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PAPER

Positive solutions to the prey–predator equations with dormancy of predators

Novrianti[1,*], Okihiro Sawada[2] and Naoki Tsuge[3]

[1]Faculty of Engineering, STITEKNAS Jambi, Jl. Lintas Timur RT. 15, Mendalo Darat, Jambi, Indonesia, [2]Faculty of Engineering, Kitami Institute of Technology, 165 Koen-cho Kitami, Japan and [3]Faculty of Education, Gifu University, 1-1 Yanagido Gifu, Japan
*Correspondence author. Email: novrianti@stiteknas.ac.id

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Abstract
The time-global unique classical positive solutions to the reaction–diffusion equations for prey–predator models with dormancy of predators are constructed. The feature appears on the nonlinear terms of Holling type II functional response. The crucial step is to establish time-local positive classical solutions by using a new approximation associated with time-evolution operators. Although the system does not equip usual comparison principle for solutions to partial differential equation, a priori bounds are derived by enclosing and renormalising arguments of solutions to the corresponding ordinary differential equations. Furthermore, time-global existence, invariant regions and asymptotic behaviours of solutions follow from such a priori bounds.

1. Introduction

Some systems of reaction–diffusion equations have attracted much interest as a prototype model for oscillation and pattern formation in the book by Murray [1] and the references therein. The main purpose of this paper is to present mathematical tools for studying the positivity of solutions of reaction–diffusion systems. So, we deal with the following reaction–diffusion equations in the whole space $\mathbb{R}^n$ for $n \in \mathbb{N}$.

$$\begin{cases} \partial_t u = \delta \Delta u + r\,(1 - u/k)\, u - \gamma uv/(u+h), \\ \partial_t v = d\Delta v + \mu uv/(u+h) + \alpha w - \theta v - \iota v - \beta v^2, \\ \partial_t w = \nu uv/(u+h) + \theta v - \alpha w - \tilde{\iota} w. \end{cases} \tag{LV}$$

This is a system of Lotka–Volterra type equations with diffusions. More precisely, this is a prey–predator model with dormancy of predators in [2, 3]. Here, $u := u(x,t)$, $v := v(x,t)$ and $w := w(x,t)$ stand for the density of prey, the density of active predator and the density of dormant predator, respectively, as the unknown scalar positive (or, nonnegative) functions at $x \in \mathbb{R}^n$ and $t > 0$. To avoid effects from boundaries, the Cauchy problem is considered, in what follows. We have denoted the nonnegative constants by

- δ the diffusion coefficient of prey
- h the constant of foraging efficiency and handling time
- d the diffusion coefficient of active predator

α the rate of awakening
r the growth rate of prey
β the mortality rate by competitions of active predators
k the capacity of prey
$\tilde{\iota}$ the mortality rate of dormant predator
γ the mortality rate of prey
ι the mortality rate of active predator
θ the rate of sleeping

Also, $\mu := \mu(u)$ and $\nu := \nu(u)$ are smooth positive functions of u denoting growth rates of active and dormant predators, respectively. In some mathematical research, $\mu(u)$ is given as a sigmoid function as

$$\mu(u) := \gamma\{1 + \tanh \xi(u - \eta)\}/2 \in (0, \gamma)$$

with some constants ξ and η, besides, $\nu(u) := \gamma - \mu(u)$; see e.g. [2]. In addition, we have used the notations of differentiation

$$\partial_t := \frac{\partial}{\partial t}, \quad \Delta := \sum_{i=1}^{n} \partial_i^2 \quad \text{with} \quad \partial_i := \frac{\partial}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, n.$$

By change of variables and constants, we may replace by $\delta = 1$, $k = 1$, $r = 1$ and $\beta = 1$. For the simplicity of notations, we put $m := \theta + \iota$, $\rho := \alpha + \tilde{\iota}$, in addition, assume that μ and ν are positive constants independent of u. So, we consider the initial value problem:

$$\begin{cases} \partial_t u = \Delta u + (1 - u)u - \gamma uv/(u + h) & \text{in } \mathbb{R}^n \times (0, \infty), \\ \partial_t v = d\Delta v + \mu uv/(u + h) + \alpha w - (m + v)v & \text{in } \mathbb{R}^n \times (0, \infty), \\ \partial_t w = \nu uv/(u + h) + \theta v - \rho w & \text{in } \mathbb{R}^n \times (0, \infty), \\ (u, v, w)\big|_{t=0} = (u_0, v_0, w_0) & \text{in } \mathbb{R}^n. \end{cases} \tag{P}$$

The bifurcation between stability and instability of stationary solutions to (LV) was concerned with some specific parameters, associated with numerical investigation [3]. Furthermore, a numerical study of Turing instability on (LV) was done [2]. Besides, in this paper, we focus on the mathematical theory for the existence of time-global nonnegative unique classical solutions to (P) with nonnegative initial data, invariant regions and asymptotic behaviours. To do so, we estimate a priori bounds for solutions to (P) by enclosing and renormalising arguments of solutions to the corresponding ordinary differential equations.

This paper is organised as follows. In Section 2, we will present the main results of this paper and related works. In Section 3, we define function spaces and recall some properties of the heat semigroup and time-evolution operators. Section 4 will be devoted to the proof of the time-local existence of nonnegative unique classical solutions with nonnegative initial data. We will discuss the time-global solvability in Section 5, deriving a priori estimates of solutions and their derivatives, due to renormalisation arguments. In Section 6, some invariant regions and asymptotic behaviours of solutions to (p) will be argued.

Throughout this paper, we denote positive constants by C the value of which may differ from one occasion to another.

2. Main results

We will state the main results in this paper. For the definition and properties of the set of all bounded and uniformly continuous functions BUC, see Section 3, as well as BUC^1.

Theorem 1. *Let $n \in \mathbb{N}$, $d, h > 0$, and let $m, \theta, \rho, \alpha, \gamma, \mu, \nu \geq 0$. If $u_0, v_0 \in BUC(\mathbb{R}^n)$ and $w_0 \in BUC^1(\mathbb{R}^n)$ are nonnegative, then there exists a nonnegative time-global unique classical solutions to (P).*

Remark 1. (i) We can find at most five stationary constant states, including the trivial solution $(0, 0, 0)$. The trivial solution is always unstable, if $u_0 > 0$. Besides, the stabilities of non-trivial constant states depend on parameters; see Theorem 2 and Remark 4 in below. (ii) Even if μ and ν are positive smooth functions of u, the same time-global solvability can be proved. Here, we may relax the condition $\gamma = \mu + \nu$, at least mathematically. (iii) In the case of $d = 0$, we may obtain the same assertion, whenever $v_0 \in \mathrm{BUC}^1$. (iv) When $u_0, v_0 \in L^\infty$, we may also get the similar assertion, although there is a lack of continuity of solutions in t at $t = 0$.

We will explain the strategy of the proof of Theorem 1, briefly. Using the heat semigroups, (P) is written as the forms of integral equations:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left[(1-u)\,u - \frac{\gamma uv}{u+h}\right](s)\,ds, \tag{1}$$

$$v(t) = e^{dt\Delta}v_0 + \int_0^t e^{d(t-s)\Delta}\left[\frac{\mu uv}{u+h} + \alpha w - (m+\nu)v\right](s)\,ds, \tag{2}$$

$$w(t) = e^{-\rho t}w_0 + \int_0^t e^{-\rho(t-s)}\left[\frac{\nu uv}{u+h} + \theta v\right](s)\,ds. \tag{3}$$

Once we obtain the existence of solutions to (1)–(3), the uniqueness and the regularity of solutions follow from these forms. However, it is not easy to get the existence of solutions, at least directly. Because, the nonnegativity of solutions or its approximation seems to be not ensured, in general. In fact, the following standard iteration scheme is often employed:

$$\bar{u}_{\ell+1}(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left[(1-\bar{u}_\ell)\,\bar{u}_\ell - \frac{\gamma\bar{u}_\ell\bar{v}_\ell}{\bar{u}_\ell + h}\right](s)\,ds.$$

See, for example, the book by Smoller [7]. With this approximation, it is not clear how to show $\bar{u}_\ell > -h$ for $\ell \geq 2$, unfortunately. Thus, we have to look for the another approximation or integral forms for proving the existence of nonnegative solutions.

There are many mathematical articles to prove the existence of partial differential equation (PDE) with the Holling type II nonlinear terms. As far as the authors know, the exiting techniques are as follows.

- Under the a priori assumption of the positivity of solutions or approximation, the existence of solutions is easily proved. However, it is not clear how to get the positivity.
- Apply the fixed point theorem of the mapping in the set of positive functions $K_+ := \{u, v, w > 0\}$. However, we have to verify its domain and range in K_+.
- Consider the modified system taking the absolute value to u or $\bar{u}_\ell$ in the denominator. To do so, we obtain weak solutions. However, it is not clear how to show that the weak solutions satisfy (P) in the classical sense, that is, $u \in C((0, T); C^2)$.
- By standard arguments, the corresponding ordinary differential equation (ODE) admit time-local solutions. So, the solution to (P) is estimated from below by that to ODE. However, the comparison principle can be applicable, after showing the existence of classical solutions.

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Each approach in above has a flaw. Hence, we employ the same sprits in [6]. To overcome difficulties, we will construct the solutions as the limits of the following successive approximations of abstract forms:

$$\begin{aligned}
\partial_t u_{\ell+1} &= \Delta u_{\ell+1} - \left(u_\ell + \frac{\gamma v_\ell}{u_\ell + h} \right) u_{\ell+1} + u_\ell, \quad u_{\ell+1}|_{t=0} = u_0, \\
\partial_t v_{\ell+1} &= d\Delta v_{\ell+1} - (m + v_\ell)\, v_{\ell+1} + \frac{\mu u_\ell v_\ell}{u_\ell + h} + \alpha w_\ell, \quad v_{\ell+1}|_{t=0} = v_0, \\
\partial_t w_{\ell+1} &= -\rho w_{\ell+1} + \frac{\nu u_\ell v_\ell}{u_\ell + h} + \theta v_\ell, \quad w_{\ell+1}|_{t=0} = w_0
\end{aligned}$$

for $\ell \in \mathbb{N}$. Our idea is to involve the coefficients of negative terms into the generators. We can rewrite them as

$$u_{\ell+1}(t) = U_\ell(t, 0)u_0 + \int_0^t U_\ell(t, s)\big[u_\ell\big](s)\, ds, \tag{4}$$

$$v_{\ell+1}(t) = V_\ell(t, 0)v_0 + \int_0^t V_\ell(t, s) \left[\frac{\mu u_\ell v_\ell}{u_\ell + h} + \alpha w_\ell \right](s)\, ds, \tag{5}$$

$$w_{\ell+1}(t) = e^{-\rho t} w_0 + \int_0^t e^{-\rho(t-s)} \left[\frac{\nu u_\ell v_\ell}{u_\ell + h} + \theta v_\ell \right](s)\, ds \tag{6}$$

for $\ell \in \mathbb{N}$. Here, we have used the time-evolution operators $\{U_\ell(t, s)\}$ and $\{V_\ell(t, s)\}$ associated with

$$A_\ell := \Delta - u_\ell - \gamma v_\ell/(u_\ell + h) \quad \text{and} \quad B_\ell := d\Delta - m - v_\ell$$

for regarding u_ℓ, v_ℓ and w_ℓ as given nonnegative functions, respectively, starting at

$$u_1(t) := e^{t\Delta} u_0, \quad v_1(t) := e^{t(d\Delta - m)} v_0 \quad \text{and} \quad w_1(t) := e^{-\rho t} w_0. \tag{7}$$

The precise definition and estimates of time-evolution operators are given in Section 3. These approximations enable us to show the nonnegativities of u_ℓ, v_ℓ and w_ℓ for all $\ell \in \mathbb{N}$, as well as its limit u, v and w. We will derive the estimates $\|u_\ell, v_\ell, w_\ell\|_\infty$ by (4)–(6), inductively, in the fixed point arguments. Besides, for estimates $\|\partial_i u_\ell, \partial_i v_\ell, \partial_i w_\ell\|_\infty$, we apply the heat semigroup representation of solutions. Once we derive uniform bounds of u_ℓ, v_ℓ, w_ℓ, $\partial_i u_\ell$, $\partial_i v_\ell$ and $\partial_i w_\ell$, we can easily see that the limit (u, v, w) becomes a classical solution to (P).

On the other hand, it is rather standard to extend the obtained solutions time-globally, deriving a priori estimates of solutions. The key idea is to apply the maximum principle to the classical solutions. We can also investigate asymptotic behaviours of solutions, more precisely. Via analysis of solutions to the system of corresponding ODE, we obtain invariant regions.

Before stating results, we provide two stationary solutions. It is easy to verify that $(1, \overline{v}, \overline{w})$ is a stationary solution to (P), where

$$\begin{aligned}
\overline{v} &:= \mu/(1 + h) + \alpha(\nu + \theta + \theta h)/(\rho + \rho h) - m, \\
\overline{w} &:= (\nu + \theta + \theta h)\overline{v}/(\rho + \rho h).
\end{aligned}$$

Furthermore, $(\underline{u}, \underline{v}, \underline{w})$ is also a stationary solution to (P), where

$$\begin{aligned}
\underline{u} &:= (1 - h)/2 + \sqrt{(1 + h)^2 - 4\gamma \overline{v}}/2, \\
\underline{v} &:= \mu \underline{u}/(\underline{u} + h) + \alpha \nu \underline{u}/(\rho \underline{u} + \rho h) + \alpha\theta/\rho - m, \\
\underline{w} &:= \nu \underline{u}\, \underline{v}/(\rho \underline{u} + \rho h) + \theta \underline{v}/\rho.
\end{aligned}$$

Theorem 2. *Assume that* (u, v, w) *is a solution to* (P). (i) *If* $\overline{v} \le 0$, *and if* $u_0 \not\equiv 0$, *then* $(u, v, w) \to (1, 0, 0)$ *as* $t \to \infty$. *Besides, if* $u_0 \equiv 0$, *then* $(u, v, w) \to (0, 0, 0)$ *as* $t \to \infty$. (ii) *If* $\overline{v} > 0$, *then for any* $0 < \varepsilon \ll 1$, *there exists a* $T_\varepsilon \ge 0$ *such that*

$$(u, v, w) \in R_\varepsilon := [0, 1 + \varepsilon) \times [0, \overline{v} + \varepsilon) \times [0, \overline{w} + \varepsilon)$$

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for $x \in \mathbb{R}^n$ and $t \geq T_\varepsilon$. Moreover,

$$(u_0, v_0, w_0) \in R_* := [0,1] \times [0, \overline{v}] \times [0, \overline{w}] \;\Rightarrow\; (u, v, w) \in R_*$$

for $x \in \mathbb{R}^n$ and $t > 0$. (iii) *Let $\overline{v} > 0$, $\underline{u} > 0$, $\underline{v} > 0$, and let $\underline{w} > 0$. If $u, v, w \geq c_\star$ for $x \in \mathbb{R}^n$ at $t = t_\star \geq 0$ with some $c_\star > 0$, then for $0 < \varepsilon \ll 1$, there exists a $T'_\varepsilon \geq t_\star$ such that*

$$(u, v, w) \in R'_\varepsilon := (\underline{u} - \varepsilon, 1 + \varepsilon) \times (\underline{v} - \varepsilon, \overline{v} + \varepsilon) \times (\underline{w} - \varepsilon, \overline{w} + \varepsilon)$$

for $x \in \mathbb{R}^n$ and $t \geq T'_\varepsilon$. Moreover,

$$(u_0, v_0, w_0) \in R_\natural := [\underline{u}, 1] \times [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}] \;\Rightarrow\; (u, v, w) \in R_\natural$$

for $x \in \mathbb{R}^n$ and $t > 0$.

The sets R_* and $R_\natural$ are invariant regions. The reader may find another (narrower) invariant regions for each individual parameter. Theorem 2 implies that an absorbing set always exists in R_* or $R_\natural$. Our conjecture is that we can also obtain the similar results in several domains with suitable boundary conditions.

3. Semigroups and time-evolution operators

In this section, we recall the definitions of function spaces and properties of the heat semigroup, as well as time-evolution operators.

Let $n \in \mathbb{N}$, $1 \leq p < \infty$, and let $L^p := L^p(\mathbb{R}^n)$ be the space of all pth integrable functions in $\mathbb{R}^n$ with the norm $\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$. We often omit the notation of the domain ($\mathbb{R}^n$), if no confusion occurs. We do not distinguish scalar-valued functions and vector, as well as function spaces. Let L^∞ be the space of all bounded functions with the norm $\|f\| := \|f\|_\infty := \text{ess.sup}_{x \in \mathbb{R}^n} |f(x)|$; BUC as the space of all bounded uniformly continuous functions. Note that L^p, L^∞ and BUC are Banach spaces. For $k \in \mathbb{N}$, let $W^{k,\infty}$ be a set of all bounded functions whose k-th derivatives are also bounded. Furthermore, define

$$\mathrm{BUC}^k := \left\{ f \in W^{k,\infty};\, \partial_i^j f \in \mathrm{BUC} \quad \text{for} \quad 1 \leq i \leq n,\; 0 \leq j \leq k \right\}.$$

In the whole space $\mathbb{R}^n$, for $\vartheta_0 \in L^\infty(\mathbb{R}^n)$, the heat equation

$$\begin{cases} \partial_t \vartheta = \Delta \vartheta & \text{in } \mathbb{R}^n \times (0, \infty), \\ \vartheta|_{t=0} = \vartheta_0 & \text{in } \mathbb{R}^n \end{cases} \tag{H}$$

admits a time-global unique smooth solution

$$\begin{aligned} \vartheta &:= \vartheta(t) := \vartheta(x,t) := (e^{t\Delta}\vartheta_0)(x) := (H_t * \vartheta_0)(x) \\ &:= \int_{\mathbb{R}^n} (4\pi t)^{-n/2} \exp\left(-|x-y|^2/4t\right) \vartheta_0(y) dy \end{aligned}$$

in $C_w((0,\infty); L^\infty(\mathbb{R}^n))$, that is, $\vartheta \in C([\tau, \infty); L^\infty(\mathbb{R}^n))$ for any small $\tau > 0$. Here, $H_t := H_t(x) := (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right)$ is the heat kernel. Since $\|H_t\|_1 = 1$ for $t > 0$, by Young's inequality we have $\|\vartheta(t)\| \leq \|H_t\|_1 \|\vartheta_0\| = \|\vartheta_0\|$ for $t > 0$. In particular, if $\vartheta_0(x) \geq 0$ for $x \in \mathbb{R}^n$, then $\vartheta(x,t) \geq 0$ holds true for $x \in \mathbb{R}^n$ and $t > 0$, so-called the maximum principle. Furthermore, if additionally $\vartheta_0 \in \mathrm{BUC}(\mathbb{R}^n)$ and $\vartheta_0 \not\equiv 0$, then $\vartheta(x,t) > 0$ for $x \in \mathbb{R}^n$ and $t > 0$, so-called the strong maximum principle. For $\vartheta_0 \in L^\infty(\mathbb{R}^n)$, there is a lack of the continuity of solutions to (H) in time at $t = 0$, in general. Note that $e^{t\Delta}\vartheta_0 \to \vartheta_0$ in L^∞ as $t \to 0$, if and only if $\vartheta_0 \in \mathrm{BUC}(\mathbb{R}^n)$. The reader may find its proof in [5]. Indeed, if $\vartheta_0 \in \mathrm{BUC}(\mathbb{R}^n)$, then $\vartheta \in C([0,\infty); \mathrm{BUC}(\mathbb{R}^n))$.

We can easily see that for $j \in \mathbb{N}$, splitting the heat semigroup into j-th parts, there exists a positive constant $C_\sharp := \pi^{-1/2} < 1$ such that

$$\|\partial_i^j e^{t\Delta}\vartheta_0\| = \|\left(\partial_i e^{\frac{t}{j}\Delta}\right) \cdots \left(\partial_i e^{\frac{t}{j}\Delta}\right) \vartheta_0\| \leq C_\sharp^j j^{j/2} t^{-j/2} \|\vartheta_0\|$$

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for $t > 0$ and $1 \le i \le n$. So, $\vartheta(t) \in \mathrm{BUC}^j(\mathbb{R}^n)$ for any $j \in \mathbb{N}$ and $t > 0$, which implies that $\vartheta(t) \in C^\infty(\mathbb{R}^n)$ for $t > 0$. Moreover, $\vartheta \in C^\infty(\mathbb{R}^n \times (0, \infty))$ by using (H).

In what follows, we recall some properties and estimates for time-evolution operators. Let us consider the following autonomous Cauchy problem with non-constant coefficients.

$$\begin{cases} \partial_t\varphi = d\Delta\varphi - \psi(x,t)\varphi & \text{in } \mathbb{R}^n \times (0,\infty), \\ \varphi|_{t=0} = \varphi_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{$\mathrm{P_A}$}$$

Here, $d > 0$ is a constant, and $\psi(x, t)$ is a given bounded function. We establish the time-local solvability of $(\mathrm{P_A})$ with upper bounds of φ.

Lemma 1 ([6]). *Let $n \in \mathbb{N}$, d, $T > 0$ and $\psi \in L^\infty([0, T]; W^{1,\infty}(\mathbb{R}^n))$. If $\varphi_0 \in BUC(\mathbb{R}^n)$, then there exist a $T_* \in (0, T]$ and a time-local unique classical solution to $(\mathrm{P_A})$, having $\|\varphi(t)\| \le \dfrac{4}{3}\|\varphi_0\|$ for $t \in [0, T_*]$. Moreover, if $\varphi_0 \ge 0$, then $\varphi \ge 0$.*

Although the proof is written in [6], we give it here. The idea is to use the standard iteration. Let $\varphi_1(t) := e^{dt\Delta}\varphi_0$, and let

$$\varphi_{\ell+1}(t) := e^{dt\Delta}\varphi_0 - \int_0^t e^{d(t-s)\Delta}\,[\psi\varphi_\ell]\,(s)\,ds$$

for each $\ell \in \mathbb{N}$, successively. It is easy to see that for $\ell \in \mathbb{N}$, $\|\varphi_\ell(t)\| \le \dfrac{4}{3}\|\varphi_0\|$ for $t \in [0, T_*]$ with some $T_* > 0$ independent of ℓ. We can easily show that $\{\varphi_\ell\}_{\ell=1}^\infty$ is a Cauchy sequence in $C([0, T_*]; \mathrm{BUC}(\mathbb{R}^n))$. So, the limit $\varphi := \lim_{\ell\to\infty}\varphi_\ell$ exists and satisfies $(\mathrm{P_A})$, having the estimate $\|\varphi(t)\| \le \dfrac{4}{3}\|\varphi_0\|$ for $t \in [0, T_*]$. It is rather straightforward to obtain the uniqueness and regularity of φ. Moreover, the nonnegativity of φ easily follows from the maximum principle.

Note that if $\|\varphi_0\| \le L$ and $\sup_{0\le t\le T}\|\psi(t)\| \le L$ with some $L > 0$, then we may derive the estimate $T_* \ge C/L$ with $C > 0$. The solution to $(\mathrm{P_A})$ can be rewritten as $\varphi(t) = U(t, 0)\varphi_0$, using time-evolution operators $\{U(t, s)\}_{t\ge s\ge 0}$ associated with $A := A(x, t) := d\Delta - \psi(x, t)$, see the book by Tanabe [8]. The boundedness of solutions φ implies that $\|U(t, 0)\|_{L^\infty\to L^\infty} \le 4/3$ for $t \in [0, T_*]$, and then $\|U(t, s)\|_{L^\infty\to L^\infty} \le 4/3$ for $0 \le s \le t \le T_*$. Here, we have used the notation of an operator-norm $\|\mathcal{O}\|_{X\to Y} := \sup_{x\in X}\|\mathcal{O}x\|_Y/\|x\|_X$.

4. Time-local solvability

We give a proof of the time-local solvability on (P) in this section. Recall $\|\cdot\| := \|\cdot\|_\infty$, and put $M := \max\{\|u_0\|, \|v_0\|, \|w_0\|, \|\partial_i w_0\|\}$.

Proposition 1. *Let $n \in \mathbb{N}$, $d > 0$, and let those other parameters be nonnegative. Assume that u_0, $v_0 \in BUC(\mathbb{R}^n)$ and $w_0 \in BUC^1(\mathbb{R}^n)$ are nonnegative, then there exist a $T_0 > 0$ and a time-local unique classical solutions to (P), having $0 \le u, v, w \le 2M$ for $x \in \mathbb{R}^n$ and $t \in [0, T_0]$. Furthermore, $T_0 \ge C_*/(M^4 + 1)$ holds with some constant $C_* > 0$ independent of M.*

Proof. For the sake of simplicity, we assume that all parameter is positive. Making the approximation sequences, we begin with (7). For $\ell \in \mathbb{N}$, we successively define $u_{\ell+1}$, $v_{\ell+1}$ and $w_{\ell+1}$ by (4)–(6). So, $u_{\ell+1}$, $v_{\ell+1}$ and $w_{\ell+1}$ also satisfy their abstract equations for $x \in \mathbb{R}^n$ and $t > 0$ with nonnegative functions u_0, v_0, w_0, u_ℓ, v_ℓ and w_ℓ, formally.

In what follows, we estimate u_ℓ, v_ℓ, w_ℓ, $\partial_i u_\ell$, $\partial_i v_\ell$ and $\partial_i w_\ell$. Put

$$\begin{aligned} K_{1,\ell} &:= \sup_{0\le t\le T}\|u_\ell(t)\|, & K_{2,\ell} &:= \sup_{0\le t\le T}\|v_\ell(t)\|, \\ K_{3,\ell} &:= \sup_{0\le t\le T}\|w_\ell(t)\|, & K_{4,\ell} &:= \sup_{0\le t\le T}t^{1/2}\|\partial_i u_\ell(t)\|, \\ K_{5,\ell} &:= \sup_{0\le t\le T}(dt)^{1/2}\|\partial_i v_\ell(t)\|, & K_{6,\ell} &:= \sup_{0\le t\le T}\|\partial_i w_\ell(t)\| \end{aligned}$$

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for $T > 0$, $\ell \in \mathbb{N}$ and $1 \leq i \leq n$. To derive uniform estimates, we argue the induction of ℓ, taking T small.

$\ell = 1$ For $0 \leq u_0(x), v_0(x), w_0(x) \leq M$, by the maximum principle and the fact that $e^{t(d\Delta - m)} = e^{-mt}e^{dt\Delta}$, we can easily see that

$$0 \leq u_1(x,t) \leq \|u_0\|, \quad 0 \leq v_1(x,t) \leq \|v_0\|, \quad 0 \leq w_1(x,t) \leq \|w_0\|$$

for $x \in \mathbb{R}^n$ and $t > 0$ by $m, \rho \geq 0$. In addition, it is easy to obtain that

$$t^{1/2}\|\partial_i u_1(t)\| \leq \|u_0\|, \quad (dt)^{1/2}\|\partial_i v_1(t)\| \leq \|v_0\|, \quad \|\partial_i w_1(t)\| \leq \|\partial_i w_0\|$$

for $t > 0$ and $1 \leq i \leq n$ by the estimate of the heat kernel. Here and hereafter, we replace the constant $C_\sharp := \pi^{-1/2} < 1$ by 1, for the sake of simplicity. Thus, we have

$$K_{j,1} \leq M \quad \text{for} \quad T > 0, \quad 1 \leq j \leq 6 \quad \text{and} \quad 1 \leq i \leq n. \tag{8}$$

$\ell = 2$ Before estimating u_2 and v_2, we will confirm bounds for time-evolution operators U_1 and V_1. By $u_1 \geq 0$ and (8), it holds that

$$\|\eta_1(t)\| \leq M + \frac{\gamma M}{h} =: \overline{\eta}_1 \quad \text{with} \quad \eta_1(x,t) := u_1(x,t) + \frac{\gamma v_1(x,t)}{u_1(x,t) + h}$$

for $t > 0$. By Lemma 1, for $\{U_1(t,s)\}_{t \geq s \geq 0}$ with $A_1(x,t) := \Delta - \eta_1(x,t)$, we thus see that $0 \leq U_1(t,s)u_0 \leq \frac{4}{3}\|u_0\|$ for $x \in \mathbb{R}^n$ and $0 \leq s \leq t \leq T_2'$ with some $T_2' > 0$ depending only on $\overline{\eta}_1$. By (4) with $\ell = 1$, we have

$$0 \leq u_2(t) \leq \|U_1(t,0)u_0\| + \int_0^t \|U_1(t,s)\zeta_1(s)\| ds \leq 2M$$

with $\zeta_1(x,t) := u_1(x,t)$ and $0 \leq \zeta_1(x,s) \leq \overline{\zeta}_1 := M$, provided $0 \leq s \leq t \leq T_2^\dagger$ with $T_2^\dagger := \min\{T_2', 1/2\}$. Similarly, since

$$\|\xi_1(t)\| \leq m + M =: \overline{\xi}_1 \quad \text{with} \quad \xi_1(x,t) := m + v_1(x,t)$$

for $x \in \mathbb{R}^n$ and $t > 0$, we may define the time-evolution operator $\{V_1(t,s)\}_{t \geq s \geq 0}$ associated with $B_1(x,t) := d\Delta - \xi_1(x,t)$, having a uniform bound. Applying Lemma 1, we see that $0 \leq V_1(t,s)v_0 \leq \frac{4}{3}\|v_0\|$ for $0 \leq s \leq t \leq T_2^\sharp$ with some $T_2^\sharp > 0$ depending only on $\overline{\xi}_1$. By (5)

$$0 \leq v_2(t) \leq \|V_1(t,0)v_0\| + \int_0^t \|V_1(t,s)\chi_1(s)\| ds \leq 2M$$

hold with $\chi_1(x,t) := \mu u_1(x,t)v_1(x,t)/\{u_1(x,t) + h\} + \alpha w_1(x,t)$ and $0 \leq \chi_1(x,s) \leq \overline{\chi}_1 := (\mu M/h + \alpha)M$, provided if $0 \leq s \leq t \leq T_2^\flat$ with $T_2^\flat := \min\{T_2^\dagger, T_2^\sharp, h/(2\mu M + 2\alpha h)\}$. For the estimate of w_2, we obtain

$$0 \leq w_2(t) \leq \|e^{-\rho t}w_0\| + \int_0^t e^{-\rho(t-s)}\|\nu u_1 v_1/(u_1 + h) + \theta v_1\| ds \leq 2M$$

for $0 \leq s \leq t \leq T_2^\natural$ with $T_2^\natural := \min\{T_2^\flat, h/(\nu M + h\theta)\}$. To derive the estimate for $\partial_i u_2$, we use the heat semigroup expression:

$$u_2(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left[\zeta_1 - \eta_1 u_2\right](s)\, ds$$

by rewriting (4). Hence, it holds that

$$t^{1/2}\|\partial_i u_2(t)\| \leq \|u_0\| + t^{1/2}\int_0^t (t-s)^{-1/2}\left[\overline{\zeta}_1 + \overline{\eta}_1\|u_2\|\right] ds \leq 2M$$

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for $t \in (0, T_2^{\heartsuit}]$ with $T_2^{\heartsuit} := \min\left\{T_2^{\natural}, h/(2h + 4hM + 4\gamma M)\right\}$. As the similar way, for $\partial_i v_2$, we appeal to the heat semigroup expression again:

$$\begin{aligned}(dt)^{1/2}\|\partial_i v_2(t)\| &\le (dt)^{1/2}\|\partial_i e^{dt\Delta}v_0\| \\ &\quad + (dt)^{1/2}\int_0^t \|\partial_i e^{d(t-s)\Delta}\left[\chi_1 - \xi_1 v_2\right](s)\| ds \\ &\le \|v_0\| + t^{1/2}\int_0^t (t-s)^{-1/2}\left[\overline{\chi}_1 + \overline{\xi}_1 2M\right] ds \\ &\le 2M\end{aligned}$$

for $t \in \left(0, T_2^{\diamondsuit}\right]$ with $T_2^{\diamondsuit} := \min\left\{T_2^{\heartsuit}, h/(2\mu M + 2\alpha h + 4hm + 4hM)\right\}$. Furthermore,

$$\partial_i w_2(t) = e^{-\rho t}\partial_i w_0 + \int_0^t e^{-\rho(t-s)}\left[\frac{\nu h(\partial_i u_1)v_1 + \nu u_1(\partial_i v_1)(u_1 + h)}{(u_1 + h)^2} + \theta\,\partial_i v_1\right] ds$$

holds true, and this implies that

$$\begin{aligned}\|\partial_i w_2(t)\| &\le M + \int_0^t \left\{\frac{\nu h\sqrt{d}M + \nu M(M+h)}{h^2} + \theta\right\} M(ds)^{-1/2} ds \\ &\le 2M\end{aligned}$$

for $t \in [0, T_2]$ with

$$T_2 := \min\left\{T_2^{\diamondsuit}, dh^4/\left[4\nu h\sqrt{d}M + 4\nu M^2 + 4\nu hM + 4h^2\theta\right]^2\right\}.$$

Therefore, it is shown that $u_2, v_2, w_2 \ge 0$ and

$$K_{j,2} \le 2M \quad \text{for} \quad t \in (0, T_2], \quad 1 \le j \le 6 \quad \text{and} \quad 1 \le i \le n. \tag{9}$$

$\ell = 3$ We stand for the time-evolution operator $\{U_2(t,s)\}_{t\ge s\ge 0}$ associated with $A_2(x,t) := \Delta - \eta_2(x,t)$ and

$$\eta_2(x,t) := u_2(x,t) + \gamma v_2(x,t)/\{u_2(x,t) + h\}.$$

By Lemma 1, $U_2(t,s)u_0 \ge 0$ holds and $\|U_2(t,s)\|_{L^\infty\to L^\infty} \le 4/3$ for $0 \le s \le t \le T_3'$ with some $T_3' > 0$, since $0 \le \eta_2(x,t) \le \overline{\eta} := 2M + 2\gamma M/h$ by (9). So, we get

$$0 \le u_3(x,t) \le \|U_2(t,0)u_0\| + \int_0^t \|U_2(t,s)\zeta_2(s)\| ds \le 2M$$

for $x \in \mathbb{R}^n$ and $t \in [0, T_3^{\dagger}]$ with $T_3^{\dagger} := \min\{T_3', 1/4\}$. Here, we used that

$$0 \le \zeta_2(x,t) := u_2(x,t) \le \overline{\zeta} := 2M.$$

Similarly, we denote the time-evolution operator by $\{V_2(t,s)\}_{t\ge s\ge 0}$ associated with $B_2(x,t) := d\Delta - \xi_2(x,t)$, where

$$0 \le \xi_2(x,t) := m + v_2(x,t) \le \overline{\xi} := m + 2M.$$

We seek that $V_2(t,s)v_0 \ge 0$ and $\|V_2(t,s)\|_{L^\infty\to L^\infty} \le 4/3$ for $0 \le s \le t \le T_3^{\sharp}$ with some $T_3^{\sharp} > 0$ by Lemma 1. Hence, we can see that

$$0 \le v_3(x,t) \le \|V_2(t,0)v_0\| + \int_0^t \|V_2(t,s)\chi_2(s)\| ds \le 2M$$

for $x \in \mathbb{R}^n$ and $t \in [0, T_3^{\flat}]$ with $T_3^{\flat} := \min\{T_3^{\dagger}, T_3^{\sharp}, h/(8\mu M + 4\alpha h)\}$. Here, we have used

$$\begin{aligned}0 &\le \chi_2(x,t) := \mu u_2(x,t)v_2(x,t)/\{u_2(x,t) + h\} + \alpha w_2(x,t) \\ &\le \overline{\chi} := 4\mu M^2/h + 2\alpha M\end{aligned}$$

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by (9). It is also easy to show that

$$0 \le w_3(x,t) \le \|w_0\| + \int_0^t \|\nu u_2 v_2/(u_2+h) + \theta v_2\| ds \le 2M$$

for $x \in \mathbb{R}^n$ and $t \in [0, T_3^\natural]$ with $T_3^\natural := \min\{T_3^\flat, h/(4\nu M + 2h\theta)\}$. By the heat semigroup expression, we obtain that

$$t^{1/2}\|\partial_i u_3(t)\| \le \|u_0\| + t^{1/2} \int_0^t (t-s)^{-1/2} \left[\|\zeta_2\| + \|\eta_2 u_3\|\right] ds \le 2M$$

for $t \in (0, T_3^\heartsuit]$ with $T_3^\heartsuit := \min\{T_3^\natural, h/(4h + 8hM + 8\gamma M)\}$. As the similar way, we derive

$$(dt)^{1/2}\|\partial_i v_3(t)\| \le \|v_0\| + t^{1/2} \int_0^t (t-s)^{-1/2} \left[\|\chi_2\| + \|\xi_2 v_3\|\right] ds \le 2M$$

for $t \in (0, T_3^\diamondsuit]$ with $T_3^\diamondsuit := \min\{T_3^\heartsuit, h/(4hm + 8hM + 8\mu M + 4\alpha h)\}$. For $\partial_i w_3$, see

$$\begin{aligned}\|\partial_i w_3(t)\| &\le M + \int_0^t \left\| \frac{\nu h(\partial_i u_2)v_2 + \nu u_2(\partial_i v_2)(u_2+h)}{h^2} + \theta \partial_i v_2 \right\| ds \\ &\le 2M\end{aligned}$$

for $t \in (0, T_0]$ with

$$T_0 := \min\left\{T_3^\diamondsuit, dh^4 \Big/ \left[8\nu h\sqrt{d}M + 16\nu M^2 + 8\nu hM + 4h^2\theta\right]^2\right\}.$$

Note that the estimate $T_0 \ge C/(M^4+1)$ is yielded with some $C > 0$.

Therefore, we see that $u_3, v_3, w_3 \ge 0$ and

$$K_{j,3} \le 2M \quad \text{for} \quad t \in (0, T_0], \quad 1 \le i \le n \quad \text{and} \quad 1 \le j \le 6.$$

$\ell = 4, 5, \ldots$ Let $\ell \ge 4$. We assume that $u_\ell, v_\ell, w_\ell \ge 0$ and

$$K_{j,\ell} \le 2M \quad \text{for} \quad t \in (0, T_0], \quad 1 \le j \le 6 \quad \text{and} \quad 1 \le i \le n \tag{10}$$

hold true. We will compute estimates for $u_{\ell+1}$, $v_{\ell+1}$ and $w_{\ell+1}$. Note that $\eta_\ell \le \overline{\eta}$, $\zeta_\ell \le \overline{\zeta}$, $\xi_\ell \le \overline{\xi}$ and $\chi_\ell \le \overline{\chi}$ hold, independently of $\ell \ge 3$. So, as the same discussion in the case $\ell = 3$ above, we can see that $u_{\ell+1}$, $v_{\ell+1}$, $w_{\ell+1} \ge 0$ and

$$K_{j,\ell+1} \le 2M \quad \text{for} \quad t \in (0, T_0], \quad 1 \le j \le 6 \quad \text{and} \quad 1 \le i \le n.$$

The detail is omitted here. Hence, the nonnegativities of approximations and (10) hold true for all $\ell \in \mathbb{N}$.

We can see that u_ℓ, v_ℓ and w_ℓ are continuous in $t \in [0, T_0]$ for $\ell \in \mathbb{N}$. And also, it is easy to see that $\left\{u_\ell, v_\ell, w_\ell, t^{1/2}\partial_i u_\ell, t^{1/2}\partial_i v_\ell, \partial_i w_\ell\right\}_{\ell=1}^\infty$ are Cauchy sequences in $C([0, T_0]; \mathrm{BUC})$, choosing T_0 small again, if necessary. Let

$$(u, v, w, \hat{u}, \hat{v}, \hat{w}) := \lim_{\ell\to\infty} \left(u_\ell, v_\ell, w_\ell, t^{1/2}\partial_i u_\ell, t^{1/2}\partial_i v_\ell, \partial_i w_\ell\right)$$

in the topology of $C([0, T_0]; \mathrm{BUC})$. Obviously, the coincidences $\hat{u} = t^{1/2}\partial_i u$, $\hat{v} = t^{1/2}\partial_i v$ and $\hat{w} = \partial_i w$ hold by construction. Furthermore, it is also ensured that

$$0 \le u(x,t), v(x,t), w(x,t) \le 2M \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{and} \quad t \in [0, T_0].$$

The uniqueness follows from (1)–(3) and Gronwall's inequality, directly. If fact, let (u, v, w) and (u^*, v^*, w^*) be solutions to (P) in $[0, T_0]$ with the same initial data, then $u \equiv u^*$, $v \equiv v^*$ and $w \equiv w^*$ simultaneously hold. Thanks to the boundedness of the first derivatives, it is easy to control the second derivatives in x of u and v for $t \in (0, T_0]$, as well as the first derivatives in t of solutions. So, we see that (u, v, w) is a time-local unique classical solution to (P). This completes the proof of Proposition 1.

Remark 2. (i) (u, v, w) is smooth in x and t, if w_0 is smooth.(ii) The instability of the trivial solution $(0, 0, 0)$ is easily obtained with $u_0 \not\equiv 0$. Moreover, by strong maximum principle for solutions to the heat

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equation, $u > 0$ for $x \in \mathbb{R}^n$ and $t \in (0, T_0]$. This means that $\operatorname{supp} u(t) = \mathbb{R}^n$ for any small $t > 0$, even if $\operatorname{supp} u_0$ is compact. That is, the propagation speed of solutions to (P) is infinite, as the same as the heat equation. In addition, $v > 0$ and $w > 0$ for $x \in \mathbb{R}^n$ and $t > 0$, if either $v_0 \not\equiv 0$ or $w_0 \not\equiv 0$.

5. Global well-posedness

In this section, we will derive a priori bounds of solutions and their derivatives. To do so, our first task is to obtain upper bounds of solutions to (P) with large initial data. For the case when $\|u_0\| \le 1$, we will discuss in Remark 3 (ii) below and Section 6.

Proposition 2. *Suppose the assumption of Proposition 1. If $\|u_0\| > 1$, then $0 < u < \|u_0\|$, $0 \le v \le \widetilde{v}$, and $0 \le w \le \widetilde{w}$ hold for $x \in \mathbb{R}^n$ and $t > 0$ with some positive constants $\widetilde{v}$ and $\widetilde{w}$ depending on $\|u_0\|$, $\|v_0\|$ and $\|w_0\|$, as long as the classical solutions exist.*

If $v_0 \equiv 0$ and $w_0 \equiv 0$, then $v \equiv w \equiv 0$ for $t > 0$. Assume either $v_0 \not\equiv 0$ or $w_0 \not\equiv 0$. So, as seen in Remark 2 (iii), we have $u, v, w > 0$. For observing the behaviour of u, we consider the following logistic equation:

$$\kappa' = (1-\kappa)\kappa, \quad \kappa(0) = \kappa_0 > 1, \tag{11}$$

where $\kappa_0 = \|u_0\|$. By maximum principle, $u(x, t) \le \kappa(t)$ holds for $x \in \mathbb{R}^n$ and $t > 0$, as long as the classical solution u exists. Since

$$\kappa(t) = \kappa_0 / \left(\kappa_0 + e^{-t} - \kappa_0 e^{-t}\right) < \kappa_0$$

for $t > 0$, it is clear that $u < \kappa_0$.

Next, we investigate on upper bounds of v and w. We will use renormalising arguments to ODE. Let a pair $\sigma = \sigma(t)$ and $\omega = \omega(t)$ be solutions to

$$\begin{cases} \sigma' = \alpha\omega - (m_\star + \sigma)\sigma, & \sigma(0) = \sigma_0 := \|v_0\|, \\ \omega' = \theta_\star \sigma - \rho\, \omega, & \omega(0) = \omega_0 := \|w_0\|. \end{cases} \tag{12}$$

Here, $m_\star := m - \mu\kappa_0/(\kappa_0 + h)$ and $\theta_\star := \theta + \nu\kappa_0/(\kappa_0 + h)$. Since $(e^{\rho t}\omega)' = \theta_\star e^{\rho t}\sigma$, we have

$$\omega(t) = e^{-\rho t}\omega_0 + e^{-\rho t}\theta_\star \int_0^t e^{\rho s}\sigma(s) ds \le \omega_0 + (\theta_\star/\rho) \sup\nolimits_{0 \le s \le t} \sigma(s)$$

for $t > 0$. Inserting it into the first equation of (12), it holds that

$$\sigma' \le \alpha \left\{\omega_0 + (\theta_\star/\rho) \sup\nolimits_{0 \le s \le t} \sigma(s)\right\} - m_\star\sigma - \sigma^2$$

for $t > 0$. Therefore, we can see that $\sigma(t) < \widetilde{v} := \max\{\sigma_0, \bar{\sigma}\} + 1$ for $t > 0$, where

$$\bar{\sigma} := \alpha\theta_\star/2\rho - m_\star/2 + \sqrt{\alpha\omega_0 + (\alpha\theta_\star/\rho - m_\star)^2/4}.$$

Note that $\bar{\sigma}$ satisfies $\alpha(\omega_0 + (\theta_\star/\rho)\bar{\sigma}) - m_\star\bar{\sigma} - \bar{\sigma}^2 = 0$. Indeed, if there exists some $t_\star > 0$ such that $\sigma(t_\star) = \widetilde{v} \ge \bar{\sigma} + 1$ and $\sigma(t) < \widetilde{v}$ for $t \in [0, t_\star)$, then $\sigma'(t_\star) \ge 0$. This contradicts $\sigma'(t_\star) < 0$. We can similarly deduce $\omega(t) \le \widetilde{w}$ holds for $t > 0$, where $\widetilde{w} := \max\{\omega_0, \theta_\star\widetilde{v}/\rho\} + 1$.

We will use enclosing arguments, that is, applying the comparison principle between solutions to PDE and those to ODE. Put $V := \sigma - v$ and $W := \omega - w$. Hence, $V(0) \ge 0$ and $W(0) \ge 0$. Also, we see

$$\begin{aligned} \partial_t V &= d\Delta V + \alpha W - mV + \mu\kappa_0\sigma/(\kappa_0 + h) - \mu uv/(u+h) - \sigma^2 + v^2 \\ &= d\Delta V + \alpha W - (m + \sigma + v)\, V \\ &\quad + \frac{\mu}{(\kappa_0 + h)(u+h)}\left[(u+h)\kappa_0 V + hv(\kappa_0 - u)\right] \end{aligned}$$

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and

$$\begin{aligned}\partial_t W &= \theta V - \rho W + \nu\kappa_0\sigma/\left(\kappa_0+h\right) - \nu uv/(u+h)\\ &= \theta V - \rho W + \frac{\nu}{(\kappa_0+h)(u+h)}\left[(u+h)\kappa_0 V + hv\left(\kappa_0 - u\right)\right].\end{aligned}$$

We thus find the fact that $V \geq 0$ and $W \geq 0$ for $t > 0$, as the same discussion in the proof of Proposition 1. This implies that

$$v(x,t) \leq \sigma(t), \quad w(x,t) \leq \omega(t) \tag{13}$$

for x and t. Therefore, we conclude that $0 \leq v \leq \widetilde{v}$ and $0 \leq w \leq \widetilde{w}$. □

Remark 3. (i) By definitions of $\overline{v}$ and $\overline{w}$, it is clear that $\widetilde{v} \geq \overline{v}$ and $\widetilde{w} \geq \overline{w}$, if $|u_0\| \geq 1$. Besides, $\widetilde{v} \leq \overline{v}$ and $\widetilde{w} \leq \overline{w}$, if $|u_0\| \leq 1$, $\|v_0\| \leq \overline{v}$ and $\|w_0\| \leq \overline{w}$; see Section 6. (ii) Even if $\|u_0\| \leq 1$, then uniform bounds on v and w are obtained; $v \leq \widetilde{v}$ and $w \leq \widetilde{w}$ hold, replacing $m_\star$ by $m_1 := m - \mu/(1+h)$ and $\theta_\star$ by $\theta_1 := \theta + \nu/(1+h)$. (iii) Although we take the maximum values of solutions to ODE (12) by the comparison method (finding t as $\sigma'(t) = 0$ or $\omega'(t) = 0$), such critical points do not always give the maximum values of solutions to PDE, in general. Hence, we have to use enclosing and renormalising arguments in above.

In what follows, we give the a priori estimate for $\|\partial_i w(t)\|$, which may grow in t. As seen in Proposition 2, and by using definitions of $\overline{v}$ and $\overline{w}$ in Theorem 2, we prove that $0 \leq u, v, w \leq N$ as long as the classical solutions exist, if N is chosen as

$$N := \max\left\{1, \|u_0\|, \overline{v}, \widetilde{v}, \|v_0\|, \overline{w}, \widetilde{w}, \|w_0\|\right\}. \tag{14}$$

Proposition 3. *Let T, $N > 0$. If $0 \leq u, v, w \leq N$ for $x \in \mathbb{R}^n$ and $t \in [0,T]$, then there exists a $C > 0$ independent of N and T such that*

$$\|\partial_i w(t)\| \leq \|\partial_i w_0\| + C\left(N^4+N\right)\left(t^{1/2}+t^{3/2}\right), \quad t \in [0,T], \ \ 1 \leq i \leq n.$$

We first derive the estimate for $\partial_i u$. By (1), we have

$$\begin{aligned}\|\partial_i u(t)\| &\leq \|u_0\| t^{-1/2} + \int_0^t (t-s)^{-1/2}\left\|(1-u)u - \frac{\gamma uv}{u+h}\right\| ds\\ &\leq C\left(N^2+N\right)\left(t^{-1/2}+t^{1/2}\right)\end{aligned}$$

for $t \in [0,T]$ and $1 \leq i \leq n$ with some C. Similarly, by (2), we seek

$$\begin{aligned}\|\partial_i v(t)\| &\leq \|v_0\|(dt)^{-1/2} + \int_0^t (dt-ds)^{-1/2}\left\|\frac{\mu uv}{u+h} + \alpha w - (m+v)v\right\| ds\\ &\leq C\left(N^2+N\right)\left(t^{-1/2}+t^{1/2}\right)\end{aligned}$$

with some C. Finally, by (3) and estimates above, it turns out that

$$\begin{aligned}\|\partial_i w(t)\| &\leq \|\partial_i w_0\| + \int_0^t \left\|\frac{\nu h(\partial_i u)v + \nu u(\partial_i v)(u+h)}{(u+h)^2} + \theta\partial_i v\right\| ds\\ &\leq \|\partial_i w_0\| + C\left(N^4+N\right)\int_0^t \left(s^{-1/2}+s^{1/2}\right) ds\\ &\leq \|\partial_i w_0\| + C\left(N^4+N\right)\left(t^{1/2}+t^{3/2}\right)\end{aligned}$$

for $t \in [0,T]$ and $1 \leq i \leq n$ with some positive constant C depending on parameters, however, independent of N and T.

Note that the proof of Theorem 1 is now complete. In fact, Theorem 1 follows from Propositions 1, 2, 3 and $T_0 \geq C_*/(M^4+1)$ in Proposition 1, since we can extend the obtained unique classical solutions time-globally, repeating the construction.

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6. Invariant regions

This section will be devoted to observing invariant regions. The proof of Theorem 2 (i) is easy, since $(1, 0, 0)$ is only one stable constant state. So, we skip it here.

We are now in position to give a proof of Theorem 2 (ii). The key step is to deduce a priori bounds of solutions, due to the maximum principle and comparison with solutions to the system of corresponding ordinary differential equations of κ, σ and ω given by (11) and (12). Let us recall the assumptions:

$$\begin{aligned}\overline{v} &:= \mu/(1+h) + \alpha(\nu+\theta+\theta h)/(\rho+\rho h) - m > 0,\\ \overline{w} &:= (\nu+\theta+\theta h)\overline{v}/(\rho+\rho h) > 0\end{aligned}$$

and $R_* := [0, 1] \times [0, \overline{v}] \times [0, \overline{w}]$.

Proof. We first show that R_* is an invariant region. Let $(u_0, v_0, w_0) \in R_*$. By construction of time-local solutions in Proposition 1, the nonnegativity of solutions is clarified. Note that $(0, 0, 0)$ and $(1, 0, 0)$ are classical solutions in R_*. If $u_0 \equiv 0$, then $u \equiv 0$, in addition, $v \in [0, \overline{v}]$ and $w \in [0, \overline{w}]$, since $v^\flat := \alpha\theta/\rho - m \leq \overline{v}$ and $w^\flat := \theta(\alpha\theta - m\rho)/\rho^2 \leq \overline{w}$. Also, it is easy to see that $v \equiv 0$ and $w \equiv 0$ hold for $t > 0$, provided if $v_0 \equiv 0$ and $w_0 \equiv 0$.

Let $u_0 \not\equiv 0$ and either $v_0 \not\equiv 0$ or $w_0 \not\equiv 0$. As seen in Remark 2 (iii), it is clear that the classical solutions u, v, w never touch 0, as long as they exist. Moreover, with $u_0 \leq 1$, we observe that $u(\tau) < 1$ for small $\tau > 0$ by the strong maximum principle. Similarly, it turns out that $v(\tau) < \overline{v}$ by $v_0 \leq \overline{v}$, as well as $w(\tau) < \overline{w}$. So, regarding τ as the initial time, we can assume $(u_0, v_0, w_0) \in R_*^\circ := (0, 1) \times (0, \overline{v}) \times (0, \overline{w}) = R_* \setminus \partial R_*$, without loss of generality.

Put $\hat{t} \in (0, T_0]$ is the first time when u touches 1 at $\hat{x} \in \mathbb{R}^n$. We may assume $|\hat{x}| < \infty$ by Oleinik's argument on the maximum principle; see e.g. [4]. Since $u(\hat{x}, \hat{t}) = 1$ is the local maximum, at $(\hat{x}, \hat{t})$ we see that $\partial_t u \geq 0$, $\Delta u \leq 0$, $(1 - u)u = 0$ and $-\gamma uv/(u + h) < 0$ by $v > 0$. This contradicts to that u is a solution to (P). Hence, u never touches 1.

The same argument works on v and w. Indeed, let $0 < u < 1$, $0 < w < \overline{w}$, and if there exists $(\check{x}, \check{t}) \in \mathbb{R}^n \times (0, T_0]$ such that $\check{t}$ is the first time when v touches $\overline{v}$ at $\check{x}$. So, at $(\check{x}, \check{t})$, we see that $\partial_t v \geq 0$, $d\Delta v \leq 0$ and

$$\frac{\mu uv}{u+h} + \alpha w - (m+v)v < \frac{\mu\overline{v}}{1+h} + \alpha\overline{w} - (m+\overline{v})\,\overline{v} = 0.$$

So, v never touches $\overline{v}$. As the same as above, we can confirm that w never touches $\overline{w}$ as long as classical solutions exist. This means that the solutions always remain in R_*°.

Next, we show the asymptotic behaviour of solutions, briefly. Even if $\|u_0\| > 1$, by $u(x, t) \leq \kappa(t)$, then there exists a $T_\varepsilon^* > 0$ such that $\|u(t)\| < 1 + \varepsilon$ for $t > T_\varepsilon^*$. From this and the comparison $v(x, t) \leq \sigma(t)$, there exists $T_\varepsilon^\sharp > T_\varepsilon^*$ such that $\|v(t)\| < \overline{v} + \varepsilon$ for $t > T_\varepsilon^\sharp$. Finally, we can also show that there exists $T_\varepsilon > T_\varepsilon^\sharp$ such that $\|w(t)\| < \overline{w} + \varepsilon$ for $t > T_\varepsilon$, by the similar way. This completes the proof of Theorem 2 (ii). □

The proof of Theorem 2 (iii) is essentially similar to above. So, we omit it here.

Remark 4. The stability of non-trivial constant states to the system of corresponding ODE can be easily obtained. For example, if

$$\mu = \nu = \frac{\gamma}{2},\ m = \theta = 0,\ \alpha = \rho = \frac{1}{4},\ \gamma = h + \frac{1}{2}$$

are chosen, then the bifurcation occurs, that is, the stability of a constant state $(u, v, w) = (1/2, 1/2, 1/2)$ is changed in h at 0. Indeed, the constant state $(1/2, 1/2, 1/2)$ is stable for any $h > 0$, while this is unstable for any $-1/2 < h < 0$. The authors believe that such stability is still valid for solutions to (P). For studying the Turing instability, we need to deal with more complicated situation, for example, when μ and ν are sigmoid functions of u.

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