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DEPENDENCE OF EIGENVALUES OF SIXTH-ORDER BOUNDARY VALUE PROBLEMS ON THE BOUNDARY

SUQIN GE✉, WANYI WANG and QIUXIA YANG

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Abstract

In this paper, we consider the dependence of eigenvalues of sixth-order boundary value problems on the boundary. We show that the eigenvalues depend not only continuously but also smoothly on boundary points, and that the derivative of the nth eigenvalue as a function of an endpoint satisfies a first-order differential equation. In addition, we prove that as the length of the interval shrinks to zero all higher eigenvalues of such boundary value problems march off to plus infinity. This is also true for the first (that is, lowest) eigenvalue.

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1. Introduction

For the second-order Sturm–Liouville differential equation

$$(-py')' + qy = \lambda wy,$$

with $p(t) \geq k > 0$ and $p, q, w \in C^\infty$, Dauge and Helffer in [8, 9] showed that its Neumann eigenvalues, as functions of an endpoint, satisfy a differential equation of the form

$$\lambda' = u^2(q - \lambda w).$$

They also found the equation satisfied by Dirichlet eigenvalues,

$$\lambda' = -pu'^2,$$

and, more generally, the equation for the eigenvalues of any self-adjoint separated boundary condition at b parameterised by, say, β,

$$\lambda' = u^2[-\beta/p + (q - \lambda w)].$$

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In addition, these authors showed that the lowest Neumann eigenvalue is, in general, not a decreasing function of the endpoints but, nevertheless, has a finite limit as the endpoints approach each other. On the other hand, they showed that the lowest Dirichlet eigenvalue is a decreasing function of the endpoints and thus must have a finite or infinite limit as the endpoints approach each other, but left open the question of whether this limit is finite or infinite. In [13] the authors showed that it is infinite. Some of the conclusions above were extended to the fourth-order case [10].

In this paper, we consider the dependence of eigenvalues of sixth-order boundary value problems (BVPs) on the boundary. The sixth-order BVPs are known to arise in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modelled by sixth-order BVPs [4, 20, 21]. In [11], Glatzmaier also notes that dynamo action in some stars may be modelled by such equations. Moreover, when an infinite horizontal layer [19] of fluid is heated from below and is subjected to the action rotation, instability sets in. When this instability is like ordinary convection the ordinary differential equation is sixth-order. Further discussions of sixth-order BVPs are given in [2, 3] and in a book by Chandrasekhar [6].

We show that the eigenvalues of sixth-order BVPs considered here depend not only continuously but also smoothly on boundary points, and that the eigenvalues, as functions of the endpoint b, satisfy a differential equation of the form

$$(p\lambda')(b) = -3(pu''')^2(b, b)$$

and all higher eigenvalues march off to plus infinity; this is also true for the first (that is, lowest) eigenvalue. Although we use the same method of proof as in [13] to get our main results, the specific process of calculation and proof is not completely the same as in [13]. Besides theoretical importance, the dependence of the eigenvalues on the interval is fundamental from the numerical point of view (see, for example, [1, 5, 7–10, 12–18, 22, 23]).

In Section 2 we summarise some of the basic results needed later and establish the notation. The main results are given in Sections 3 and 4, followed by a example in Section 5.

2. Notation and basic results

Consider the differential equation

$$(-py''')'''(t) + q(t)y(t) = \lambda w(t)y(t) \quad \text{on } (A, B),\ -\infty \le A < B \le \infty,\ \text{with } \lambda \in \mathbb{R}, \tag{2.1}$$

where

$$p, q, w : I = (A, B) \to \mathbb{R},\quad 1/p, q, w \in L_{\mathrm{loc}}(I),\ w > 0 \text{ a.e. on } I. \tag{2.2}$$

From [22], we know that there are three basic types of self-adjoint boundary conditions: separated, coupled and mixed. In the separated case, there are many forms for the sixth-order problems. In this paper, we study one form of them.

Let

$$J = [a, b], \quad A < a < b < B, \tag{2.3}$$

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and consider boundary conditions

$$\cos\alpha y(a) - \sin\alpha y'(a) = 0, \tag{2.4}$$

$$\cos\alpha y''(a) - \sin\alpha (py''')(a) = 0, \tag{2.5}$$

$$\cos\alpha (py''')'(a) - \sin\alpha (py''')''(a) = 0, \quad 0 \le \alpha < \pi, \tag{2.6}$$

$$\cos\beta y(b) - \sin\beta y'(b) = 0, \tag{2.7}$$

$$\cos\beta y''(b) - \sin\beta (py''')(b) = 0, \tag{2.8}$$

$$\cos\beta (py''')'(b) - \sin\beta (py''')''(b) = 0, \quad 0 < \beta \le \pi. \tag{2.9}$$

In this paper, we fix p, q, w and the boundary condition (constants) and one endpoint and study the dependence of the eigenvalues and eigenfunctions on the other endpoint.

By a solution of (2.1) on I we mean a function $y \in AC_{\mathrm{loc}}(I)$ such that

$$y', y'', (py'''), (py''')', (py''')'' \in AC_{\mathrm{loc}}(I),$$

and (2.1) is satisfied almost everywhere on I. Here $AC_{\mathrm{loc}}(I)$ denotes the set of functions which are absolutely continuous on all compact subintervals of I.

It is well known that the sixth-order BVP consisting of (2.1) together with boundary conditions (2.4)–(2.9) is a regular sixth-order self-adjoint BVP which has an infinite but countable number of only real eigenvalues. If $p \ge 0$ a.e. on $J = [a, b]$, then the eigenvalues are bounded below and can be ordered to satisfy

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \quad \text{and} \quad \lambda_n \to +\infty \quad \text{as } n \to +\infty. \tag{2.10}$$

If u_n is an eigenfunction of λ_n, then u_n can be chosen real, and is unique up to constant multiples.

3. Differential equations for eigenvalues

In this section, we first show the continuity properties of eigenvalues and eigenfunctions of the BVP (2.1)–(2.9), then obtain the differentiability of the eigenvalues and establish differential equations satisfied by them.

By a normalised eigenfunction u of the BVP (2.1)–(2.9) we mean one that satisfies

$$\int_a^b |u|^2 w = 1. \tag{3.1}$$

For fixed a and fixed boundary condition constants α, β we abbreviate the notation of Section 2 to $\lambda_n(b)$ and study $\lambda_n(b)$ as a function of b for fixed $n \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$, as b varies in the interval (a, B).

We now present a continuity result for the eigenvalues and eigenfunctions.

THEOREM 3.1. *Let self-adjoint BVPs be described as (2.1)–(2.9). Fix the boundary conditions and the endpoint a. Fix $n \in \mathbf{N}_0$. Let $\lambda_n = \lambda_n(b)$ for $b \in (a, B)$.*

(1) *$\lambda_n(b)$ is a continuous function of b for $b \in (a, B)$.*

(2) *If $\lambda_n(b)$ is simple for some $b \in (a, B)$ then $\lambda_n(b)$ is simple for every $b \in (a, B)$.*

(3) *There exists a normalised eigenfunction $u_n(\cdot, b)$ of $\lambda_n(b)$ for $b \in (a, B)$ such that $u_n(\cdot, b), u_n'(\cdot, b), u_n''(\cdot, b), (pu_n''')(\cdot, b), (pu_n''')'(\cdot, b)$ and $(pu_n''')''(\cdot, b)$ are uniformly convergent in b on any compact subinterval of (a, B), that is,*

$$\begin{aligned} u_n(\cdot, b+h) &\to u_n(\cdot, b), \quad u_n'(\cdot, b+h) \to u_n'(\cdot, b),\\ u_n''(\cdot, b+h) &\to u_n''(\cdot, b), \quad (pu_n''')(\cdot, b+h) \to (pu_n''')(\cdot, b),\\ (pu_n''')'(\cdot, b+h) &\to (pu_n''')'(\cdot, b), \quad (pu_n''')''(\cdot, b+h) \to (pu_n''')''(\cdot, b), \end{aligned} \tag{3.2}$$

and this convergence is uniform on any compact subinterval of (a, B).

Proof. (1) The continuity of $\lambda_n(b)$ as a function of b follows from [13, Theorem 3.1]. Although the proof is given there for second-order Sturm–Liouville case, it extends readily to our case.

(2) The fact that the multiplicity of $\lambda_n(b)$ is constant in b for $b \in (a, B)$ is a consequence of the spectral theorem for self-adjoint operators in Hilbert space.

(3) Firstly we show that there exist (not necessarily normalised) eigenfunctions $u_n(\cdot, b)$, $u_n(\cdot, b+h)$ for h sufficiently small such that (3.2) hold uniformly on any compact subinterval of (a, B). For any solution y of (2.1) and eigenfunction $u(\cdot, b)$, let

$$\mathbf{Y} = (y, y', y'', (py'''), (py''')', (py''')'')^T,$$
$$\mathbf{U} = (u, u', u'', (pu'''), (pu''')', (pu''')'')^T$$

where T denotes the transpose. Choose eigenfunctions $u = u_n(\cdot, b+h)$ for small h, all satisfying the same initial condition at a. Then the uniform convergence $\mathbf{U}(\cdot, b+h) \to \mathbf{U}(\cdot, b)$ on compact subintervals follows from part 1 and from the continuous dependence of solutions y and their quasi-derivatives $y', y'', (py'''), (py''')', (py''')''$ on the parameter λ.

By normalising the eigenfunctions we complete the proof. □

It turns out that the eigenvalues are differential functions of the endpoints satisfying first-order differential equations. The following lemmas are used to obtain these differential equations.

Lemma 3.2. *Assume u and v are solutions of (2.1) with $\lambda = \mu$ and $\lambda = \nu$, respectively. Then*

$$\begin{aligned} [u, v]_a^b &= [u, v](b) - [u, v](a)\\ &= [-(pu''')''\bar{v} + (pu''')'\bar{v}' - (pu''')\bar{v}'' + (p\bar{v}''')u'' - (p\bar{v}''')'u' + (p\bar{v}''')''u](b)\\ &\quad - [-(pu''')''\bar{v} + (pu''')'\bar{v}' - (pu''')\bar{v}'' + (p\bar{v}''')u'' - (p\bar{v}''')'u' + (p\bar{v}''')''u](a)\\ &= (\mu - \nu)\int_a^b u\bar{v}w. \end{aligned} \tag{3.3}$$

Proof. This follows from integration by parts. □

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Lemma 3.3. *Assume a real-valued function $f \in L_{\mathrm{loc}}(A, B)$. Then*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f = f(t) \quad \textit{a.e. in } (A, B).$$

Proof. See the proof given in [13]. □

Theorem 3.4. *Fix $n \in \mathbf{N}_0$, and consider the BVP (2.1)–(2.9) with $0 \le \alpha < \pi$ and $\beta = \pi$. Letting $\lambda = \lambda_n$ be the eigenvalue and $u = u_n$ be the corresponding eigenfunction, we have the following differential equation:*

$$(p\lambda')(b) = -3(pu''')^2(b, b) \quad \textit{a.e. in } (a, B). \tag{3.4}$$

In particular, if p is continuous at $b \in [a, B)$ and $p(b) \ne 0$, then (3.4) holds at b.

Proof. For small h, in (3.3) we choose $\mu = \lambda(b)$, $\nu = \lambda(b + h)$, and $u = u(\cdot, b)$, $v = u(\cdot, b + h)$. From (3.3) and the boundary conditions (2.4)–(2.9), noting that $[u, v](a) = 0$, $u(b, b) = u''(b, b) = (pu''')'(b, b) = 0$, we have

$$\begin{aligned}&[\lambda(b) - \lambda(b + h)] \int_a^b u(s, h)u(s, b + h)w(s)\, ds \\ &\quad = -(pu''')''(b, b)u(b, b + h) - (pu''')(b, b)u''(b, b + h) - u'(pu''')'(b, b + h),\end{aligned} \tag{3.5}$$

$$\begin{aligned} u(b, b + h) &= u(b, b + h) - u(b + h, b + h) \\ &= -\int_b^{b+h} u'(s, b + h)\, ds \\ &= -\int_b^{b+h} u'(s, b)\, ds + \int_b^{b+h} [u'(s, b) - u'(s, b + h)]\, ds. \end{aligned}$$

By Theorem 3.1 and the normalisation (3.1),

$$\lim_{h\to 0} \frac{u(b, b + h)}{h} = -u'(b, b). \tag{3.6}$$

Also

$$\begin{aligned} u''(b, b + h) &= u''(b, b + h) - u''(b + h, b + h) \\ &= -\int_b^{b+h} \frac{1}{p(s)} (pu''')(s, b + h)\, ds \\ &= -\int_b^{b+h} \frac{1}{p(s)} (pu''')(s, b)\, ds \\ &\quad + \int_b^{b+h} \frac{1}{p(s)} [(pu''')(s, b) - (pu''')(s, b + h)]\, ds. \end{aligned}$$

By Theorem 3.1 and the normalisation (3.1),

$$\lim_{h\to 0} \frac{u''(b, b + h)}{h} = -\frac{1}{p(b)} (pu''')(b, b). \tag{3.7}$$

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Similarly,

$$\lim_{h\to 0}\frac{(pu''')'(b,b+h)}{h} = -(pu''')''(b,b). \tag{3.8}$$

Observe that

$$\int_a^b u(s,b)u(s,b+h)w(s)\,ds \to \int_a^b u^2(s,b)w(s)\,ds = 1, \quad \text{as } h\to 0. \tag{3.9}$$

Plugging (3.6)–(3.9) into (3.5) and noting that $h \to 0$, we get

$$-\lambda'(b) = 2[(pu''')''u'](b,b) + \frac{1}{p(b)}(pu''')^2(b,b).$$

Noting that

$$\begin{aligned}[(pu''')''u'](b,b) &= \Big(\int_{b_0}^b [(pu''')''u'](s,b)\,ds\Big)' \\ &= \Big(\int_{b_0}^b u'(s,b)d(pu''')'(s,b)\Big)' \\ &= \Big(\int_{b_0}^b \frac{1}{p(s)}(pu''')^2(s,b)\,ds\Big)' \\ &= \frac{1}{p(b)}(pu''')^2(b,b),\end{aligned}$$

we get (3.4). The second part of the theorem follows from the above.

Theorem 3.5. *Fix $n \in \mathbf{N}_0$, and consider the BVP (2.1)–(2.9) with $0 \le \alpha < \pi$ and $\beta = \pi/2$. Letting $\lambda = \lambda_n$ be the eigenvalue and $u = u_n$ be the corresponding eigenfunction, we have the following differential equation:*

$$\lambda^{'}(b) = u^2(b,b)(q(b) - \lambda(b)w(b)) \quad \text{a.e. in } (a,B). \tag{3.10}$$

In particular, if q and w are continuous at $b \in [a,B)$, then (3.10) holds at b.

Proof. The proof is similar to that of Theorem 3.4. For small h, we choose μ, ν and u, v as in the proof of Theorem 3.4. From (3.3) and the boundary conditions (2.4)–(2.9), noting that

$$[u,v](a) = 0, u'(b,b) = (pu''')(b,b) = (pu''')''(b,b) = 0,$$

we have

$$\begin{aligned}[\lambda(b) - \lambda(b+h)]&\int_a^b u(s,h)u(s,b+h)w(s)\,ds \\ &= (pu''')'(b,b)u'(b,b+h) \\ &\quad + u''(b,b)(pu''')(b,b+h) + u(b,b)(pu''')''(b,b+h),\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
(pu''')''(b,b+h) &= (pu''')''(b,b+h) - (pu''')''(b+h,b+h)\\
&= -\int_b^{b+h} (pu''')'''(s,b+h)\,ds\\
&= -\int_b^{b+h} [q(s)u(s,b+h) - \lambda(b+h)u(s,b+h)w(s)]\,ds\\
&= -\int_b^{b+h} q(s)u(s,b)\,ds + \int_b^{b+h} q(s)[u(s,b) - u(s,b+h)]\,ds\\
&\quad + \lambda(b+h)\int_b^{b+h} u(s,b)w(s)\,ds\\
&\quad - \lambda(b+h)\int_b^{b+h} [u(s,b) - u(s,b+h)]w(s)\,ds.
\end{aligned}
$$

By Theorem 3.1 and Lemma 3.3,

$$\lim_{h\to 0}\frac{(pu''')''(b,b+h)}{h} = -(q(b) - \lambda(b)w(b))u(b,b). \tag{3.12}$$

In a similar way,

$$\lim_{h\to 0}\frac{(pu''')(b,b+h)}{h} = -(pu''')'(b,b), \tag{3.13}$$

when $h \to 0$, noting that

$$\int_a^b u(s,b)u(s,b+h)w(s)\,ds \to \int_a^b u^2(s,b)w(s)\,ds = 1. \tag{3.14}$$

Plugging (3.12)–(3.14) into (3.11), we obtain (3.10). The second part of the theorem follows from the above. □

THEOREM 3.6. *Fix $n \in \mathbf{N}_0$, and consider the BVP (2.1)–(2.9) with $0 \le \alpha < \pi, 0 < \beta \le \pi$. Letting $\lambda = \lambda_n$ be the eigenvalue and $u = u_n$ be the corresponding eigenfunction, we have the following differential equation:*

$$
\begin{aligned}
\lambda'(b) = &-\frac{1}{p(b)}(pu''')^2(b,b) + (q(b) - \lambda(b)w(b))u^2(b,b)\\
&- 2(pu''')''(b,b)u'(b,b) + 2(pu''')'(b,b)u''(b,b) \quad \text{a.e. in } (a,B).
\end{aligned} \tag{3.15}
$$

Furthermore, if $\beta \neq \pi$, then

$$
\begin{aligned}
\lambda'(b) = &-\frac{\cot^2\beta}{p(b)}(u'')^2(b,b) + (q(b) - \lambda(b)w(b))u^2(b,b)\\
&- 2\cot^2\beta(pu''')'(b,b)u(b,b) + 2(pu''')'(b,b)u''(b,b) \quad \text{a.e. in } (a,B).
\end{aligned} \tag{3.16}
$$

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If $\beta \neq \pi/2$, then

$$\begin{aligned}\lambda'(b) = &-\frac{1}{p(b)}(pu''')^2(b,b) + (q(b) - \lambda(b)w(b))\tan^2\beta(u')^2(b,b)\\ &- 2(pu''')''(b,b)u'(b,b)\\ &+ 2\tan^2\beta(pu''')''(b,b)(pu''')(b,b) \quad \text{a.e. in } (a,B).\end{aligned} \tag{3.17}$$

In particular, if p, q and w are continuous at b and $p(b) \neq 0$, then (3.15)–(3.17) hold at b.

Proof. The proof is more complicated, but consists basically of combining the techniques in the proofs of Theorems 3.4 and 3.5. For small h, we choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$, and $u = u(\cdot, b)$, $v = u(\cdot, b+h)$. From (3.3) and the boundary conditions (2.4)–(2.9), noting that $[u, v](a) = 0$, we have

$$\begin{aligned}([\lambda(b) - \lambda(b+h)]&\int_a^b u(s,h)u(s,b+h)w(s)\,ds\\ &= -(pu''')''(b,b)u(b,b+h) + (pu''')'(b,b)u'(b,b+h)\\ &\quad - (pu''')(b,b)u''(b,b+h) + (pu''')(b,b+h)u''(b,b)\\ &\quad - (pu''')'(b,b+h)u'(b,b) + (pu''')''(b,b+h)u(b,b).\end{aligned} \tag{3.18}$$

Now dividing (3.18) by h and taking the limit as $h \to 0$, plugging (3.7), (3.8), (3.12) and (3.13) into (3.18), and using the continuity of λ at b, the uniform convergence of $u(\cdot, b+h)$ to $u(\cdot, b)$, and Lemma 3.3, we obtain (3.15). In addition, from the boundary conditions (2.4)–(2.9) we note that if $\beta \neq \pi$, then

$$u'(b,b) = \cot\beta u(b,b), (pu''')(b,b) = \cot\beta u''(b,b), (pu''')''(b,b) = \cot\beta(pu''')'(b,b),$$

and if $\beta \neq \pi/2$, then

$$u(b,b) = \tan\beta u'(b,b), u''(b,b) = \tan\beta(pu''')(b,b), (pu''')'(b,b) = \tan\beta(pu''')''(b,b).$$

Plugging these into (3.15) we obtain (3.16) and (3.17), respectively. The rest part of the theorem follows from the above.

It is easy to see that Theorem 3.6 includes Theorems 3.4 and 3.5. □

4. Behaviour of the eigenvalues of sixth-order boundary value problems

Based on the differential equations we obtained in the previous section, we discuss the behaviour of the eigenvalues as functions of the endpoint b.

Theorem 4.1. *Consider the BVP (2.1)–(2.9). Let (2.2) hold, $\alpha = 0$, $\beta = \pi$. Fix a and consider the eigenvalues $\lambda_n^D(b) = \lambda_n^D(b)(0, \pi, a, b)$ for b in (a, B). If*

$$p \geq 0 \text{ a.e.} \quad \text{and} \quad \frac{q^2}{w}, \quad \frac{1}{p^2} \in L_{\text{loc}}(A,B),$$

then, for $n \in \mathbf{N}_0$, $\lambda_n(b)$ is strictly decreasing on (a, B) and

$$\lambda_n^D(b) \to +\infty \quad \text{as } b \to a^+. \tag{4.1}$$

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PROOF. The decreasing property of λ_n^D as a function of b follows directly from Theorem 3.4. Assume that (4.1) is false. Then, by Theorem 3.2, $\lambda(b) = \lambda_0^D$ has a finite limit, say $\lambda^+(a)$, as $b \to a^+$ and hence is bounded on $(a, B_1]$ for $B_1 < B$. Let $u = u_0(\cdot, b)$ be an eigenfunction of $\lambda(b)$ normalised to satisfy

$$\int_a^b u^2 w = 1. \tag{4.2}$$

Next we show that

$$(pu''')''(a,b) \to 0,\ (pu''')(a,b) \to 0,\ u'(a,b) \to 0, \quad \text{as } b \to a^+. \tag{4.3}$$

Noting that $(pu''')'(a,b) = (pu''')'(b,b) = 0$, according to Rolle's theorem we know that there exists at least one point $c_3 \in (a,b)$ such that $(pu''')''(c_3,b) = 0$. In addition, using the boundedness of λ and the Schwarz inequality, we get

$$\begin{aligned}
[(pu''')''(a,b)]^2 &= [-(pu''')''(a,b) + (pu''')''(c_3,b)]^2 \\
&= \Big[\int_a^{c_3} (pu''')'''\Big]^2 \\
&= \Big[\int_a^{c_3} (q - \lambda w)u\Big]^2 \\
&= \Big[\int_a^{c_3} (qw^{-1/2} - \lambda w^{1/2})w^{1/2}u\Big]^2 \\
&\le \int_a^{c_3} (qw^{-1/2} - \lambda w^{1/2})^2 \int_a^{c_3} u^2 w \\
&\le \int_a^b \Big(\frac{q^2}{w} - 2\lambda q + \lambda^2 w\Big) \int_a^b u^2 w \to 0 \quad (\text{as } b \to a^+).
\end{aligned}$$

There exists at least one point $c_2 \in (a,b)$ such that $(pu''')(c_2,b) = 0$ since $u''(a,b) = u''(b,b) = 0$. For $(pu''')(a,b)$, we similarly have

$$\begin{aligned}
[(pu''')(a,b)]^2 &= [-(pu''')(a,b) + (pu''')(c_2,b)]^2 \\
&= \Big[\int_a^{c_2} (pu''')'\Big]^2 \\
&= \Big[\int_a^{c_2} \int_a^{s} (pu''')''\, dt\, ds\Big]^2 \\
&\le \Big[\int_a^{c_2} \int_a^{s} \int_{c_3}^{t} (pu''')'''\, d\eta\, dt\, ds\Big]^2 \\
&\le (b-a)^4 \Big[\int_{c_3}^{t} (pu''')'''\, d\eta\Big]^2 \\
&\le (b-a)^4 \int_a^b \Big(\frac{q^2}{w} - 2\lambda q + \lambda^2 w\Big) \int_a^b u^2 w \to 0 \quad (\text{as } b \to a^+).
\end{aligned}$$

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For $u'(a,b)$, note that there exists at least one point $c_1 \in (a,b)$ such that $u'(c_1,b) = 0$ since $u(a,b) = u(b,b) = 0$. In addition, using the boundedness of λ and the Schwarz inequality, we get

$$\begin{aligned}[u'(a,b)]^2 &= [-u'(a,b) + u'(c_1,b)]^2\\ &= \Big[\int_a^{c_1} u''\,ds\Big]^2\\ &= \Big[\int_a^{c_1}\int_a^s \frac{1}{p(t)}\int_{c_2}^t\int_a^\eta\int_{c_3}^\tau (pu''')'''\,dm\,d\tau\,d\eta\,dt\,ds\Big]^2\\ &\le (b-a)^7\int_a^b \frac{1}{p^2(t)}\,dt\int_a^b\Big(\frac{q^2}{w} - 2\lambda q + \lambda^2 w\Big)\int_a^b u^2 w \to 0 \quad (\text{as } b\to a^+).\end{aligned}$$

Noting that $\lambda(b) \to \lambda^+(a)$ as $b \to a^+$, by (4.3) and the continuous dependence of solutions (2.1) on initial conditions and on the parameter, we conclude that $u(\cdot,b) \to 0$ uniformly on any compact subinterval of $[a,B)$. Therefore, for $\varepsilon > 0$, there exists a $b_0 \in (a,B)$ such that

$$|u(t,b)| < \varepsilon, \quad t \in [a,b],\ a < b < b_0. \tag{4.4}$$

This implies that

$$\int_a^b u^2 w < \varepsilon^2\int_a^b w. \tag{4.5}$$

For ε sufficiently small this contradicts the normalisation (4.2) and completes the proof. □

5. Example

To illustrate the result of Theorem 4.1 we give a simple example. Consider the differential equation

$$y^{(6)}(x) = \lambda y(x), \quad 0 \le x \le l < +\infty, \tag{5.1}$$

subject to the boundary conditions

$$y(0) = y(l) = 0, \quad y''(0) = y''(l) = 0, \quad y^{(4)}(0) = y^{(4)}(l) = 0. \tag{5.2}$$

It is easy to see the boundary value problem (5.1)–(5.2) is a regular sixth-order self-adjoint BVP which has an infinite but countable number of only real eigenvalues satisfying

$$0 < \lambda_1 \le \lambda_2 \le \cdots \quad \text{and} \quad \lambda_n \to +\infty \quad \text{as } n \to +\infty.$$

Let $\mathbb{Z}^+ = \{1,2,\cdots\}$, fix the boundary conditions, the endpoint 0, and $n \in \mathbb{Z}^+$. Let $\lambda_n = \lambda_n(l)$ for $l \in (0,+\infty)$. We now consider the dependence of eigenvalues of the BVP (5.1)–(5.2) on the boundary. For the differential equation (5.1) we know the general solutions

$$\begin{aligned}y(x) &= c_1 e^{i\alpha x} + c_2 e^{-i\alpha x} + c_3 e^{(\sqrt{3}/2+(1/2)i)\alpha x}\\ &\quad + c_4 e^{(\sqrt{3}/2-(1/2)i)\alpha x} + c_5 e^{(-\sqrt{3}/2+(1/2)i)\alpha x} + c_6 e^{(-\sqrt{3}/2-(1/2)i)\alpha x},\end{aligned}$$

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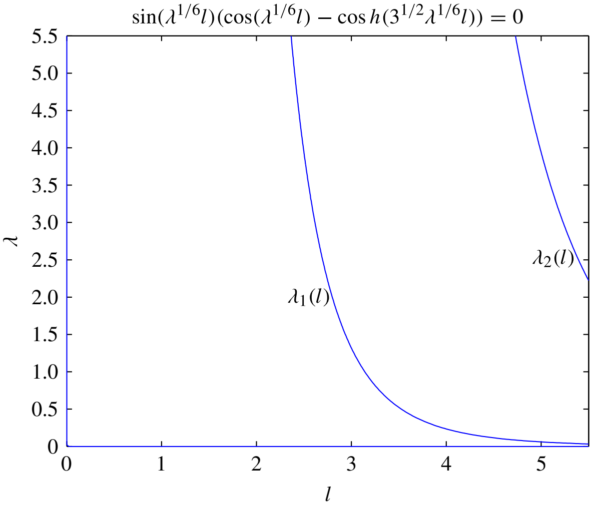

FIGURE 1. The functions $\lambda_1(l)$ and $\lambda_2(l)$.

where $i^2 = -1, \alpha^6 = \lambda, \alpha > 0$. The six homogeneous boundary conditions (5.2) yield six homogeneous equations of the form $\sum a_{ik}c_k$ $(i = 1, \ldots, 6)$ in the six quantities $c_1, c_2, c_3, c_4, c_5, c_6$.

To obtain a nontrivial solution, we set the determinant $|a_{ik}| = 0$. Calculations yield a transcendental equation for the eigenvalues λ,

$$\sin(\lambda^{1/6}l)(\cos(\lambda^{1/6}l) - \cosh(\sqrt{3}\lambda^{1/6}l)) = 0. \tag{5.3}$$

Although we cannot get exact solutions of (5.3), based on Theorem 4.1 we know that $\lambda_n(l) \to +\infty$, $n \in \mathbb{Z}^+$ as $l \to 0^+$. Noting that $\lambda_n(l) > 0$, we show the functions $\lambda_1(l)$ and $\lambda_2(l)$ in Figure 1, drawn using MATLAB.

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SUQIN GE, School of Mathematics, Physics and Biological Engineering,
Inner Mongolia University of Science and Technology,
BaoTou 014010, PR China
e-mail: 15647280518@163.com
and
School of Mathematical Sciences, Inner Mongolia University,
Hohhot 010021, PR China

WANYI WANG, School of Mathematical Sciences, Inner Mongolia University,
Hohhot 010021, PR China
e-mail: wwy@imu.edu.cn

QIUXIA YANG, College of Information and Management, Dezhou University,
Dezhou 253023, PR China
e-mail: yawenxuan@21cn.com