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p-Radial Exceptional Sets and Conformal Mappings

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Abstract. For $p > 0$ and for a given set E of type G_δ in the boundary of the unit disc $\partial\mathbb{D}$ we construct a holomorphic function $f \in \mathbb{O}(\mathbb{D})$ such that

$$\int_{\mathbb{D}\setminus[0,1]E} |f|^p \, d\mathfrak{L}^2 < \infty \quad \text{and} \quad E = E^p(f) = \left\{ z \in \partial\mathbb{D} : \int_0^1 |f(tz)|^p \, dt = \infty \right\}.$$

In particular if a set E has a measure equal to zero, then a function f is constructed as integrable with power p on the unit disc $\mathbb{D}$.

1 Preface

This paper deals mainly with radial exceptional sets of the holomorphic functions in the unit disc $\mathbb{D}$. The set

$$E^p(f) = \left\{ z \in \partial\mathbb{D} : \int_0^1 |f(tz)|^p \, dt = \infty \right\}$$

is called a p-radial exceptional set for the holomorphic function $f \in \mathbb{O}(\mathbb{D})$. The above definition was inspired by the questions posed by Peter Pflug and Jacques Chaumat. Peter Pflug[1] asked whether there existed a domain $\Omega \subset \mathbb{C}^n$, a complex subspace M in $\mathbb{C}^n$ and a function f holomorphic in Ω, square-integrable, such that $f|_{M\cap\Omega}$ is non square-integrable.

A similar question was posed by Jacques Chaumat.[2] He wondered whether there exists a function f holomorphic in the ball $\mathbb{B}^n$ such that for any subspace M which is linear and complex in $\mathbb{C}^n$, the function $f|_{M\cap\mathbb{B}^n}$ is non square-integrable.

We can find many papers [1–4, 6, 8, 9] in the literature inspired by the above questions. In particular, functions that are non-integrable along some set of complex or real subspaces are considered. We studied the exceptional sets of type G_δ for holomorphic functions in Hartogs domains [8]. We presented the construction of the holomorphic function in the unit ball which is non-integrable along a pre-selected set of complex directions of type G_δ and F_σ [6]. Due to [1, 4] we know that for a

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convex domain Ω with a boundary of class C^1, it is possible to construct a holomorphic function f which is non-integrable with square along any real manifold M of the class C^1 crossing a boundary Ω transversally.

This paper deals with functions that are non-integrable along a fixed set of real directions in the unit disc $\mathbb{D}$. Observe that if E is the p-radial exceptional set for a holomorphic function f, then E is a set of type G_δ. (Indeed, let $u_\delta(z) := \int_0^1 |f(\delta tz)|^p\, dt$. We have $u_{\frac{n}{n+1}} \le u_{\frac{n+1}{n+2}} \le \cdots \le \lim_{n\to\infty} u_{\frac{n}{n+1}} = u$ and $E_\Omega(f) = u^{-1}(\infty)$.) We present our main result which gives a complete description of the p-radial exceptional sets for the holomorphic functions in the unit disc.

Theorem 2.5 *If $E \subset \partial\mathbb{D}$ is a set of type G_δ and $p > 0$, then there exists a holomorphic function $f \in \mathbb{O}(\mathbb{D})$ such that $\int_{\mathbb{D}\setminus[0,1]E} |f|^p\, d\mathfrak{L}^2 < \infty$ and $E = E^p(f)$.*

Observe that if E is a set which has a measure 0, then a function f is square-integrable.

2 Exceptional Sets

Denote $S(E) = [0.5, 1]E$. Each pair (i, j) is assigned to a natural number $\lfloor i, j\rfloor \ge 1$ so that

$$\lfloor i, j\rfloor < \lfloor k, l\rfloor \Leftrightarrow \begin{cases} i + j < k + l & \text{where } i + j \ne k + l, \\ i < k & \text{where } i + j = k + l \end{cases}$$

Lemma 2.1 *Fix $p \ge 1$. If $E = \bigcap_{i\in\mathbb{N}} U_i \subset \cdots \subset U_{i+1} \subset U_i \subset \cdots \subset \partial\mathbb{D}$, where $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of open sets in $\partial\mathbb{D}$, then there exist the sequences of compact sets $\{T_{i,j}\}_{i,j\in\mathbb{N}}$, $\{D_{i,j}\}_{i,j\in\mathbb{N}}$ in $\partial\mathbb{D}$ such that*

(i) $U_i = \bigcup_{j\in\mathbb{N}} T_{i,j}$,
(ii) $T_{i,j} \cap D_{i,j} = \varnothing$,
(iii) $\partial\mathbb{D} \setminus E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\lfloor i,j\rfloor \ge n} D_{i,j}$,
(iv) $\sum_{j\in\mathbb{N}} (\mathfrak{L}^2(S(\partial\mathbb{D} \setminus (E \cup D_{i,j}))))^{\frac{1}{p}} \le 9(\mathfrak{L}^2(S(U_i \setminus E)))^{\frac{1}{p}} + 2^{-i}$.

Proof Consider a sequence $\{r_{i,j}\}_{i,j\in\mathbb{N}}$ such that $0 < \cdots < 2r_{i,j+1} < r_{i,j}$. Denote

$$T_{i,j} := \{z \in U_i : \; r_{i,j+1} \le \inf_{w\in\partial U_i} \|z - w\| \le r_{i,j}\},$$

$$D_{i,j} := \{z \in \partial\mathbb{D} : \; r_{i,j+1} - r_{i,j+2} \le \inf_{w\in T_{i,j}} \|z - w\|\},$$

$$G_{i,j} := S\big(U_i \cap \big(\overline{U_i \setminus \bigcup_{1\le m\le j} T_{i,j}}\big)\big),$$

$$H_{i,j} := S(\partial\mathbb{D} \setminus (D_{i,j} \cup E)).$$

Assume $T_{i,-1} = T_{i,0} = \varnothing$. Select a sequence $\{r_{i,j}\}_{i,j\in\mathbb{N}}$. Let $r_{i,1} = 2$. Moreover, let $r_{i,2}$ be so small that $\mathfrak{L}^2(G_{i,1}) < \frac{1}{9}2^{-i-1}$ and $0 < 2r_{i,2} < r_{i,1}$. The other numbers $r_{i,j+1}$

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are selected so that $\mathfrak{L}^2(G_{i,j}) < \frac{1}{9}2^{-i-j}$ and $0 < 2r_{i,j+1} < r_{i,j}$. As $S(T_{i,j+1}) \subset G_{i,j}$, therefore we have

$$\sum_{j\in\mathbb{N}}\big(\mathfrak{L}^2(S(T_{i,j}\setminus E))\big)^{\frac{1}{p}} < \big(\mathfrak{L}^2(S(U_i\setminus E))\big)^{\frac{1}{p}} + \frac{1}{9}\sum_{j=1}^{\infty}2^{-i-j}. \tag{2.1}$$

We show that the sets $T_{i,j}$ and $D_{i,j}$ fulfill conditions (i)–(iv).

Conditions (i) and (ii) result directly from the definition. Moreover, it can be easily seen that $\partial\mathbb{D}\setminus U_i \subset D_{i,j}$.

Step 1: If $k - j \geq 2$, then we have the inequality $\|z-w\| \geq r_{i,j+1} - r_{i,j+2}$ for $z \in T_{i,j}$ and $w \in T_{i,k}$. Assume that $z \in T_{i,j}$, $w \in T_{i,k}$ and $\|z-w\| < r_{i,j+1} - r_{i,j+2}$. In this case there exists a point $u \in \partial U_i$ such that $\|u-w\| \leq r_{i,k} \leq r_{i,j+2}$. We can estimate

$$r_{i,j+1} \leq \|u-z\| \leq \|u-w\| + \|w-z\| < r_{i,j+2} + r_{i,j+1} - r_{i,j+2} \leq r_{i,j+1},$$

which is impossible.

Step 2: If $|k-j| \geq 2$, then $T_{i,k} \subset D_{i,j}$. Assume that $x \in T_{i,k}\setminus D_{i,j}$. Then there exists a point $y \in T_{i,j}$ such that $\|x-y\| < r_{i,j+1} - r_{i,j+2}$. If $k-j \geq 2$, then we get inconsistency with the inequality from Step 1. If $j-k \geq 2$, then

$$\|x-y\| < r_{i,j+1} - r_{i,j+2} < r_{i,j+1} < r_{i,k+2} < r_{i,k+1} - r_{i,k+2},$$

which is also impossible on the basis of Step 1.

Step 3: We have property (iii). Fix $z \in \partial\Omega\setminus E$. If $z \notin U_0$, then $z \in D_{i,j}$ for any $i,j \in \mathbb{N}$, as $\partial\Omega\setminus U_i \subset D_{i,j}$ and $U_{i+1} \subset U_i$. If $z \in U_0$, then there exists $m \in \mathbb{N}$ such that $z \notin U_i$ for $i \geq m$ and $z \in U_i$ for $i < m$. Moreover, there exist numbers k_i for $i < m$ such that $z \in T_{i,k_i}$ for $i < m$. Let $n = 2 + \max\{m, k_1, \dots, k_m\}$. From Step 2 it follows that $z \in D_{i,j}$, when $i + j > n$. If $\lfloor i,j\rfloor > \lfloor n,1\rfloor$, then $i+j \geq n+1$. Therefore $z \in \bigcup_{n\in\mathbb{N}}\bigcap_{\lfloor i,j\rfloor>\lfloor n,1\rfloor} D_{i,j}$, which finishes the proof of Step 3.

Step 4: We have the estimation

$$\sum_{j\in\mathbb{N}}(\mathfrak{L}^2(H_{i,j}))^{\frac{1}{p}} \leq 9\big(\mathfrak{L}^2(S(U_i\setminus E))\big)^{\frac{1}{p}} + 2^{-i},$$

which is property (iv). As $T_{i,k} \subset D_{i,j}$, when $|k-j| \geq 2$ (Step 2) and $\partial\mathbb{D}\setminus U_i \subset D_{i,j}$, therefore $\partial\mathbb{D}\setminus D_{i,j} \subset \bigcup_{|k-j|\leq 1} T_{i,k}$. In particular $H_{i,j} \subset \bigcup_{|k-j|\leq 1} S(T_{i,k}\setminus E)$. Observe that if $0 \leq x_i, a_i$ and $x_i \leq a_{i-1} + a_i + a_{i+1}$, then

$$\sum_{i\in\mathbb{N}} x_i^{\frac{1}{p}} \leq \sum_{i\in\mathbb{N}}(a_{i-1}+a_i+a_{i+1})^{\frac{1}{p}} \leq \sum_{i\in\mathbb{N}}\big(3\max\{a_{i-1},a_i,a_{i+1}\}\big)^{\frac{1}{p}}$$
$$\leq 3\sum_{i\in\mathbb{N}}\big(a_{i-1}^{\frac{1}{p}} + a_i^{\frac{1}{p}} + a_{i+1}^{\frac{1}{p}}\big) \leq 9\sum_{i\in\mathbb{N}} a_{i-1}^{\frac{1}{p}}.$$

Using the inequality (2.1) we can estimate the following:

$$\sum_{j\in\mathbb{N}}(\mathfrak{L}^2(H_{i,j}))^{\frac{1}{p}} \le 9\sum_{j\in\mathbb{N}}\mathfrak{L}^2(S(T_{i,j}\setminus E))^{\frac{1}{p}}$$
$$\le 9\big(\mathfrak{L}^2(S(U_i\setminus E))\big)^{\frac{1}{p}} + 2^{-i}.$$ ■

Lemma 2.2 *If $T=\overline{T}\subset\partial\mathbb{D}$, then there exists a function $h\in\mathbb{O}(\mathbb{D})\cap C^\infty(\overline{\mathbb{D}})$ such that*

(i) $|h(z)|\le|z|$ *for* $z\in\overline{\mathbb{D}}$;
(ii) $|h(z)|=1$ *if and only if* $z\in T$;
(iii) $|h'(z)|<2$ *for* $z\in\overline{\mathbb{D}}$.

Proof There exists a domain U convex with a boundary of the class C^∞ such that $T=\partial\mathbb{D}\cap\partial U$, $\overline{\mathbb{D}}\setminus T\subset U$. Let g be a conformal mapping $g\colon U\to\mathbb{D}$ such that $g(0)=0$. On the basis of [10, Theorems 2.6, 3.6], we know that there exists an extension to a homeomorphism $g\colon\overline{U}\to g(\overline{U})=\overline{\mathbb{D}}$ of the class C^∞ in such a way that $g'(z)\neq 0$ for $z\in\overline{U}$. Therefore, there exists a natural number m such that $|g'(z)|<\sqrt{m}-1$ and $\left|\frac{g(z)}{z}\right|>\frac{1}{\sqrt{m}}$ for $z\in\overline{\mathbb{D}}$. We define

$$h(z):=\left(\frac{g(z)}{z}\right)^{\frac{1}{m}}z.$$

As $g^{-1}(0)=\{0\}$ and $g'(0)\neq 0$, therefore the function h is a properly defined holomorphic function on $\mathbb{D}$. Moreover $h\in C^\infty(\overline{\mathbb{D}})$ and $|h(z)|\le|z|$ for $z\in\mathbb{D}$. It can also be easily observed that $|h(z)|=1$ if and only if $z\in T$. We can estimate

$$|h_m'(z)|\le\frac{1}{m}\left|\frac{g(z)}{z}\right|^{\frac{1}{m}-1}\left|\frac{g'(z)z+g(z)}{z^2}\right||z|+\left|\frac{g(z)}{z}\right|^{\frac{1}{m}}<\frac{m}{m}+1=2$$

for $z\in\overline{\mathbb{D}}$, which finishes the proof. ■

Theorem 2.3 *Fix $p>0$. If $T=\overline{T}\subset\partial\mathbb{D}$, then for $\varepsilon>0$ and for each closed set D contained in $\overline{\mathbb{D}}\setminus T$ there exists a function $f\in\mathbb{O}(\mathbb{D})\cap C(\overline{\mathbb{D}})$ such that*

(i) $\int_0^1|f(zt)|^p\,dt>1$ *for* $z\in T$;
(ii) $|f(z)|\le\varepsilon$ *for* $z\in D$;
(iii) $\int_0^1|f(zt)|^p\,dt\le 2$ *for* $z\in\partial\mathbb{D}$.

Proof Fix a set D which is closed and such that $D\subset\overline{\mathbb{D}}\setminus T$ and the number $\varepsilon>0$.

On the basis of Lemma 2.2, there exists the function $h\in\mathbb{O}(\mathbb{D})\cap C^\infty(\overline{\mathbb{D}})$ and $\delta\in(1,2)$ such that

$$|h(z)|\le|z| \text{ for } z\in\overline{\mathbb{D}},\quad |h(z)|=1\iff z\in T,\quad |h'(z)|<\delta \text{ for } z\in\overline{\mathbb{D}}.$$

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In particular

$$\begin{aligned} h(z) - h(w) &= \int_0^1 \frac{d}{dt} h(zt + (1-t)(w-z))\, dt \\ &= (z-w) \int_0^1 h'(zt + (1-t)(w-z))\, dt \end{aligned}$$

and

$$|h(z) - h(w)| \le \delta |z-w|$$

Obviously $|h(z)| < 1$ when $z \in D$. In particular, there exists a natural number n such that $(2np+2)^{\frac{1}{p}} |h(z)|^n \le \varepsilon$ for $z \in D$. Let $f(z) = (2np+2)^{\frac{1}{p}} h^n(z)$.

Obviously $f \in \mathbb{O}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and $|f(z)| \le \varepsilon$ for $z \in D$.

If $z \in T$, then $|h(z)| = 1$ and $1 - |h(zt)| \le |h(z) - h(zt)| \le \delta(1-t)$ for $t \in [0,1]$. In particular, $1 - \delta + \delta t \le |h(zt)|$ for $z \in T$ and $t \in [0,1]$. We can estimate

$$\begin{aligned} \int_0^1 |f(zt)|^p\, dt &= (2np+2) \int_0^1 |h(zt)|^{np}\, dt \\ &> (2np+2) \int_{1-\frac{1}{\delta}}^1 (1 - \delta + \delta t)^{np}\, dt \\ &= \frac{2}{\delta} \big[(1-\delta+\delta t)^{np+1} \big]_{1-\frac{1}{\delta}}^{1} = \frac{2}{\delta} > 1, \end{aligned}$$

for $z \in T$. Moreover,

$$\int_0^1 |f(zt)|^p\, dt = (2np+2) \int_0^1 |h^n(zt)|^p\, dt \le (2np+2) \int_0^1 t^{np}\, dt = 2.$$

for $z \in \partial\mathbb{D}$, which finishes the proof. ■

Proposition 2.4 *If $K \subset \partial\mathbb{D}$, the function u is any non-negative measurable function and $S(K)$ is a measurable set, then we have the following inequality*

$$\int_{S(K)} u\, d\mathfrak{L}^2 \le 4\mathfrak{L}^2(S(K)) \sup_{w \in K} \int_0^1 u(wt)\, dt.$$

Proof There exists a set $\Theta \subset [0, 2\pi]$ such that

$$\begin{aligned} \int_{S(K)} u\, d\mathfrak{L}^2 &= \int_\Theta \int_{0.5}^1 u(re^{i\theta}) r\, dr d\theta \le \int_\Theta \int_{0.5}^1 u(re^{i\theta})\, dr d\theta \\ &\le \int_\Theta \sup_{\theta \in \Theta} \int_0^1 u(te^{i\theta})\, dt d\theta \\ &\le 4 \sup_{\theta \in \Theta} \Big(\int_0^1 u(te^{i\theta})\, dt \Big) \int_\Theta \int_{0.5}^1 r\, dr d\theta \\ &\le 4\mathfrak{L}^2(S(K)) \sup_{w \in K} \int_0^1 u(wt)\, dt. \end{aligned}$$

■

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Theorem 2.5 *Fix $p > 0$. If E is a set of type G_δ in $\partial\mathbb{D}$, then there exists a holomorphic function $f \in \mathbb{O}(\mathbb{D})$ such that $E = E^p(f)$ and $\int_{\mathbb{D}\setminus[0,1]E} |f|^p d\mathfrak{L}^2 < \infty$.*

Proof If $p > 1$, then let $q = p$. If $0 < p \le 1$, then $q = 1$. There exist open sets U_i in $\partial\mathbb{D}$ such that $E = \bigcap_{i\in\mathbb{N}} U_i \subset \cdots \subset U_{i+1} \subset U_i$ and $\mathfrak{L}^2(S(U_i \setminus E)) \le 2^{-qi}$. On the basis of Lemma 2.1, there exist two sequences of compact sets $\{T_{i,j}\}_{i,j\in\mathbb{N}}$, $\{D_{i,j}\}_{i,j\in\mathbb{N}}$ in $\partial\mathbb{D}$ such that

- $U_i = \bigcup_{j\in\mathbb{N}} T_{i,j}$;
- $T_{i,j} \cap D_{i,j} = \varnothing$;
- $\partial\mathbb{D} \setminus E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\lfloor i,j\rfloor \ge n} D_{i,j}$;
- $\sum_{j\in\mathbb{N}} (\mathfrak{L}^2(S(\partial\mathbb{D} \setminus (E \cup D_{i,j}))))^{\frac{1}{q}} \le 9(\mathfrak{L}^2(S(U_i \setminus E)))^{\frac{1}{q}} + 2^{-i}$.

For the sets $T_{i,j}, D_{i,j}$ we select functions $f_{i,j} \in \mathbb{O}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and real numbers $a_{i,j}, b_{i,j}$ such that

(i) $0 \le a_{i,j} < b_{i,j} < a_{k,l} < \lim_{\lfloor n,m\rfloor\to\infty} a_{n,m} = 1$ when $\lfloor i,j\rfloor < \lfloor k,l\rfloor$;
(ii) $|f_{i,j}(z)|^p \le 2^{-2q\lfloor i,j\rfloor}$ for $z \in K_{i,j} := \overline{a_{i,j}\mathbb{D} \cup [0,1]D_{i,j}}$;
(iii) $\int_{a_{i,j}}^{b_{i,j}} |f_{i,j}(zt)|^p\, dt > (1 - 2^{-2\lfloor i,j\rfloor})^q$ for $z \in T_{i,j}$;
(iv) $\int_{b_{i,j}}^{1} |f_{i,j}(zt)|^p\, dt \le 2^{-2q\lfloor i,j\rfloor}$ for $z \in \partial\mathbb{D}$;
(v) $\int_0^1 |f_{i,j}(zt)|^p\, dt \le 2$ for $z \in \partial\mathbb{D}$.

Let $a_{1,1} = 0$. On the basis of Theorem 2.3, we select the function $f_{1,1} \in \mathbb{O}(\Omega) \cap C(\overline{\Omega})$ (for the set $T_{1,1}$) such that the conditions (ii), (iii), and (v) are fulfilled (for $b_{1,1} = 1$).

As $f_{1,1} \in C(\overline{\mathbb{D}})$, therefore there exists a number $b_{1,1} \in (a_{1,1}, 1)$ such that

$$\int_{a_{1,1}}^{b_{1,1}} |f_{1,1}(zt)|^p\, dt > (1 - 2^{-2\lfloor 1,1\rfloor})^q$$

for $z \in T_{1,1}$ and

$$\int_{b_{1,1}}^{1} |f_{1,1}(zt)|^p\, dt \le 2^{-2q\lfloor 1,1\rfloor}$$

for $z \in \partial\mathbb{D}$. Therefore a triplet $(a_{1,1}, b_{11}, f_{1,1})$ was properly selected.

Now fix indices i, j. Assume that we have already selected triplets $(a_{k,l}, b_{k,l}, f_{k,l})$ such that conditions (i)–(v) are fulfilled when $\lfloor k,l\rfloor < \lfloor i,j\rfloor$. Let $a_{i,j} \in (0,1)$ be such that $b_{k,l} < a_{i,j}$ and $2(1 - a_{i,j}) \le 1 - a_{k,l}$ when $\lfloor k,l\rfloor < \lfloor i,j\rfloor$. On the basis of Theorem 2.3, there exists a holomorphic function $f_{i,j} \in \mathbb{O}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

- $\int_0^1 |f_{i,j}(zt)|^p\, dt > 1$ for $z \in T_{i,j}$;
- $|f_{i,j}(z)|^p \le 2^{-2q\lfloor i,j\rfloor}$ for $z \in K_{i,j}$;
- $\int_0^1 |f_{i,j}(zt)|^p\, dt \le 2$ for $z \in \partial\mathbb{D}$.

As $f_{i,j} \in C(\overline{\mathbb{D}})$ and $a_{i,j}\mathbb{D} \subset K_{i,j}$, therefore there exists $b_{i,j} \in (a_{i,j}, 1)$ such that

$$\int_{a_{i,j}}^{b_{i,j}} |f_{i,j}(zt)|^p \, dt > (1 - 2^{-2\lfloor 1,1 \rfloor})^q$$

for $z \in T_{i,j}$, and

$$\int_{b_{i,j}}^{1} |f_{i,j}(zt)|^p \, dt \leq 2^{-2q\lfloor i,j \rfloor}$$

for $z \in \partial\mathbb{D}$. Observe that a triplet $(a_{i,j}, b_{i,j}, f_{i,j})$ has the properties (i)–(v).

We show that the function f defined by the formula $f(z) = \sum_{i,j\in\mathbb{N}} f_{i,j}(z)$ fulfills required conditions. As $\lim_{\lfloor i,j \rfloor\to\infty} a_{i,j} = 1$, therefore $\bigcup_{i,j\in\mathbb{N}} K_{i,j} = \mathbb{D}_{i,j}$. In particular, condition (ii) implies that f is a holomorphic function.

Let $z \in E$. If $z \in T_{i,j}$, then using the conditions (ii)–(iv) we can estimate as follows:

$$\begin{aligned}
\Big(\int_{a_{i,j}}^{b_{i,j}} |f(zt)|^p \, dt\Big)^{\frac{1}{q}} &\geq \Big(\int_{a_{i,j}}^{b_{i,j}} |f_{i,j}(zt)|^p \, dt\Big)^{\frac{1}{q}} - \sum_{\lfloor k,l \rfloor < \lfloor i,j \rfloor} \Big(\int_{a_{i,j}}^{b_{i,j}} |f_{k,l}(zt)|^p \, dt\Big)^{\frac{1}{q}} \\
&\quad - \sum_{\lfloor k,l \rfloor > \lfloor i,j \rfloor} \Big(\int_{a_{i,j}}^{b_{i,j}} |f_{k,l}(zt)|^p \, dt\Big)^{\frac{1}{q}} \\
&> 1 - 2^{-2\lfloor i,j \rfloor} - \sum_{\lfloor k,l \rfloor < \lfloor i,j \rfloor} \Big(\int_{b_{k,l}}^{1} |f_{k,l}(zt)|^p \, dt\Big)^{\frac{1}{q}} \\
&\quad - \sum_{\lfloor k,l \rfloor > \lfloor i,j \rfloor} \sup_{z\in a_{k,l}\mathbb{D}} |f_{k,l}(z)|^{\frac{p}{q}} \\
&\geq 1 - 2^{-2\lfloor i,j \rfloor} - \sum_{\lfloor k,l \rfloor < \lfloor i,j \rfloor} 2^{-2\lfloor k,l \rfloor} - \sum_{\lfloor k,l \rfloor > \lfloor i,j \rfloor} 2^{-2\lfloor k,l \rfloor} \\
&> 1 - \sum_{m=1}^{\infty} 2^{-2m} = \frac{2}{3}.
\end{aligned}$$

There exists a sequence $\{k_i\}_{i\in\mathbb{N}}$ such that $z \in T_{i,k_i}$ for $i \in \mathbb{N}$. In particular, we can estimate

$$\int_0^1 |f(zt)|^p \, dt \geq \sum_{i=1}^{\infty} \int_{a_{i,k_i}}^{b_{i,k_i}} |f(zt)|^p \, dt > \sum_{k=1}^{\infty} (\frac{2}{3})^q = \infty.$$

Fix $z \in \partial\mathbb{D} \setminus E$. As $\partial\mathbb{D} \setminus E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\lfloor i,j \rfloor \geq n} D_{i,j}$, therefore there exists $m \in \mathbb{N}$ such that $z \in D_{i,j}$ when $\lfloor i, j \rfloor > m$. Using condition (ii) we may estimate as follows:

$$\begin{aligned}
\Big(\int_0^1 |f(zt)|^p \, dt\Big)^{\frac{1}{q}} &\leq \sum_{\lfloor i,j \rfloor < m} \Big(\int_0^1 |f_{i,j}(zt)|^p \, dt\Big)^{\frac{1}{q}} + \sum_{\lfloor i,j \rfloor \geq m} \Big(\int_0^1 |f_{i,j}(zt)|^p \, dt\Big)^{\frac{1}{q}} \\
&\leq \sum_{\lfloor i,j \rfloor < m} \Big(\int_0^1 |f_{i,j}(zt)|^p \, dt\Big)^{\frac{1}{q}} + \sum_{\lfloor i,j \rfloor > m} 2^{-2\lfloor i,j \rfloor} < \infty
\end{aligned}$$

for $z \in \partial\mathbb{D} \setminus E$. So $E = E^p(f)$.

We show also that $\int_{\mathbb{D}\setminus S(E)} |f(zt)|^p \, d\mathfrak{L}^2 < \infty$. Let $H_{i,j} := \partial\mathbb{D} \setminus (D_{i,j} \cup E)$. On the basis of Proposition 2.4 and due to property (v), we can estimate as follows:

$$\int_{S(H_{i,j})} |f_{i,j}|^p d\mathfrak{L}^2 \leq 4\mathfrak{L}^2(S(H_{i,j})) \sup_{w\in H_{i,j}} \int_0^1 |f_{i,j}(wt)|^p dt \leq 8\mathfrak{L}^2(S(H_{i,j})).$$

Now it is enough to prove that $\int_{S(\partial\mathbb{D}\setminus E)} |f|^p d\mathfrak{L}^2 < \infty$. On the basis of property (ii) it follows that

$$\begin{aligned}
\Big(\int_{S(\partial\mathbb{D}\setminus E)} |f|^p \, d\mathfrak{L}^2\Big)^{\frac{1}{q}} &\leq \sum_{i,j\in\mathbb{N}} \Big(\int_{S(\partial\mathbb{D}\setminus E)} |f_{i,j}|^p \, d\mathfrak{L}^2\Big)^{\frac{1}{q}} \\
&\leq \sum_{i,j\in\mathbb{N}} \Big(\int_{S(D_{i,j})} |f_{i,j}|^p d\mathfrak{L}^2 + \int_{S(H_{i,j})} |f_{i,j}|^p d\mathfrak{L}^2\Big)^{\frac{1}{q}} \\
&\leq 2\sum_{i,j\in\mathbb{N}} \Big(\int_{S(D_{i,j})} |f_{i,j}|^p d\mathfrak{L}^2\Big)^{\frac{1}{q}} + 2\sum_{i,j\in\mathbb{N}} \Big(\int_{S(H_{i,j})} |f_{i,j}|^p \, d\mathfrak{L}^2\Big)^{\frac{1}{q}} \\
&\leq 2\sum_{i,j\in\mathbb{N}} 2^{-2\lfloor i,j\rfloor} + 2\sum_{i,j\in\mathbb{N}} (8\mathfrak{L}^2(S(H_{i,j})))^{\frac{1}{q}} \\
&\leq 2 + 2\sum_{i\in\mathbb{N}} 72(\mathfrak{L}^2(S(U_i \setminus E)))^{\frac{1}{q}} + 2^{-i} \\
&\leq 2 + 145\sum_{i\in\mathbb{N}} 2^{-i} < \infty.
\end{aligned}$$

■

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