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ON EQUIVALENCE OF ANALYTIC FUNCTIONS TO RATIONAL REGULAR FUNCTIONS

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Abstract

We give sufficient conditions for an analytic function from $\mathbf{R}^n$ to $\mathbf{R}$ to be analytically equivalent to a rational regular function.

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1. Introduction

We say that two functions $f_1, f_2 : \mathbf{R}^n \to \mathbf{R}$ are analytically equivalent if $f_2 = f_1 \circ \sigma$ for some analytic diffeomorphism $\sigma : \mathbf{R}^n \to \mathbf{R}^n$.

A function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is said to be rational regular or, simply, regular if it can be written as $\varphi = \lambda/\mu$, where λ and μ are polynomial functions on $\mathbf{R}^n$ and μ does not vanish on $\mathbf{R}^n$.

In this paper we study the following problem.

PROBLEM 1.1. Under what conditions is a given analytic function $f : \mathbf{R}^n \to \mathbf{R}$ analytically equivalent to a regular function?

Some variations of this problem have been investigated in [1], [2], [5] and [6]. It was Thom's paper [6], which gave an impulse for research in this direction.

First let us observe that if f is analytically equivalent to a regular function, then for each point x in $\mathbf{R}^n$ the germ f_x of f at x is locally analytically equivalent to a germ of a regular function, that is, $f_x \circ \sigma_x$ is a germ of a regular function for

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some local analytic diffeomorphism $\sigma_x : (\mathbf{R}^n, x) \to (\mathbf{R}^n, x)$. The following example shows that even the nicest local behavior of f does not guarantee analytic equivalence of f to a regular function.

EXAMPLE 1.2. Let $f: \mathbf{R} \to \mathbf{R}$ be an analytic function with no critical point which has two distinct horizontal asymptotes, for example, $f(x) = \arctan x$. Clearly, for each point x in $\mathbf{R}$, the geremf$_x$ is locally analytically equivalent to the germ of the identity. However, f is not analytically equivalent to a regular function.

The only obstruction which prevents the function f of Example 1.2 from being analytically equivalent to a regular function is its "bad" behavior at "infinity" (cf. Theorem 1.5). To avoid this, we impose some restrictions on analytic functions under consideration.

Let S^n be the unit n-dimensional sphere and let $a = (0, \ldots, 0, 1) \in S^n$.

DEFINITION 1.3. An analytic function $f: \mathbf{R}^n \to \mathbf{R}$ is said to be meromorphic at infinity if there exists an analytic diffeomorphism $\tau: S^n \setminus \{a\} \to \mathbf{R}^n$ such that $f \circ \tau$ extends to a meromorphism function on S^n, that is, there exist a connected neighborhood U of a in S^n and analytic functions $u, v: U \to \mathbf{R}$ such that v is not identically equal to 0 on U and $f \circ \tau = u/v$ on $U \setminus v^{-1}(0)$ (it is well-known that u and v can be selected with $v^{-1}(0) = \{a\}$).

Definition 1.3 is quite natural in the context of this paper. Indeed, we have

PROPOSITION 1.4. *If an analytic function $f: \mathbf{R}^n \to \mathbf{R}$ is analytically equivalent to a regular function, thenit is meromorphic at infinity.*

PROOF. Choose an analytic diffeomorphism $\sigma: \mathbf{R}^n \to \mathbf{R}^n$ such that $f \circ \sigma$ is a regular function. Let $\rho: S^n \setminus \{a\} \to \mathbf{R}^n$ be the stereographic projection from a,

$$\rho(x_1, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right).$$

Clearly, $f \circ \sigma \circ \rho$ can be written as $f \circ \sigma \circ \rho = \lambda/\mu$, where λ and μ are polynomial functions on $\mathbf{R}^{n+1}$ with μ nonvanishing on $S^n \setminus \{a\}$. It follows that f is meromorphic at infinity.

Summarizing, every analytic function from $\mathbf{R}^n$ to $\mathbf{R}$, analytically equivalent to a regular function, is locally analytically equivalent to a germ of a regular function and is meromorphic at infinity. It is an interesting question to what extent the converse is true.

Before we formulate our main result, we need to recall a few concepts. Given a point x in $\mathbf{R}^n$, we denote by $\mathcal{O}_x$ the ring of all analytic function-germs $(\mathbf{R}^n, 0) \to \mathbf{R}$. If f_x belongs to $\mathcal{O}_x$, then $\Delta(f_x)$ denotes the ideal of $\mathcal{O}_x$ generated by the first

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partial derivatives of f_x. The Milnor number of f_x is the dimension of the $\mathbf{R}$-vector space $\mathcal{O}_x/\Delta(f_x)$. It is well-known that if the Milnor number of f_x is finite, then given any analytic germ g_x in $\mathcal{O}_x$ with $g_x - f_x$ being k-flat at x, one can find a local analytic diffeomorphism $\sigma_x: (\mathbf{R}^n, x) \to (\mathbf{R}^n, x)$ satisfying $g_x = f_x \circ \sigma_x$, provided that k is sufficiently large [7]. In particular, f_x is locally analytically equivalent to the germ at x of a polynomial function.

THEOREM 1.5. *Let $f: \mathbf{R}^n \to \mathbf{R}$ be an analytic function whose germ at each point in $\mathbf{R}^n$ has a finite Milnor number. Then f is analytically equivalent to a regular function if and only if the set of critical poins of f is finite and f is meromorphic at infinity.*

We conclude this section by recalling that the set of all analytic functions $f: \mathbf{R}^n \to \mathbf{R}$ such that for each point x in $\mathbf{R}^n$ the germ f_x has a finite Milnor number is very large [7].

2. Proof of Theorem 1.5

Let M be a C^∞ manifold. Denote by $\mathscr{E}(M)$ the ring of C^∞ functions on M. Let $X_1, \dots, X_n$ be C^∞ vector fields on M generating the $\mathscr{E}(M)$-module of all C^∞ vector fields on M. Given an element (f_1, f_2) in $\mathscr{E}(M)^2$, we define $I(f_1, f_2)$ to be the ideal of $\mathscr{E}(M)$ generated by all 2×2 minors of the matrix

$$\begin{pmatrix} f_1 & X_1 f_1 & \cdots & X_n f_1 \\ f_2 & X_1 f_1 & \cdots & X_n f_2 \end{pmatrix}$$

and $S(f_1, f_2)$ to be the set of zeros of $I(f_1, f_2)$.

We shall need the following two auxiliary results.

LEMMA 2.1. *Let M be a compact C^∞ manifold. Let (f_1, f_2) be an element in $\mathscr{E}(M)^2$ and let Z be a subset of M. Assume that the ideal $I(f_1, f_2)^2$ is closed in the C^∞ topology and the ideal $I(Z)$ of all functions in $\mathscr{E}(M)$ vanishing on Z is finitely generated. Then there exists a neighborhood $\mathscr{V}$ of 0 in $\mathscr{E}(M)$ such that for every element (g_1, g_2) in $\mathscr{E}(M)^2$ with*

$$g_i - f_i \in I(Z) I(f_1, f_2)^2 \cap \mathscr{V}, \qquad i = 1, 2,$$

one can find a C^∞ diffeomorphism $\sigma: M \to M$ and an element u in $\mathscr{E}(M)$ satisfying $\sigma(x) = x$ for all x in Z, $u > 0$ and $(g_1, g_2) = (u(f_1 \circ \sigma), u(f_2 \circ \sigma))$.

PROOF. If Z is the empty set, then Lemma 2.1 is a particular case of [2], Theorem 2.1 with, in the notation of [2], $p = 2$ and G being the subgroup of $Gl(2, \mathbf{R})$ of all diagonal matrices having the diagonal of the form (λ, λ), where

$\lambda > 0$. In the general case only an obvious modification of the proof of [2], Theorem 2.1 is necessary.

LEMMA 2.2. *Let M be a compact real analytic manifold. Let (f_1, f_2) be an element of $\mathscr{E}(M)^2$. Assume that the set $S(f_1, f_2)$ is discrete and f_i is analytic in a neighborhood of $S(f_1, f_2)$ for $i = 1, 2$. Then there exist a neighborhood $\mathscr{V}$ of 0 in $\mathscr{E}(M)$ and a positive integer k such that for each element (g_1, g_2) in $\mathscr{E}(M)^2$ with $g_i - f_i$ belonging to $\mathscr{V}$ and being k-flat at $S(f_1, f_2)$, $i = 1, 2$, one has $S(g_1, g_2) = S(f_1, f_2)$.*

PROOF. Since one can choose finitely many analytic vector fields on M generating the $\mathscr{E}(M)$-module of all C^∞ vector fields on M, the ideal $I(f_1, f_2)$ is generated by finitely many C^∞ functions which are analytic in a neighborhood of $S(f_1, f_2)$. Now the conclusion is a consequence of the following observation.

Let $u: \mathbf{R}^n \to \mathbf{R}$ be an analytic function with $u^{-1}(0) = \{0\}$. By the Lojasiewicz inequality [7], there exist positive real numbers c and ε and a positive integer l such that

$$|u(x)| \geqslant c\|x\|^l \quad \text{for } \|x\| \leqslant \varepsilon.$$

If $v: \mathbf{R}^n \to \mathbf{R}$ is a C^∞ function such that $v - u$ is l-flat at 0 in $\mathbf{R}^n$, then, by the Taylor formula,

$$\begin{aligned} |v(x)| &\geqslant |u(x)| - |v(x) - u(x)| \\ &\geqslant c\|x\|^l - \left(\sup\left\{\sum_\alpha \frac{1}{\alpha!}|D^\alpha(v-u)(x)| : \|x\| \leqslant \varepsilon\right\}\right)\|x\|^{l+1}. \end{aligned}$$

We shall also need a special local version of Lemma 2.1. Let $(f_1, f_2): (\mathbf{R}^n, 0) \to \mathbf{R}^2$ be a C^∞ map-germ. Denote by $I_0(f_1, f_2)$ the ideal of the ring $\mathscr{E}_0$ of C^∞ function-germs $(\mathbf{R}^n, 0) \to \mathbf{R}$ generated by all 2×2 minors of the matrix

$$M_0(f_1, f_2) = \begin{pmatrix} f_1 & \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ f_2 & \dfrac{\partial f_2}{\partial x_1} & \cdots & \dfrac{\partial f_2}{\partial x_n} \end{pmatrix}.$$

LEMMA 2.3. *Let $(f_1, f_2): (\mathbf{R}^n, 0) \to \mathbf{R}^2$ be an analytic map-germ. Assume that the set-germ of zeros of $I_0(f_1, f_2)$ is contained in $\{0\}$. Then there exists a positive integer k such that for each analytic map-germ $(g_1, g_2): (\mathbf{R}^n,) \to \mathbf{R}^2$ with $g_i - f_i$ being k-flat at 0 in $\mathbf{R}^n$, $i = 1, 2$, one can find a local C^1 orientation preserving diffeomorphism $\sigma: (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^1 function-germ $u: (\mathbf{R}^n, 0) \to \mathbf{R}$, both analytic off the origin and satisfying $u > 0$ and $(g_1, g_2) = (u(f_1 \circ \sigma), u(f_2 \circ \sigma))$.*

PROOF. Let k be a positive integer and let $(g_1, g_2): \mathbf{R}^n \to \mathbf{R}^2$ be an analytic map-germ such that $g_i - f_i$ is k-flat at 0 in $\mathbf{R}^n$ for $i = 1, 2$. Define $F(x,t) = (F_1(x,t), F_2(x,t))$ by $F_i(x,t) = (1-t)f_i(x) + tg_i(x)$ for t in $[0,1]$ and $i = 1, 2$. Let $\mathcal{O}_n[0,1]$ be the ring of analytic function-germs $(\mathbf{R}^n \times \mathbf{R}, \{0\} \times [0,1]) \to \mathbf{R}$. Set

$$M_0(F_1, F_2) = \begin{pmatrix} F_1 & \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ F_2 & \dfrac{\partial F_2}{\partial x_1} & \cdots & \dfrac{\partial F_2}{\partial x_n} \end{pmatrix}$$

and

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If Δ is a 2×2 minor of $M_0(F_1, F_2)$, then Δe_i is a linear combination of the columns of $M_0(F_1, F_2)$ with coefficients in $\mathcal{O}_n[0,1]$. Let δ be the sum of squares of all 2×2 minors of $M_0(F_1, F_2)$. Then

$$-\delta(x,t)\frac{\partial F}{\partial t}(x,t) = \sum_{j=1}^{n} \xi_j(x,t)\frac{\partial F}{\partial x_j}(x,t) + \eta(x,t)F(x,t)$$

for some ξ_j, η in $\mathcal{O}_n[0,1]$. Moreover, $\xi_j(\cdot,t)$ and $\eta(\cdot,t)$ are k-flat at 0 in $\mathbf{R}^n$ for all t in $[0,1]$. Notice that $\delta(\cdot,0)$ is the sum of squares of all 2×2 minors of the matrix $M_0(f_1, f_2)$. Thus the set-germ of zeros of $\delta(\cdot,0)$ is contained in $\{0\}$. By the Lojasiewicz inequality [7],

$$\delta(x,0) \geqslant c\|x\|^l \quad \text{for } \|x\| < \alpha,$$

where $c, \alpha > 0$ and l is a positive integer. Now assume that k has been chosen satisfying $k \geqslant 2(l+1)$. Since $\delta(\cdot,t) - \delta(\cdot,0)$ is $(k-1)$-flat at 0 in $\mathbf{R}^n$, it follows that

$$\delta(x,t) \geqslant \frac{c}{2}\|x\|^l \quad \text{for } \|x\| < \beta \quad \text{and} \quad t \in [0,1],$$

where $\beta > 0$. By choice of k, for each $j = 1, \ldots, n$, ξ_j/δ and η/δ have C^1 extensions ξ'_j and η' on a neighborhood of $\{0\} \times [0,1]$ vanishing on $\{0\} \times [0,1]$. One also has

$$-\frac{\partial F}{\partial t}(x,t) = d_2F(x,t)\big(\xi'(x,t)\big) + \eta'(x,t)F(x,t),$$

where

$$\xi'(x,t) = \sum_{j=1}^{n} \xi'(x,t)\frac{\partial}{\partial x_j}.$$

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Let $\tau:(\mathbf{R}^n \times \mathbf{R}, \{0\} \times [0,1]) \to (\mathbf{R}^n, 0)$ be a C^1 map-germ satisfying

$$\begin{cases} \dfrac{\partial \tau}{\partial t}(x,t) = \xi'(\tau(x,t),t), \\ \tau(x,0) = x \end{cases}$$

and let

$$v(x,t) = \exp\left(-\int_0^t \eta'(\tau(x,s),s)\,ds\right).$$

Notice that

$$\frac{\partial}{\partial t}(F(\tau(x,t),t)) = -\eta'(\tau(x,t),t)F(\tau(x,t),t)$$

and

$$\frac{\partial}{\partial t}(v(x,t)F(x,0)) = -\eta'(\tau(x,t),t)(v(x,t)F(x,0)).$$

By the uniqueness of the solution of ordinary differential equations, one obtains $F(\tau(x,t),t) = v(x,t)F(x,0)$. It suffices to set $\sigma^{-1}(x) = \tau(x,1)$ and $u(x) = v(\sigma(x),1)$. Clearly, σ and u are analytic off the origin and the conclusion follows.

Let M and N be C^∞ manifolds and let $f:M \to N$ be a C^∞ map. The set of critical points of f will be denoted by $\Sigma(f)$ and the germ of f at a point x in M by f_x. If X is a subset of $\mathbf{R}^n$, then by a polynomial function on X we shall mean the restriction to X of a polynomial function on $\mathbf{R}^n$.

Proof of Theorem 1.5. It follows from the assumption that the set $\Sigma(f)$ is discrete. Assume that $f \circ \sigma$ is a regular function for some analytic diffeomorphism $\sigma:\mathbf{R}^n \to \mathbf{R}^n$. Clearly, the set $\Sigma(f \circ \sigma)$ is semi-algebraic. Since every semi-algebraic set has only finitely many connected components [3], the set $\Sigma(f \circ \sigma)$ and, hence also $\Sigma(f)$, is finite. By Proposition 1.4, f is meromorphic at infinity.

Now suppose that $\Sigma(f)$ is finite and f is meromorphic at infinity. Let $\tau: S^n \setminus \{a\} \to \mathbf{R}^n$ be an analytic diffeomorphism and let $u_1, u_2: U \to \mathbf{R}$ be analytic functions defined on a neighborhood U of a in S^n such that u_2 does not vanish on $U \setminus \{a\}$ and $f \circ \tau = u_1/u_2$ on $U \setminus \{a\}$. We may assume that $u_2 \geqslant 0$ on U (replace u_1 and u_2 by u_1u_2 and u_2^2, respectively, if necessary). It is easy to construct two C^∞ functions $f_1, f_2: S^n \to \mathbf{R}$ such that $f_i = u_i$ on a neighborhood of a, $i = 1,2$, $f_2 \geqslant 0$, f_2 does not vanish on $S^n \setminus \{a\}$ and $f \circ \tau = f_1/f_2$ on $S^n \setminus \{a\}$. Notice that

$$S(f_1, f_2) \subseteq \sum(f \circ \tau) \cup \{a\}.$$

Let $\mathscr{V}$ be a small neighborhood of 0 in $\mathscr{E}(S^n)$ and let k be a large positive integer (how small $\mathscr{V}$ is and how large k is will be clear later on). For each $i = 1,2$, one can find a polynomial map $\varphi_i: S^n \to \mathbf{R}$ such that $f_i - \varphi_i$ belongs to

$\mathscr{V}$ and $f_i - \varphi_i$ is k-flat at $\Sigma(f \circ \tau) \cup \{a\}$ (cf. [1], Corollary 1). By Lemma 2.2, $S(\varphi_1, \varphi_2) = S(f_1, f_2)$. Moreover, for each point x in $\Sigma(f \circ \tau)$ one can find a local C^∞ orientation preserving diffeomorphism $\alpha_x : (S^n, x) \to (S^n, x)$ and a C^∞ function-germ $v_x : (S^n, x) \to \mathbf{R}$ such that $v_x > 0$ and $(\varphi_{1x}, \varphi_{2x}) = (v_x(f_{1x} \circ \alpha_x), v_x(f_{2x} \circ \alpha_x))$ (this follows from the fact that the Milnor number of f at $\tau(x)$ is finite and from [7], p. 169, Theorem 3.11 applied for $p = 2$ and the subgroup G of $Gl(2, \mathbf{R})$ consisting of all diagonal matrices having the diagonal of the form (λ, λ), where $\lambda > 0$). By Lemma 2.3, there exists a local C^1 orientation preserving diffeomorphism $\alpha_a : (S^n, a) \to (S^n, a)$ and a C^1 function-germ $v_a : (S^n, a) \to \mathbf{R}$ such that $v_a > 0$, α_a and v_a are analytic off a and $(\varphi_{1a}, \varphi_{2a}) = (v_a(f_{1a} \circ \alpha_a), v_a(f_{2a} \circ \alpha_a))$. Now one can find a C^1 diffeomorphism $\beta : S^n \to S^n$ and a C^1 function $w : S^n \to \mathbf{R}$ such that β and w are of class C^∞ on $S^n \setminus \{a\}$, $\beta_a = \alpha_a$, $w_a = v_a$, $w > 0$ on S^n and for each x in $\Sigma(f \circ \tau)$, $\beta_x = \alpha_x$ and $w_x = v_x$. Set $g_i = w(f_i \circ \beta)$ for $i = 1, 2$. Clearly, g_i is a C^∞ function on S^n and $g_i = \varphi_i$ on a neighborhood of $\Sigma(f \circ \tau) \cup \{\alpha\}$. Notice that the ideal $I(g_1, g_2)^2 = I(\varphi_1, \varphi_2)^2$ is closed in $\mathscr{E}(S^n)$. Indeed, the ideal $I(\varphi_2, \varphi_2)^2$ can be generated by polynomial functions (choose the appropriate vector fields on S^n) and, hence, is closed [7]. Fix a polynomial function λ in $I(g_1, g_2)^2$ with $\lambda^{-1}(0) = S(g_1, g_2)$. For each $i = 1, 2$, we can find a C^∞ function $h_i : S^n \to \mathbf{R}$ satisfies

$$g_i - \varphi_i = h_i \lambda \quad \text{and} \quad h_i(a) = 0.$$

Let μ_i be a polynomial approximation to h_i with $\mu_i(a) = 0$. By Lemma 2.1 (with $Z = \{a\}$), there exist a C^∞ diffeomorphism $\gamma : S^n \to S^n$ and a C^∞ function $u : S^n \to \mathbf{R}$ such that $u > 0$, $\gamma(a) = a$ and $(\psi_1, \psi_2) = (u(g_1 \circ \gamma), u(g_2 \circ \gamma))$, where $\psi_i = \varphi_i + \mu_i \lambda$ for $i = 1, 2$. It follows that $f \circ \tau \circ \beta \circ \gamma \circ \rho^{-1} = (\psi_1/\psi_2) \circ \rho^{-1}$ on $\mathbf{R}^n$, where $\rho : S^n \setminus \{a\} \to \mathbf{R}^n$ is the stereographic projection. Notice that $\tau \circ \beta \circ \gamma \circ \rho^{-1}$ is a C^∞ diffeomorphism and $(\psi_1/\psi_2) \circ \rho^{-1}$ is a regular function. By [4], Theorem 8.4, f and $(\psi_1/\psi_2) \circ \rho^{-1}$ are analytically equivalent.

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