phi(x + th) - \phi(x) - t\phi'(x)(h) \leqslant C\,|t|^{1+\alpha} \quad \text{for all } |t| < \delta, \quad h \in S_X \quad \text{and all } x \in U.$$

THEOREM 2.5. *Consider a continuous convex function ϕ on an open convex subset A of a Banach space X. If ϕ has a directionally Hölder right-hand derivative on a residual* *subset A_w of A, then ϕ is locally uniformly Hölder differentiable at each point of a dense open subset of A.*

PROOF: Suppose that the set of points where ϕ is Fréchet differentiable is not residual in A. Then by Proposition 2.4 there is a separable subspace Y of X and an open subset G of A such that $A_w \cap G$ is residual in G and $\phi|_Y$ is not Fréchet differentiable at any point of $G \cap Y$. However, by Mazur's Theorem [**17**, p.12], $\phi|_Y$ is Gateaux differentiable on a dense G_δ subset G_1 of $G \cap Y$. Now $A_w \cap G_1 \neq \emptyset$ and $\phi|_Y$ is Hölder differentiable at each point of $A_w \cap G_1$, which contradicts $\phi|_Y$ being not Fréchet differentiable at any point of $G \cap Y$. Therefore ϕ is Fréchet differentiable on a residual subset D of A and ϕ is Hölder differentiable on the residual subset $A_w \cap D$. Consider $x_0 \in A$ and choose $\delta > 0$ such that $x_0 + \delta B_X \subseteq A$. For each $n \in \mathbb{N}$, write

$$F_n \equiv \Big\{x \in x_0 + \delta B_X : \phi(x+h) - \phi(x) - \phi'(x)(h) \leqslant n \|h\|^{1+(1/n)} \text{ for all } \|h\| \leqslant \frac{1}{n}\Big\}.$$

Now F_n is closed and ϕ is Hölder differentiable at $x \in x_0 + \delta B_X$ if and only if $x \in F_n$ for some $n \in \mathbb{N}$. Therefore $\bigcup_{n\in\mathbb{N}} F_n$ is residual in $x_0 + \delta B_X$. Since $\bigcup_{n\in\mathbb{N}} F_n$ is second category there exists $n_0 \in \mathbb{N}$ such that F_{n_0} contains a nonempty open set on which ϕ satisfies (2.7) with $\alpha = 1/n_0$. ∎

The variant of Gregory's Theorem given in Proposition 2.4 enabled us in Theorem 2.5 to assume that the differentiability property of ϕ occurs only on a residual subset of A rather than on all of A. Proposition 2.4 can similarly be used to improve [**11**, Theorem 2.2] where the differentiability property of ϕ is weak Hadamard right-hand differentiability.

3. Structural implications of norms with Hölder right-hand derivatives

Given a Banach space X, for $u \in X \setminus \{0\}$, write $J(u) = \big\{f \in S_{X^*} : f(u) = \|u\|\big\}$. As in [**9**, p.49] we develop a notion of "exposed faces" of the dual ball which is dual to directional right-hand differentiability of the norm. This result can be viewed as a nonsmooth directional analogue of [**7**, Proposition 2.2] and as such has an analogous proof.

LEMMA 3.1. *Consider a Banach space X with $u \in S_X$ and $h \in X$. Then given $\alpha > 0$, the following are equivalent.*

(a) There exists $C > 0$ such that

$$\big\|u + t\tilde{h}\big\| - \|u\| - t\,\|u\|'_+\big(\tilde{h}\big) \leqslant Ct^{1+\alpha} \text{ for } t \geqslant 0 \text{ and } \tilde{h} \in \{-h, h\}.$$

(b) There exists $K > 0$ such that $\operatorname{dist}_h^{1+(1/\alpha)}\big(f, J(u)\big) \leqslant K\big[1 - f(u)\big]$ for each $f \in B_{X^*}$, where $\operatorname{dist}_h\big(f, J(u)\big) = \inf\big\{|(f-g)(h)| : g \in J(u)\big\}$.

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PROOF: (a) $\Rightarrow$ (b): If $\mathrm{dist}_h\big(f, J(u)\big) = 0$, the result is obvious. So we consider $\mathrm{dist}_h\big(f, J(u)\big) > 0$. Now, either $f(h) = \sup\big\{g(h) : g \in J(u)\big\} + \mathrm{dist}_h\big(f, J(u)\big)$ or $f(-h) = \sup\big\{g(-h) : g \in J(u)\big\} + \mathrm{dist}_h\big(f, J(u)\big)$. So we fix $\widetilde{h} \in \{h, -h\}$ such that

$$f\big(\widetilde{h}\big) = \sup\big\{g\big(\widetilde{h}\big) : g \in J(u)\big\} + \mathrm{dist}_h\big(f, J(u)\big).$$

Choose $f_u \in J(u)$ such that $\|u\|'_+ \big(\widetilde{h}\big) = f_u\big(\widetilde{h}\big)$. Then for $t \geqslant 0$, (a) implies that

$$\begin{aligned}(f - f_u)\big(t\widetilde{h}\big) - Ct^{1+\alpha} &\leqslant (f - f_u)\big(t\widetilde{h}\big) - \big\|u + t\widetilde{h}\big\| + \|u\| + f_u\big(t\widetilde{h}\big)\\ &= f\big(u + t\widetilde{h}\big) - \big\|u + t\widetilde{h}\big\| + (f_u - f)(u)\\ &\leqslant (f_u - f)(u) = 1 - f(u).\end{aligned}$$

Putting $t = \Big(\big((f - f_u)\big(\widetilde{h}\big)\big)\big/\big(C(1+\alpha)\big)\Big)^{1/\alpha}$ which is positive in the inequality yields

$$\Big|(f - f_u)\big(\widetilde{h}\big)\Big|^{1+(1/\alpha)} \leqslant K\big[1 - f(u)\big] \quad \text{where} \quad K = \big[C(1+\alpha)\big]^{1/\alpha}\Big(1 + \frac{1}{\alpha}\Big).$$

This shows that (a) implies (b) because $\mathrm{dist}_h\big(f, J(u)\big) \leqslant \Big|(f - f_u)\big(\widetilde{h}\big)\Big|$.

(b) $\Rightarrow$ (a): Fix $\widetilde{h} \in \{-h, h\}$ and $t > 0$. Choose $f_u \in J(u)$ such that $f_u\big(\widetilde{h}\big) = \|u\|'_+ \big(\widetilde{h}\big)$ and choose $f \in S_{X^*}$ such that $f\big(u + t\widetilde{h}\big) = \big\|u + t\widetilde{h}\big\|$. Then

$$\begin{aligned}0 \leqslant \big\|u + t\widetilde{h}\big\| - \|u\| - t\,\|u\|'_+ \big(\widetilde{h}\big) &= f\big(u + t\widetilde{h}\big) - f_u(u) - tf_u\big(\widetilde{h}\big)\\ &= (f - f_u)\big(t\widetilde{h}\big) - \big(1 - f(u)\big)\\ &\leqslant (f - f_u)\big(t\widetilde{h}\big) - \frac{1}{K}\mathrm{dist}_h^{1+(1/\alpha)}\big(f, J(u)\big). \qquad (3.1)\end{aligned}$$

Now, $f_u\big(\widetilde{h}\big) = \max\big\{f\big(\widetilde{h}\big) : f \in J(u)\big\}$ and so $0 \leqslant (f - f_u)\big(t\widetilde{h}\big) = \min\big\{(f - g)\big(t\widetilde{h}\big) : g \in J(u)\big\}$. Consequently, $(f - f_u)\big(t\widetilde{h}\big) = t\mathrm{dist}_h\big(f, J(u)\big)$. Putting this in (3.1) yields

$$\big\|u + t\widetilde{h}\big\| - \|u\| - t\,\|u\|'_+ \big(\widetilde{h}\big) \leqslant t\mathrm{dist}_h\big(f, J(u)\big) - \frac{1}{K}\mathrm{dist}_h^{1+(1/\alpha)}\big(f, J(u)\big) \leqslant Ct^{1+\alpha}$$

where $C = K^\alpha \big/ \alpha\big(1 + (1/\alpha)\big)^{1+\alpha}$ is independent of t and f. ∎

According to Example 2.2, the exposed faces given in Lemma 3.1(b) are not necessarily strongly exposed in the sense of [**9**, p.49]. Nevertheless this duality is sufficient to prove Lemma 3.3 which is what we shall need to establish our Asplund space property. We use the following computational property.

FACT 3.2. If for a Banach space X given $x \in S_X$ and $h \in B_X$, there exist $C_h > 0$, $\alpha_h > 0$ and $\delta > 0$ such that

$$\|x + th\| - \|x\| - t\,\|x\|'_+ (h) \leqslant C_h t^{1+\alpha_h} \textit{ for all } 0 < t < \delta$$

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then for $0 < \alpha \leqslant \alpha_h$ *there exists* $K > 0$ *such that*

$$\|x + th\| - \|x\| - t\,\|x\|'_+(h) \leqslant Kt^{1+\alpha} \text{ for all } t > 0.$$

PROOF: For all $t > 0$, $\|x + th\| - \|x\| - t\,\|x\|'_+(h) \leqslant 2t$. Then $\|x+th\| - \|x\| - t\,\|x\|'_+(h) \leqslant K_h t^{1+\alpha_h}$ for all $t > 0$ where $K_h \equiv \max\{C_h, 2/\delta^{\alpha_h}\}$. But also for $0 < t < 1$, $\|x+th\| - \|x\| - t\,\|x\|'_+(h) \leqslant K_h t^{1+\alpha}$ for $0 < \alpha < \alpha_h$. Then $\|x+th\| - \|x\| - t\,\|x\|'_+(h) \leqslant Kt^{1+\alpha}$ for $0 < \alpha < \alpha_h$ where $K \equiv \max\{2, K_h\}$. ∎

LEMMA 3.3. *If the norm on a Banach space* X *has a directionally Hölder right-hand derivative at each point on its unit sphere and* W *is a weak*-dense subset* B_{X^*}, *then for any* $u \in S_X$, *we have* $\operatorname{dist}(W, J(u)) = 0$.

PROOF: Using Fact 3.2 we have that given $u \in S_X$ and $h \in B_X$ there exist $K > 0$ and $\alpha_h > 0$ such that $\left\|u + t\tilde{h}\right\| - \|x\| - t\,\|u\|'_+\left(\tilde{h}\right) \leqslant Kt^{1+\alpha_h}$ for all $t \geqslant 0$ and $\tilde{h} \in \{-h, h\}$. Lemma 3.1 implies that for each $h \in B_X$, there exists $C_h > 0$ and $\alpha_h > 0$ satisfying

$$\operatorname{dist}_h^{1+(1/\alpha_h)}\left(f, J(u)\right) \leqslant C_h\left[1 - f(u)\right] \text{ for all } f \in B_{X^*}. \tag{3.2}$$

For each $n \in \mathbb{N}$, write

$$F_n = \left\{h \in B_X : \operatorname{dist}_h^n\left(f, J(u)\right) \leqslant n\left(1 - f(u)\right) \text{ for all } f \in B_{X^*}\right\}.$$

It can be easily checked that F_n is closed. Moreover, $\bigcup_{n\in\mathbb{N}} F_n \supset B_X/2$ because for $\|h\| \leqslant 1/2$ we have $\operatorname{dist}_h\left(f, J(u)\right) \leqslant 1$ for all $f \in B_{X^*}$, and this with (3.2) implies that

$$\operatorname{dist}_h^n\left(f, J(u)\right) \leqslant n\left[1 - f(u)\right] \quad \text{for all } f \in B_{X^*}, \text{ and } n \geqslant \max\left\{1 + \frac{1}{\alpha_h}, C_h\right\}.$$

Since $\bigcup_{n\in\mathbb{N}} F_n$ is second category there exists an $N \in \mathbb{N}$, $h_0 \in B_X$ and $\delta > 0$ such that $h_0 + \delta B_X \subset F_N$. Therefore

$$\operatorname{dist}_h^N\left(f, J(u)\right) \leqslant N\left(1 - f(u)\right) \text{ for all } h \in h_0 + \delta B_X, \text{ and all } f \in B_{X^*}. \tag{3.3}$$

Given $\varepsilon > 0$, and using the weak*-density of W in B_{X^*} we fix $x^* \in W$ such that

$$(a)\ \ x^*(h_0) > \sup\left\{f(h_0) : f \in J(u)\right\} - \frac{\delta\varepsilon}{2} \quad \text{and} \quad (b)\ \ x^*(u) > 1 - \frac{\delta^N\varepsilon^N}{2^N N}. \tag{3.4}$$

Now suppose $\operatorname{dist}\left(x^*, J(u)\right) > \varepsilon$. Because $J(u)$ is weak*-compact and convex, we use the separation theorem to choose $h \in S_X$ such that

$$x^*(h) > \sup\left\{f(h) : f \in J(u)\right\} + \varepsilon.$$

This implies that

$$\begin{aligned} x^*(\delta h) &> \delta\left(\sup\left\{f(h) : f \in J(u)\right\} + \varepsilon\right), \text{ which with (3.4a) implies} \\ x^*(h_0 + \delta h) &> \sup\left\{f(h_0 + \delta h) : f \in J(u)\right\} + \frac{\delta\varepsilon}{2}. \end{aligned}$$

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Consequently $\mathrm{dist}_{h_0+\delta h}\big(x^*, J(u)\big) > \delta\varepsilon/2$ which with (3.3) and then (3.4b) implies that

$$\frac{\delta^N\varepsilon^N}{2^N} < N\big(1 - x^*(u)\big) < \frac{\delta^N\varepsilon^N}{2^N}.$$

This contradiction shows that $\mathrm{dist}\big(x^*, J(u)\big) \leqslant \varepsilon$, and so $\mathrm{dist}\big(W, J(u)\big) \leqslant \varepsilon$. ∎

With Lemma 3.3, using ideas and results based on Simons' inequality that were discovered by Godefroy [**12**] and later refined in [**3, 13**] we can readily prove our main result. First, let us recall that a subspace $N \subset X^*$ is said to be *1-norming* if $\|x\| = \sup\big\{|f(x)| : f \in N \cap B_{X^*}\big\}$ for all $x \in X$.

Theorem 3.4. *A Banach space X where the norm has a directionally Hölder right-hand derivative at each point on its sphere is an Asplund space and its dual X^* contains no proper closed 1-norming subspaces.*

Proof: To show that X is an Asplund space, it is sufficient to show that every separable subspace Y has separable dual Y^*, [**17**, p.24]. As in the proof of [**13**, Lemma 3], consider N a closed 1-norming subspace of Y^*. Because Y is separable, we fix a countable set W that is a weak* dense subset of $N \cap B_{Y^*}$. Then W is weak*-dense in B_{Y^*} because N is 1-norming. Because the restricted norm on Y has a directionally Hölder right-hand derivative at each point on its sphere, it follows from Lemma 3.3 that $\big(W + \frac{1}{2}B_Y\big) \cap J(y) \neq \emptyset$ for each $y \in S_Y$. According to [**3**, Lemma 2.2], Y^* is the closed linear span of W. Therefore Y^* is separable and moreover $N = Y^*$. So X is Asplund and moreover Y^* has no proper closed 1-norming subspace. Because Y was an arbitrary separable subspace of X, [**13**, Lemma 4] shows that X has no proper closed 1-norming subspace. ∎

The norming result in this theorem has the following immediate consequence.

Corollary 3.5. *If a Banach space X has an equivalent dual norm on X^* with a directionally Hölder right-hand derivative at each point of its unit sphere then X is reflexive.*

We observe that not every Asplund space has an equivalent norm with a directionally Hölder right-hand derivative on its sphere.

Remark 3.6. Kunen's $C(K)$ Asplund space where K is constructed using the continuum hypothesis does not have an equivalent norm with a directionally Hölder right-hand derivative at each point on its sphere.

Proof: This follows from Theorem 3.4 and [**15**, Corollary 4.4(i)] which shows that given any equivalent norm on X, X^* has a closed proper 1-norming subspace. ∎

As a final illustration of the structural implications of directionally Hölder right-hand derivatives we sketch the following result that complements [**8**, Theorem 3.3].

Theorem 3.7. *A Banach space X is superreflexive if the norm has a directionally Hölder right-hand derivative at each point of a residual subset of S_X and the set of strongly exposed points of B_X is dense in the S_X.*

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PROOF: By Theorem 2.5, there exists an open ball U centred on S_X where the norm is uniformly Hölder differentiable. But this implies that the norm is uniformly smooth on $U \cap S_X$. Since $U \cap S_X$ contains a strongly exposed point of B_X we follow the proof of [4, Proposition V.1.3, p.188-189] to show that X is superreflexive. ∎

Comparable results from papers [**5, 16, 18**] lead us to pose the following questions.

QUESTIONS 3.8.

(a) Is a Banach space an Asplund space if it has a continuous bump function with directionally Hölder right-hand derivative?

(b) Is a Banach space superreflexive under the conditions in Corollary 3.5? Or more generally is it superreflexive if it has the RNP and a continuous bump function with a directionally Hölder right-hand derivative at each point of its domain?

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