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AREA-DIAMETER AND AREA-WIDTH RELATIONS FOR COVERING PLANE SETS

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An area-diameter relation and an area-width relation for plane lattice-point-free-convex bodies is proved. This implies relations on covering sets with respect to general lattices.

1. INTRODUCTION AND NOTATIONS

Let E^2 denote the Euclidean plane and let $\mathcal{L}^2$ denote the set of lattices $L \subset E^2$ with $\det(L) \neq 0$. Further let $\mathcal{K}^2$ denote the set of convex bodies $K \subset E^2$. For $K \in \mathcal{K}^2$, let $A(K)$, $D(K)$ and $\Delta(K)$ be the area, the diameter and the minimal width of K respectively. Further for $L \in \mathcal{L}^2$ let $\lambda_i(L)$ be the successive minima of L, that is, $\lambda_i(L) = \lambda_i(B^2, L) = \min\{\lambda > 0 \mid \dim\ \operatorname{aff}(\lambda B^2 \cap L) \geqslant i\}$ and let $\mu_i(L)$ be the covering minima of the lattice L, that is, $\mu_i(L) = \mu_i(B^2, L) = \min\{\mu > 0 \mid \mu B^2 + \mathbf{g},\ \mathbf{g} \in L$, meets every flat F of E^2 with $\dim(F) = 2 - i\}$ (for these definitions see [**4**] and [**5**]).

Note that $\lambda_1(L)$ is the length of the shortest non-zero vector of L and $2\mu_1(L)$ is the maximal distance of two adjacent lattice lines. Therefore $\det(L) = 2\mu_1\lambda_1$ and $2\mu_1(L) = 1/\lambda_1(L^*)$ where L^* is the reciprocal lattice of L. The relation $2\mu_1 \geqslant \sqrt{3}/2\lambda_1$ will be also useful in the sequel (see, for example, [**4**]).

Let $G(K, L) =$ card $((\operatorname{int} K) \cap L)$ denote the lattice point enumerator.

A convex set K is called a *lattice-point-free convex set* with respect to L, if $G(K, L) = 0$. Further K is a *covering set* if $K + L = \{K + \mathbf{g} \mid \mathbf{g} \in L\} = E^2$.

For the integer lattice $\mathbf{Z}^2$ there are several inequalities relating $\Delta(K)$, $D(K)$, $A(K)$ and the perimeter $P(K)$ of covering sets or lattice-free convex bodies (see [**2**]); but only a few results concerning arbitrary lattices [**6, 7, 10, 11**].

In this paper we generalise two results of Scott [**8, 9**] to arbitrary lattices.

2. RESULTS

Let us denote by τ the unique solution of the equation $\int_0^t \sqrt{1 - x^2}\,dx = \pi/8$ ($\tau \simeq 0.403977$ and $\tau = \sin(\phi^*)$, where ϕ^* is, as in Scott's theorem [**8**], the unique solution of $\sin(2\phi) + 2\phi = \pi/2$) then we get:

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Theorem 1. *If $K \in \mathcal{K}^2$ and $L \in \mathcal{L}^2$, with $G(K,L) = 0$, then*

$$\frac{A(K)}{D(K)} \leqslant \max\left\{2\mu_1(L),\ 2\tau\sqrt{\lambda_1^2(L) + (2\mu_1(L))^2}\right\}, \tag{1}$$

and this result is best possible.

Remark. We have $2\mu_1 > 2\tau\sqrt{\lambda_1^2 + (2\mu_1)^2}$ if and only if $2\mu_1 > \lambda_1\left(2\tau/\sqrt{1-4\tau^2}\right) \simeq 1.3711\,\lambda_1$.

Theorem 2. *If $K \in \mathcal{K}^2$ and $L \in \mathcal{L}^2$, with $G(K,L) = 0$, then*

$$2A(K)\,(\Delta(K) - 2\mu_1(L)) \leqslant \lambda_1(L)\Delta^2(K), \tag{2}$$

and equality holds if and only if K is a triangle with width $\Delta(K)$ and diameter $D(K) = (\lambda_1(L)\Delta(K))/(\Delta(K) - 2\mu_1(L))$.

Corollary 1. *Let $K \in \mathcal{K}^2$ and $L \in \mathcal{L}^2$ be given such that:*

$$\frac{A(K)}{D(K)} > k\ \max\left\{2\mu_1, 2\tau\sqrt{\lambda_1^2 + (2\mu_1)^2}\right\}, \qquad k \in \mathbf{Z}. \tag{3}$$

Then $G(K,L) \geqslant k^2$, that is, $\{K + \mathbf{g} \mid \mathbf{g} \in L\}$ is at least a k^2-fold covering of E^2.

Corollary 2. *Let $K \in \mathcal{K}^2$ and $L \in \mathcal{L}^2$ be given such that:*

$$2A(K)\,(\Delta(K) - 2\mu_1(L)) > \lambda_1(L)\Delta^2(K). \tag{4}$$

Then K is a covering set.

3. Proof of the results

Proof of Theorem 1.

Theorem 1 will be proved by reducing the problem to rectangular lattices and symmetric convex bodies.

Let $\{\mathbf{b}_1, \mathbf{b}_2\}$ be a Minkowski reduced basis of L (see [**2**, p.84]), with $\|\mathbf{b}_1\| = \lambda_1(L)$ and let θ be the acute angle between $\mathbf{b}_1$ and $\mathbf{b}_2$ (so that $2\mu_1(L) = \|\mathbf{b}_2\|\sin\theta$).

Let $\mathbf{v}_1 = \mathbf{b}_1$, and let $\mathbf{v}_2$ be a vector of length $2\mu_1$, perpendicular to $\mathbf{v}_1$. Let Λ denote the rectangular lattice determined by the basis vectors $\mathbf{v}_1$, $\mathbf{v}_2$. We shall prove the following:

Lemma 1. *If K is a convex body such that $G(K,L) = 0$, then there exists another convex body C containing no points of Λ, such that*

- (i) $A(C) = A(K)$, $D(C) \leqslant D(K)$,
- (ii) *C is symmetric about the lines $x = 1/2$, $y = 1/2$, the coordinates x and y being relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$.*

PROOF: Let K' be the convex body obtained from K by symmetrisation with respect to the line $x = 1/2$. It is well known that Steiner symmetrisation preserves convexity and areas, and does not increase diameters (see [1]). Therefore K' is convex, $A(K') = A(K)$, and $D(K') \leqslant D(K)$.

We shall show now that $G(K', \Lambda) = 0$. If K' contained a lattice point of Λ, say the point $m\mathbf{v}_1 + n\mathbf{v}_2$, then the line $y = n$, for the symmetry of K' with respect to $x = 1/2$, intersects K' in a line segment of length greater than λ_1. The same line also intersects K in a line segment of the same length and this implies that $G(K, L) > 0$, contradicting the hypothesis. Therefore $G(K', \Lambda) = 0$.

A similar argument shows that if we now symmetrise K' with respect to the line $y = 1/2$, we obtain a convex body C with the required properties. ∎

In view of Lemma 1, to deduce the inequality of Theorem 1 it suffices to prove the following:

LEMMA 2. *Let Λ be a rectangular lattice with basis $\{\lambda_1\mathbf{e}_1, \lambda_2\mathbf{e}_2\}$ (so that $\lambda_i(\Lambda) = \lambda_i$ where $i = 1, 2$). For any convex body K, symmetric with respect to the lines $x = \lambda_1/2$ and $y = \lambda_2/2$ with $G(K, \Lambda) = 0$*

$$\frac{A(K)}{D(K)} \leqslant \max\left\{\lambda_2,\ 2\tau\sqrt{\lambda_1^2 + \lambda_2^2}\right\} \tag{5}$$

and the inequality is sharp.

REMARK. For the original lattice L, Lemma 2 implies

$$\frac{A(K)}{D(K)} \leqslant \max\left\{2\mu_1(L),\ \lambda_1(L),\ 2\tau\sqrt{\lambda_1^2(L) + (2\mu_1(L))^2}\right\},$$

so that, by $2\mu_1(L) > (\sqrt{3}/2)\lambda_1(L)$, we obtain $2\tau\sqrt{\lambda_1^2(L) + (2\mu_1(L))^2} \geqslant \tau\lambda_1\sqrt{7} > \lambda_1$.

PROOF: To better utilise the symmetry of K, we translate the origin to the point $(\lambda_1/2, \lambda_2/2)$. Then the lattice Λ is changed into the grid $\Gamma = \{(\lambda_1(m + 1/2), \lambda_2(n + 1/2)) \mid m, n \in \mathbf{Z}\}$.

For the sake of brevity we write $D = D(K)$ and $A = A(K)$.

Since K is centrally symmetric, it lies within the disc $x^2 + y^2 \leqslant D^2/4$. If $D \leqslant \sqrt{\lambda_1^2 + \lambda_2^2}$, no point of Γ is interior to this disc and then:

$$A \leqslant \frac{\pi}{4}D^2 \leqslant \frac{\pi}{4}\left(\sqrt{\lambda_1^2 + \lambda_2^2}\right)D < 2\tau\left(\sqrt{\lambda_1^2 + \lambda_2^2}\right)D.$$

Therefore we may suppose $D > \sqrt{\lambda_1^2 + \lambda_2^2}$.

Let Q be the part of K lying in the quadrant $x \geqslant 0$, $y \geqslant 0$. Because of the convexity of K, Q lies below some line l through the point $P = ((\lambda_1/2), (\lambda_2/2))$ with

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non positive slope and with equation $y = (\lambda_2/2) + m(x - (\lambda_1/2))$. Let us denote by X and Y respectively the intersection of the line l with the coordinate axes and by $\mathcal{D}$ the disc $x^2 + y^2 \leqslant D^2/4$. We distinguish two cases:

(a) $X, Y \notin \mathcal{D}$;

(b) exactly one of the points X, Y is exterior to $\mathcal{D}$.

In case (a) the area of Q is given by:

$$A(Q) = \frac{\pi}{16} D^2 - 2 \int_0^{\sqrt{(D/2)^2 - q^2}} \left(\sqrt{(D/2)^2 - u^2} - q \right) du$$

where q is the distance of the line l from the origin.

We have thus:

$$\frac{A}{D} = 4q\sqrt{1 - \left(\frac{2q}{D}\right)^2} + \frac{\pi}{4} D - 2D \int_0^{\sqrt{1-(2q/D)^2}} \sqrt{1 - t^2}\, dt$$

and a short calculation shows that this function attains its maximum when $q = (1/2)\sqrt{\lambda_1^2 + \lambda_2^2}$, that is, when the line l is normal to the segment OP, and the diameter D satisfies the equation $\int_0^{\sqrt{1-(q/D)^2}} \sqrt{1 - t^2}\, dt = \pi/8$, that is, $D \simeq 1.09317\sqrt{\lambda_1^2 + \lambda_2^2}$, so that we obtain:

$$\frac{A(K)}{D(K)} = 2\tau\sqrt{\lambda_1^2 + \lambda_2^2}. \tag{6}$$

(Actually the function $A(K)/D(K)$ is an increasing function of q and $q \leqslant (1/2)\sqrt{\lambda_1^2 + \lambda_2^2}$, moreover its derivative with respect to D vanishes if and only if $D(K)$ satisfies the above equation.)

If $\lambda_2 \leqslant 2\tau\sqrt{\lambda_1^2 + \lambda_2^2}$, the previous solutions are acceptable since for this value of $D(K)$ the points X and Y are exterior to the disk $\mathcal{D}$. Otherwise the maximum value of $A(K)/D(K)$ is taken when the line l is normal to the segment OP and passes through the point Y: obviously in this case we have $A(K)/D(K) < 2\tau\sqrt{\lambda_1^2 + \lambda_2^2} < \lambda_2$.

In case (b) let us suppose that $Y \in \mathcal{D}$, so Q is a subset of the trapezium T bounded by the coordinate axes, the line l and the line $x = D/2$. This trapezium has area $A(T) = (D/8)\,[2\lambda_2 + m\,(D - 2\lambda_1)]$.

If $D \geqslant 2\lambda_1$, this area is at most $(D(K)\lambda_2)/4$ and thus

$$\frac{A(K)}{D(K)} < \lambda_2. \tag{7}$$

If $D < 2\lambda_1$, then $A(T)$ is an increasing function of m so that it is easy to see that the maximum of the area of the region Q is taken when the point Y belongs to the line l.

In this case we have $A(K)/D(K) \leqslant 2\tau\sqrt{\lambda_1^2 + \lambda_2^2}$. □

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REMARK. Inequality (7) could seem too wide, but the example of a rectangle with diagonal–length D and an edge–length λ_2 shows that this bound is best possible when $D \to \infty$.

PROOF OF THEOREM 2.

For the sake of brevity we write $\Delta = \Delta(K)$, $A = A(K)$, $\lambda_1 = \lambda_1(L)$ and $\mu_1 = \mu_1(L)$.

First we observe that the inequality in Theorem 2 can be written

$$\frac{\lambda_1}{2A} - \frac{\Delta - 2\mu_1}{\Delta^2} \geqslant 0.$$

Therefore we can take K to be the set realising the minimum of the left-hand side of this inequality.

Because of $\Delta \leqslant 2\mu_1 + (\sqrt{3}/2)\lambda_1 \leqslant 4\mu_1$ (see [10]), $(\Delta - 2\mu_1)/\Delta^2$ is an increasing function of Δ and hence we choose K with A and Δ as large as possible.

It is clear that K must be one of the following sets:

(a) a triangle with one (or two) of its sides on a lattice line;
(b) a triangle with one lattice point on each of its sides;
(c) a quadrilateral with one lattice point on each of its sides.

Moreover it is easy to see that in cases (a) and (c), K circumscribes a parallelogram (which is a cell of L) with one side of length λ_1 and altitude $2\mu_1$, and in case (b) K circumscribes a triangle with one side of length λ_1 and relative altitude $2\mu_1$.

Let K be a triangle (cases (a) and (b)).

In this case we have

$$A = \frac{1}{2}D\Delta \leqslant \frac{\lambda_1\Delta^2}{2(\Delta - 2\mu_1)} \tag{8}$$

where the second inequality follows immediately from $(\Delta - 2\mu_1)D \leqslant \lambda_1\Delta$ proved in [11] and where equality holds if and only if K is a triangle of width $\Delta(K)$ and diameter $D(K) = \lambda_1(L)\Delta(K)/(\Delta(K) - 2\mu_1(L))$.

Thus it is sufficient to establish (2) in case (c).

Let K be the quadrilateral $XYZT$ and let O, B, C, E be lattice points such that $O \in [X,Y]$, $B \in [Y,Z]$, $C \in [Z,T]$, $E \in [T,X]$, $\overline{OB} = \overline{EC} = \lambda_1$, $\widehat{BOE} = \widehat{BCE} = \varphi$ $(\varphi \leqslant \pi/2)$, $\overline{OE}\sin\varphi = \overline{BC}\sin\varphi = 2\mu_1$. Let $m = \overline{TY}$ and let n be the length of the width of K in the direction normal to TY and let us put $\overline{OE} = \nu$. Further let ϑ be the angle between the lines EC and XZ.

By computing the area of K and the areas of its component parts we obtain:

$$2A = mn = \begin{cases} \nu n\cos\vartheta + \lambda_1 m\sin(\varphi - \vartheta) & \text{if } \vartheta \leqslant \pi/2, \\ \nu n\cos\vartheta + \lambda_1 m\sin(\varphi + \vartheta) & \text{if } \vartheta > \pi/2. \end{cases}$$

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As $\vartheta > \pi/2$ implies $\varphi + \vartheta < \pi - \varphi$, we have $mn \leqslant \nu n + \lambda_1 m \sin\varphi$ in either case, and equality holds if the line XZ is parallel to the line OB. Thus we can suppose $2A = mn = \nu n + \lambda_1 m \sin\varphi$. Let $m \geqslant n$ so that $n \leqslant \lambda_1 \sin\varphi + \nu$. Then

$$2A = mn = \frac{\nu n^2}{n - \lambda_1 \sin\varphi} \leqslant \max\left\{\frac{\nu\Delta^2}{\Delta - \lambda_1 \sin\varphi}, (\lambda_1 \sin\varphi + \nu)^2\right\}.$$

Let $m < n$ so that $m \leqslant \lambda_1 \sin\varphi + \nu$ and further let us suppose that the lines XT and YZ are parallel or meet in the half-plane containing Z and determined by XY. (In the other cases the proof is similar.) Then $\Delta \leqslant m \sin\left(\widehat{XTY}\right) \leqslant m \sin\varphi$. Since

$$2A = mn = \frac{m^2 \lambda_1 \sin\varphi}{m - \nu}$$

is a decreasing function of m, then

$$2A \leqslant \max\left\{\frac{\nu\Delta^2}{\Delta - \lambda_1 \sin\varphi}, (\lambda_1 \sin\varphi + \nu)^2, \frac{\lambda_1\Delta^2}{\Delta - \nu\sin\varphi}\right\}.$$

Now it is a straightforward calculation to show that

$$\max\left\{\frac{\nu\Delta^2}{\Delta - \lambda_1 \sin\varphi}, (\lambda_1 \sin\varphi + \nu)^2, \frac{\lambda_1\Delta^2}{\Delta - \nu\sin\varphi}\right\} = \frac{\lambda_1\Delta^2}{\Delta - \nu\sin\varphi}$$

so that Theorem 2 follows. □

PROOF OF COROLLARIES.

As Corollary 2 is an obvious consequence of Theorem 2, we shall only prove Corollary 1.

The idea of the proof follows an analogous argument given by Hammer in [3] (see also [**11**]) which we repeat here for completeness.

Let us suppose $k \geqslant 1$ (if $k = 0$ the result is obvious) and consider the similarity transformation $K \to K' = (1/k)K$.

Obviously $A(K') = (1/k^2)A(K)$ and $D(K') = (1/k)D(K)$. Now let $\{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of L with $|\mathbf{b}_i| = \lambda_i$ and let $Q = q_1\mathbf{b}_1 + q_2\mathbf{b}_2$ be a lattice point with $0 \leqslant q_i \leqslant (k-1)\lambda_i$ $(i = 1, 2)$. Then for the translate K'' of K' given by $K'' = K' - (1/k)Q$ we have

$$\frac{A(K'')}{D(K'')} = \frac{A(K')}{D(K')} = \frac{1}{k}\frac{A(K)}{D(K)} > \max\left\{2\mu_1,\ 2\tau\sqrt{\lambda_1^2 + (2\mu_1)^2}\right\}.$$

Thus, by Theorem 1, K'' contains a lattice point T. Then K' contains the point $T + (1/k)Q$, so that K contains the point $U = k(T + (1/k)Q) = kT + Q$. Since Q can be chosen in k^2 different ways, by selecting each of q_1, q_2 in k different ways we have k^2 distinct lattice points in K. □

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