The weak closure of the set of singular elements in a Banach algebra

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In this note it is proved that for a certain class of infinite dimensional Banach algebras the set of singular elements (the non-units) is dense in the weak topology.

It is well known and easily proven (Rickart, [2], p. 12), that in any (complex) Banach algebra $B$, with identity, the set $S$ of singular elements (the non-units) is closed in the norm topology. In some recent work of the author on a generalization of the operational calculus for Banach algebras it became important to know something of the topological nature of $S$ when $B$ is equipped with the weak topology. This topology has as a basis sets of the form

$$\{ \xi \in B : |x^*(x) - x^*(\xi)| < \varepsilon ; x^* \in A \}$$

where $x \in B$, $\varepsilon > 0$ and $A$ is a finite set in the dual space $B^*$ of continuous linear functionals on $B$. If $B$ (as a vector space) has finite dimension, the weak and the norm topology coincide, and so, in this case, $S$ is closed in the weak topology.

For a certain class of algebras we have a partial converse to this result.

**Theorem.** Suppose $B$ is an infinite dimensional, semi-simple, commutative Banach algebra with identity, for which the Gelfand map is surjective. Then $S$ is weakly dense in $B$.

**Proof.** To see that this is a partial converse to the above statement

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we note that $S$ is always a proper subset of $B$, and so, if it is dense it is not closed. The Gelfand map is that well known homomorphism $B \rightarrow C(X)$ of $B$ into the Banach algebra $C(X)$ of all continuous, complex-valued functions on the compact Hausdorff space $X$ of maximal ideals of $B$. As $B$ is semi-simple, this homomorphism is injective, and thus, by assumption, bijective. By the open mapping theorem we conclude that it is a homeomorphism. Therefore, by Theorem V.3.15 of Dunford and Schwartz [1], it is a homeomorphism when both $B$ and $C(X)$ have the weak topology. Thus it suffices to prove the theorem for the algebra $C(X)$. Now a function in $C(X)$ is singular if and only if it vanishes at some point of $X$. The problem is then: given $f \in C(X)$ (which we may assume not to be identically zero), show that every neighbourhood of $f$ in the weak topology contains a function which vanishes somewhere in $X$. It suffices therefore to exhibit a net $\{f_\alpha\}$ of singular elements with $\lim_{\alpha} f_\alpha = f$.

$f$ may be written in a unique way as $\phi + i \psi$, where $\phi, \psi : X \rightarrow \mathbb{R}$ are continuous. Now, as $C(X)$ is infinite dimensional, $X$ is an infinite set, and as it is also compact, we conclude that there is a point $p \in X$ which is not isolated. Let $U$ be a neighbourhood basis for $p$ - indexed by some well-ordered set $\Gamma'$, so $U = \{U_\alpha' : \alpha \in \Gamma'\}$. Choose an $\alpha_0 \in \Gamma'$ and define $U_{\alpha_0} = U_\alpha'$. If $\beta \geq \alpha_0$ and $U_\beta$ has been defined, define inductively $U_{\beta+} = U_\beta' \cup U_\beta$, where $\beta+$ is the least element of the set $\{\gamma \in \Gamma' : \gamma > \beta\}$. Then the family $\{U_\alpha : \alpha \geq \alpha_0\}$ is also a neighbourhood basis of open sets for $p$. Furthermore, if $\beta > \alpha \geq \alpha_0$ we have $U_\beta \supset U_\alpha$, and each $U_\alpha$ contains a point other than $p$ (as $p$ is not isolated). Next, because $X$ is Hausdorff, we have $\cap_{\alpha \geq \alpha_0} U_\alpha = \{p\}$. For convenience we let $\Gamma$ be the directed set $\{\alpha \in \Gamma' : \alpha \geq \alpha_0\}$. For each $\alpha \in \Gamma$ let $F_\alpha = X - U_\alpha$, so that $F_\alpha$ is closed. Also, let $\hat{F}_\alpha$ be the closed set $F_\alpha \cup \{p\}$. For $\alpha \in \Gamma$ we may, by construction, choose a point $p_\alpha \in U_\alpha$ in such a way that $p_\alpha \not\in U_{\alpha+}$. $X$, being a compact Hausdorff space, is also normal, and hence, by Urysohn's Lemma, for each $\alpha \in \Gamma$, we may choose a continuous real-valued
function $g_a$ on $X$ so that $0 \leq g_a \leq 1$; $g_a(p_a) = 1$ and $g_a$ vanishes on $\hat{F}_a$. Now define $\phi_a : X \to R$ by

$$\phi_a(\lambda) = (1 - g_a(\lambda)) \phi(\lambda); \quad \lambda \in X.$$ 

Then $\phi_a$ is continuous; $-\phi \leq \phi_a \leq \phi$; $\phi_a(p_a) = 0$ and $\phi_a|\hat{F}_a = \phi|\hat{F}_a$. In exactly the same manner we construct a continuous function $\psi_a : X \to R$ with $-\psi \leq \psi_a \leq \psi$; $\psi_a(p_a) = 0$ and $\psi_a|\hat{F}_a = \psi|\hat{F}_a$. Write $f_a = \phi_a + i\psi_a$ so that each $f_a$ is a singular element of $C(X)$.

The net $\{f_a : a \in \Gamma\}$ will converge to $f$ in the weak topology if for each continuous linear functional $x^*$ on $C(X)$, and for each $\varepsilon > 0$, there is an $a_1 \in \Gamma$ so that $a > a_1$ implies that

$$|x^*(f) - x^*(f_a)| < \varepsilon.$$ 

The Riesz Representation Theorem ([I], Theorem IV.6.3) asserts the existence of an isometric isomorphism between $C(X)^*$ and the Banach space of regular, countably-additive, complex-valued measures on the Borel sets of $X$. Further, if $\mu$ is such a measure,

$$x^*(g) = \int_X g d\mu$$

for all $g \in C(X)$. Thus

$$|x^*(f) - x^*(f_a)| \leq \left| \int_{F_a} (f - f_a) d\mu \right| + \left| \int_{U_a} (f - f_a) d\mu \right|.$$ 

However, for each $a \in \Gamma$, the first factor above is identically zero as $f$ and $f_a$ agree on $F_a$. As for the second factor, it is

$$\leq \int_{U_a} |f - f_a| d\|\mu\|,$$

here $\|\mu\|$ represents the total variation of $\mu$, and, by Theorem III.5.12 of [I], $\|\mu\|$ is also a regular (positive) measure on the Borel sets of $X$. Now suppose that the $\|\mu\|$-measure of the point $p_a$ is zero. Then $\inf_V \|\mu\|(V) = 0$ - the infimum being taken over all open sets $V$.
containing \( p \). Thus we can choose such an open set \( V \) with
\[
\|u\|(V) < \varepsilon/2\|f\|.
\]
But \( \{U_\alpha\} \) is a basis for the open sets containing \( p \), and as it is also decreasing, there is an \( \alpha_1 \in \Gamma \) such that, if \( \alpha > \alpha_1 \) we have \( U_\alpha \subset V \) and so \( \|u\|(U_\alpha) < \varepsilon/2\|f\| \). Hence
\[
\int_{U_\alpha} |f-f_\alpha| \, d\|u\| \leq \int_{U_\alpha} \|f-f_\alpha\| \, d\|u\| \leq \|f-f_\alpha\| \cdot \|u\|(U_\alpha) \leq 2\|f\| \cdot \|u\|(U_\alpha) < \varepsilon
\]
provided \( \alpha > \alpha_1 \). Now suppose that \( \{p\} \) does not have \( \|u\| \)-measure zero, then, without loss of generality, we may assume that \( \|u\|(\{p\}) = 1 \) and that
\[
\int_{\{p\}} (f-f_\alpha) \, d\|u\| = 0 \quad \text{(remember that } f(p) = f_\alpha(p))
\]
Then
\[
\int_{U_\alpha} |f-f_\alpha| \, d\|u\| = \int_{U_\alpha-p} |f-f_\alpha| \, d\|u\| \leq 2\|f\| \cdot \|u\|(U_\alpha-p).
\]
By regularity we may choose an open set \( V \) containing \( p \) for which
\[
\|u\|(V) < 1 + \varepsilon/2\|f\| \quad \text{so that } \|u\|(V-p) < \varepsilon/2\|f\|.
\]
Arguing as before we find an \( \alpha_1 \) such that \( \|u\|(U_\alpha-p) < \varepsilon/2\|f\| \) provided \( \alpha > \alpha_1 \). This completes the proof of the theorem.

The proofs of the next results follow immediately from the theorem.

**COROLLARY 1.** Suppose \( B \) is an infinite dimensional, commutative, \( B^* \)-algebra with identity. Then \( S \) is weakly dense in \( B \).

**COROLLARY 2.** Let \( X \) be an infinite compact Hausdorff space. Suppose \( X \) contains a non-isolated point with a countable neighbourhood basis. Then \( S \) is weakly sequentially dense in \( C(X) \).

**COROLLARY 3.** Suppose the Banach algebra \( B \) satisfies the conditions of the theorem. Then, in the weak topology, the group of units of \( B \) has empty interior.

Finally, let \( B \) be the Banach algebra of bounded, complex-valued functions on a set \( \Omega \). As mentioned in [2], p. 295, the group \( G \) of units of \( B \) is open and dense in the norm topology. However, in the weak topology, \( G \) has empty interior. This follows from Corollary 3 and Theorem IV.6.18 of [1], according to which \( B \) is (isometrically isomorphic to) an algebra \( C(X) \), for a suitable compact Hausdorff space \( X \).
Singular elements in a Banach algebra

References


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