A NOTE ON SPACES $C_p(X)$ $K$-ANALYTIC-FRAMED IN $\mathbb{R}^X$

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Abstract

This paper characterizes the $K$-analyticity-framedness in $\mathbb{R}^X$ for $C_p(X)$ (the space of real-valued continuous functions on $X$ with pointwise topology) in terms of $C_p(X)$. This is used to extend Tkachuk’s result about the $K$-analyticity of spaces $C_p(X)$ and to supplement the Arkhangel’skii–Calbrix characterization of $\sigma$-compact cosmic spaces. A partial answer to an Arkhangel’skii–Calbrix problem is also provided.


Keywords and phrases: $\mathcal{P}$-framed space, cosmic space, $K$-analytic space, Lindelöf $\Sigma$-space, angelic space.

1. Preliminaries

Christensen [12] proved that a metric and separable space $X$ is $\sigma$-compact if and only if $C_p(X)$ is analytic, that is, a continuous image of the Polish space $\mathbb{N}^{\mathbb{N}}$. Calbrix [11] showed that a completely regular Hausdorff space $X$ is $\sigma$-compact if $C_p(X)$ is analytic. The converse does not hold in general; for if $\xi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $X = \mathbb{N} \cup \{\xi\}$, where the set of natural numbers $\mathbb{N}$ is considered with the discrete topology, then $C_p(X)$ is a metrizable Baire space [17] but not even $K$-analytic by [22, p. 64]. A closely related result is given in [4]: A regular cosmic space $X$ is $\sigma$-compact if and only if $C_p(X)$ is $K$-analytic-framed in $\mathbb{R}^X$, that is, there exists a $K$-analytic space $Y$ such that $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, although it was already known [18] that if $X$ is $\sigma$-bounded (that is, a countable union of functionally bounded sets), then $C_p(X)$ is $K_{\sigma\delta}$-framed in $\mathbb{R}^X$.

In this note we prove: (a) that $C_p(X)$ is $K$-analytic-framed in $\mathbb{R}^X$ if and only if $C_p(X)$ has a bounded resolution, that is a family $\{A_\alpha \mid \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets covering $C_p(X)$ with $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$ such that each $A_\alpha$ is pointwise bounded; and (b) $C_p(X)$ with
a bounded resolution is an angelic space. Then [4, Theorem 3.4] combined with (a) yields that a regular cosmic space \(X\) (that is, a continuous image of a metric separable space) is \(\sigma\)-compact if and only if \(C_p(X)\) has a bounded resolution. Hence, for metric separable \(X\) the space \(C_p(X)\) is analytic if and only if it has a bounded resolution. Part (b) implies that for any topology \(\xi\) on \(C(X)\) stronger than the pointwise one the space \((C(X), \xi)\) is \(K\)-analytic if and only if it is quasi-Souslin if and only if it admits a (relatively countably) compact resolution. This extends a recent result of Tkachuk [21] and answers a question of Bierstedt (personal communication): What about Tkachuk’s theorem for topologies on \(C(X)\) different from the pointwise one? We apply Proposition 1 (and Corollary 2) to give a partial answer to [4, Problem 1].

A topological Hausdorff space (or space for short) \(X\) is called:

(i) analytic, if \(X\) is a continuous image of the space \(\mathbb{N}^\mathbb{N}\);

(ii) \(K\)-analytic, if there is an upper semi-continuous (usc) set-valued map from \(\mathbb{N}^\mathbb{N}\) with compact values in \(X\) whose union is \(X\);

(iii) quasi-Souslin, if there exists a set-valued map \(T\) from \(\mathbb{N}^\mathbb{N}\) covering \(X\) such that if \((\alpha_n)_n\) is a sequence in \(\mathbb{N}^\mathbb{N}\) which converges to \(\alpha\) in \(\mathbb{N}^\mathbb{N}\) and \(x_n \in T(\alpha_n)\) for all \(n \in \mathbb{N}\), then the sequence \((x_n)_n\) has an adherent point in \(X\) belonging to \(T(\alpha)\);

(iv) Lindelöf \(\Sigma\) (also called \(K\)-countably determined) if there exists a usc set-valued map from a subspace of \(\mathbb{N}^\mathbb{N}\) with compact values in \(X\) covering \(X\).

It is known that a space \(X\) is Lindelöf \(\Sigma\) if and only if it has a countable network modulo some compact cover of \(X\); see [1]. Recall that analytic \(\Rightarrow\) \(K\)-analytic \(\Rightarrow\) quasi-Souslin and \(K\)-analytic \(\Rightarrow\) Lindelöf \(\Sigma\).

By Talagrand [20] every \(K\)-analytic space admits a compact resolution, although the converse does not hold in general. Talagrand [20] showed that for a compact space \(X\) the space \(C_p(X)\) is \(K\)-analytic if and only if \(C_p(X)\) has a compact resolution. Canela [5] extended this result to paracompact and locally compact spaces \(X\). Finally, Tkachuk [21] extended Talagrand’s result to any completely regular Hausdorff space \(X\).

A space \(X\) is angelic if every relatively countably compact set \(A\) in \(X\) is relatively compact and each \(x \in A\) is the limit of a sequence of \(A\). In angelic spaces (relative) compact sets, (relative) countable compact sets and (relative) sequential compact sets are the same; see [16].

2. Bounded resolutions in \(C_p(X)\) and \(K\)-analytic-framedness of \(C_p(X)\) in \(\mathbb{R}^X\)

We start with the following, where \(\overline{B}^{\mathbb{R}^X}\) denotes the closure of \(B\) in the space \(\mathbb{R}^X\).

**Lemma 1.** Let \(X\) be a nonempty set and let \(Z\) be a subspace of \(\mathbb{R}^X\). If \(Z\) has a countable network modulo a cover \(\mathcal{B}\) of \(Z\) by pointwise bounded subsets, then \(Y = \bigcup\{\overline{B}^{\mathbb{R}^X} \mid B \in \mathcal{B}\}\) is a Lindelöf \(\Sigma\)-space such that \(Z \subseteq Y \subseteq \mathbb{R}^X\).

**Proof.** Let \(\mathcal{N} = \{T_n \mid n \in \mathbb{N}\}\) be a countable network modulo a cover \(\mathcal{B}\) of \(Z\) consisting of pointwise bounded sets. Set \(\mathcal{N}_1 = \{\overline{T_n}^{\mathbb{R}^X} \mid n \in \mathbb{N}\}\), \(B_1 = \{\overline{B}^{\mathbb{R}^X} \mid B \in \mathcal{B}\}\)
and $Y = \bigcup B_1$. Clearly every element of $B_1$ is a compact subset of $\mathbb{R}^X$. We show that $\mathcal{N}_1$ is a network in $Y$ modulo the compact cover $B_1$ of $Y$. In fact, if $U$ is a neighborhood in $\mathbb{R}^X$ of $\overline{B}^X$, the regularity of $\mathbb{R}^X$ and compactness of $\overline{B}^X$ are used to obtain a closed neighborhood $V$ of $\overline{B}^X$ in $\mathbb{R}^X$ contained in $U$. Since $\mathcal{N}$ is a network modulo $B$ in $Z$, there exists $n \in \mathbb{N}$ with $B \subseteq T_n \subseteq V \cap Z$, which implies that $\overline{B}^X \subseteq \overline{T}_n^X \subseteq U$. According to Nagami’s criterion [1, Proposition IV.9.1], $Y$ is a Lindelöf $\Sigma$-space which clearly satisfies $Z \subseteq Y \subseteq \mathbb{R}^X$. □

**Proposition 1.** The following are equivalent:

(i) $C_p(X)$ admits a bounded resolution.

(ii) $C_p(X)$ is $K$-analytic-framed in $\mathbb{R}^X$ and $C_p(X)$ is angelic.

(iii) $C_p(X)$ is $K$-analytic-framed in $\mathbb{R}^X$.

(iv) For any topological vector space (tvs) $Y$ containing $C_p(X)$ there exists a space $Z$ such that $C_p(X) \subseteq Z \subseteq Y$ and $Z$ admits a resolution consisting of $Y$-bounded sets.

**Proof.** (i) implies (ii). Let $\{A_\alpha \mid \alpha \in \mathbb{N}^\omega\}$ be a bounded resolution for $C_p(X)$. Denote by $B_\alpha$ the closure of $A_\alpha$ in $\mathbb{R}^X$ and put $Z = \bigcup \{B_\alpha \mid \alpha \in \mathbb{N}^\omega\}$. Clearly each $B_\alpha$ is a compact subset of $\mathbb{R}^X$ and $Z$ is a quasi-Souslin space (see [6, Proposition 1]) such that $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$.

Since each quasi-Souslin space $Z$ has a countable network modulo a resolution $B$ of $Z$ consisting of countably compact sets (see, for instance, [14, proof of Theorem 8]) and every countable compact subset of $\mathbb{R}^X$ is pointwise bounded, then Lemma 1 ensures that $Y = \bigcup \{\overline{B}^X \mid B \in \mathcal{B}\}$ is a Lindelöf $\Sigma$-space, hence Lindelöf, such that $Z \subseteq Y \subseteq \mathbb{R}^X$. Given that every Lindelöf quasi-Souslin space $Y$ is $K$-analytic and $C_p(X) \subseteq Y \subseteq \mathbb{R}^X$, then $C_p(X)$ is $K$-analytic-framed in $\mathbb{R}^X$. Hence, by [18] the space $\nu X$ is Lindelöf $\Sigma$. Since each Lindelöf $\Sigma$ space is web-compact in the sense of Orihuela, then [19, Theorem 3] is used to deduce that $C_p(\nu X)$ is angelic. Hence, $C_p(X)$ is also angelic [8, Note 4].

(iii) implies (iv). If $L$ is a space with a compact resolution $\{A_\alpha \mid \alpha \in \mathbb{N}^\omega\}$ and $C_p(X) \subseteq L \subseteq \mathbb{R}^X$, then $\{A_\alpha \cap C_p(X) \mid \alpha \in \mathbb{N}^\omega\}$ is a bounded resolution in $Z := C_p(X)$ consisting of bounded sets in any tvs $Y$ topologically containing $C_p(X)$.

That (iv) implies (i) is obvious. □

The next theorem extends the main result in Tkachuk [21] and answers the question of [14].

**Theorem 1.** Let $\xi$ be a topology on $C(X)$ stronger than the pointwise one. The following assertions are equivalent.

(i) $(C(X), \xi)$ is $K$-analytic.

(ii) $(C(X), \xi)$ is quasi-Souslin.

(iii) $(C(X), \xi)$ admits a (relatively countably) compact resolution.
Any condition mentioned above implies (by Proposition 1) the angelicity of \( C_p(X) \). Therefore (by the angelic lemma; see [16, p. 29]) the space \((C(X), \xi)\) is angelic as well. But for angelic spaces all three conditions mentioned above are equivalent by [6, Corollary 1.1]. \( \Box \)

It is easy to see that if \( X \) is \( \sigma \)-bounded, then \( C_p(X) \) has a bounded resolution. Indeed, if \( X \) is covered by a sequence \((C_n)_n\) of functionally bounded sets, then \( \{A_\alpha \mid \alpha \in \mathbb{N}^\mathbb{N}\} \) with \( A_\alpha = \{f \in C(X) : \sup_{x \in C_n} |f(x)| \leq \alpha(n), n \in \mathbb{N}\} \) is a bounded resolution for \( C_p(X) \). If \( X \) is a locally compact group, then \( X \) is \( \sigma \)-compact if and only if \( C_p(X) \) has a bounded resolution. This easily follows from the fact that \( X \) is homeomorphic to the product \( \mathbb{R}^n \times D \times G \), where \( D \) is a discrete space and \( G \) is a compact subgroup of \( X \); see [13, Theorem 1 and Remark (ii)]. Proposition 1 combined with [4, Theorem 2.4] yields the following result.

**Corollary 1.** Let \( X \) be a regular cosmic space. Then \( X \) is \( \sigma \)-compact if and only if \( C_p(X) \) has a bounded resolution.

The corresponding variant of Corollary 1 for the weak* dual of Banach spaces does not hold in general. Let \( E \) be an infinite-dimensional separable non-reflexive Banach space. Then the weak topology \( \sigma(E, E') \) is cosmic and not \( \sigma \)-compact but the weak* dual \((E', \sigma(E', E))\) is even analytic.

If \( C_p(C_p(X)) \) has a bounded resolution, then \( X \) is angelic by Proposition 1. If \( C_p(C_p(X)) \) is \( K \)-analytic, then \( X \) is finite [1, IV.9.21]. We note the following result.

**Corollary 2.** For a realcompact space \( X \) the space \( C_p(C_p(X)) \) has a bounded resolution if and only if \( X \) is finite.

**Proof.** If \( C_p(C_p(X)) \) has a bounded resolution, it is \( K \)-analytic-framed in \( \mathbb{R}^{C(X)} \). Consequently there is a \( K \)-analytic space \( Y \) such that \( C_p(C_p(X)) \subseteq Y \subseteq \mathbb{R}^{C(X)} \). By [4, Corollary 3.4] every compact subset of \( X \) is finite. Since \( X \subseteq Y \subseteq \mathbb{R}^{C(X)} \) and \( X \) is realcompact, then \( X \) is a closed subspace of \( Y \). Hence, \( X \) is a \( K \)-analytic space whose compact sets are finite; so it must be countable [1, Proposition IV.6.15]. Consequently, \( C_p(X) \) is a separable metric space, hence a cosmic space. Again [4, Theorem 2.4] is used to deduce that \( C_p(X) \) is \( \sigma \)-compact and [2, Theorem 6.1] concludes that \( X \) is finite. \( \Box \)

**Remark 1.** Corollary 2 does not hold in general. By [3, Proposition 9.31] (see also [4, Remark]) there exists an infinite space \( X \) such that \( C_p(X) \) is \( \sigma \)-bounded; hence \( C_p(C_p(X)) \) has a bounded resolution. Recall also that [4, Corollary 2.6] shows that \( C_p(\mathbb{N}^\mathbb{N}) \) is not \( K \)-analytic-framed in \( \mathbb{R}^X \). In [4, Problem 1] Arkhangel’skii and Calbrix ask if there exists a regular analytic space \( Z \) containing \( C_p(\mathbb{N}^\mathbb{N}) \) (\( C_p(C_p(\mathbb{N}^\mathbb{N})) \)). Proposition 1 (Corollary 2) provides a partial answer. Indeed, if \( Y \) is a tvs containing \( C_p(\mathbb{N}^\mathbb{N}) \) (\( C_p(C_p(\mathbb{N}^\mathbb{N})) \)), then there does not exist a space \( Z \) with \( C_p(\mathbb{N}^\mathbb{N}) \subseteq Z \subseteq Y \) (\( C_p(C_p(\mathbb{N}^\mathbb{N})) \subseteq Z \subseteq Y \)) admitting a resolution consisting of \( Y \)-bounded sets.
Remark 2. Cascales and Orihuela [8] introduced the class $\mathcal{G}$ of locally convex spaces (lcs) $E$ for which there is a family $\{A_\alpha \mid \alpha \in \mathbb{N}\}$ of subsets of the topological dual $E'$ of $E$ covering $E'$ such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, and sequences are equicontinuous in each $A_\alpha$. Class $\mathcal{G}$ includes $(DF)$-spaces, $(LM)$-spaces (hence metrizable lcs), the space of distributions $D'($ and the space $A($ of real analytic functions for open $\Omega \subseteq \mathbb{R}^N$, and so on. From [7, Theorem 11] it follows that the weak topology $\sigma (E, E')$ of an lcs $E$ in class $\mathcal{G}$ is angelic. Now applying the argument used in the proof of Theorem 1 one concludes that if $E \in \mathcal{G}$ and $\xi$ is a topology on $E$ stronger than $\sigma (E, E')$, then $(E, \xi)$ is quasi-Souslin if and only if it is $K$-analytic if and only if it admits a (relatively countably) compact resolution. A similar result fails to hold for the weak* topology $\sigma (E', E)$ of the dual $E'$ of an lcs $E \in \mathcal{G}$. Indeed, in [15] we proved that $(E', \sigma (E', E))$ is quasi-Souslin for each $E \in \mathcal{G}$ but in [9] we provided spaces $E \in \mathcal{G}$ such that $(E', \sigma (E', E))$ is not $K$-analytic. On the other hand, by [10, Corollary 2.8] the space $C_p(X)$ belongs to class $\mathcal{G}$ only if and only if $X$ is countable; so the angelicity of $C_p(X)$ (which we used in Theorem 1) cannot be automatically deduced from Cascales and Orihuela’s result [7, Theorem 11] mentioned above.

References

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