We obtain an inequality concerning the width and diameter of a planar convex set with interior containing no point of the rectangular lattice. We then use the result to obtain a corresponding inequality for a planar convex set with interior containing exactly two points of the integral lattice.

1. Introduction

Let \( K \) be a compact, non-empty convex set in \( E^2 \) with minimal width \( w(K) = w \) and diameter \( d(K) = \delta \). Let \( K^o \) denote the interior of \( K \) and let \( \Gamma \) denote the integral lattice. A number of results are known concerning the relationship between the width and the diameter of a convex set. The following elegant result was obtained by Scott [3].

**Theorem 1.** If \( K^o \) contains no point of \( \Gamma \), then \((w - 1)(\delta - 1) \leq 1 \) with equality when and only when \( K \) is a triangle of diameter \( \delta \) and width \( w = \delta/(\delta - 1) \) (Figure 1).

![Figure 1.](https://doi.org/10.1017/S0004972900017238)

Theorem 1 has been extended to sets containing exactly one point of \( \Gamma \) in the interior [4]. The analogous result is:

**Theorem 2.** If \( K^o \) contains one point of \( \Gamma \), then \((w - \sqrt{2})(\delta - \sqrt{2}) \leq 2 \); the inequality is best possible.

The purpose of this paper is to generalise Theorem 1 to rectangular lattices and to use the result to obtain analogous inequalities for convex sets containing exactly two points of \( \Gamma \) in the interior. Let \( \Lambda_R(u, v) \) be a rectangular lattice generated by the vectors \((u, 0)\) and \((0, v)\). We prove the following two pretty results:
THEOREM 3. Suppose that \( u \leq v \) and that \( K^o \) contains no point of \( \Lambda_R(u, v) \). Then \( (w - v)(\delta - u) \leq uv \); equality is attained when and only when \( K \) is a triangle with diameter \( \delta \) and width \( w = \delta v / (\delta - u) \) (Figure 2).

THEOREM 4. If \( K^o \) contains exactly two points of \( \Gamma \) then \( (w - 2)(\delta - 1) \leq 2 \); equality is attained when and only when \( K \) is a triangle with diameter \( \delta \) and width \( w = 2\delta / (\delta - 1) \) (Figure 3).

2. THREE USEFUL LEMMAS

We shall denote lines by lower case letters: thus \( x \) is a line containing the point \( X \) of \( \Lambda_R(u, v) \). Let the slope of \( x \) be \( m_x \) and let \( d(Y, x) \) denote the perpendicular distance from the point \( Y \) to the line \( x \).

Let \( K \) be a set containing no point of \( \Lambda_R(u, v) \) in its interior. A set for which \( (w - v)(\delta - u) \) is as large as possible is called a maximal set. Clearly we may assume that \( \delta \geq \omega > v \geq u \). We first establish three lemmas which will help us narrow down the possibilities for a maximal set.

We say that a triangle circumscribes a rectangle (or equivalently, a rectangle is inscribed in a triangle) if all vertices of the rectangle lie on the sides of the triangle. Lemma 1 establishes the maximal value of \( (w - v)(\delta - u) \) where \( K \) is a triangle circumscribing a fundamental rectangular cell of \( \Lambda_R(u, v) \). Lemmas 2 and 3 will help us eliminate those cases for which \( K \) is not maximal.

Lemmas 2 and 3 will help us eliminate those cases for which \( K \) is not maximal.

**LEMMA 1.** Let \( K \) be a triangle circumscribing a fundamental rectangular cell of \( \Lambda_R(u, v) \). Then \( (w - v)(\delta - u) \leq uv \) with equality when and only when the side of the rectangular cell having length \( u \) lies on the edge of \( K \) with length \( \delta \) (Figure 4).

**PROOF:** Let the vertices of \( K \) be \( X, Y \) and \( Z \) and let \( C \) denote the fundamental rectangular cell inscribed in \( K \). Without loss of generality, let \( XY \) be the side of \( K \) containing two vertices of \( C \). Let \( XY \) have length \( b \) and let the altitude from \( Z \) to \( XY \) be \( h \).

We first let the side of \( C \) with length \( u \) lie on the edge \( XY \). Then the area of \( K \) is \( (1/2)bh = (1/2)u\delta \). The edges of \( C \) partition \( K \) into four regions. The area of \( K \)
may therefore be calculated as the sum of the areas of the four component parts (Figure 4).

\[ \frac{1}{2} \cdot w \delta = \frac{1}{2} \cdot bh = \frac{1}{2} (b - u)v + \frac{1}{2} (h - v)u + uv \]

that is,

\[ w \delta = bh = bv + hu. \]

From the identity \((\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta\), we note that the sum of two numbers with a given product is smallest when the difference between them is least. Applying this first to the pair \((bv, hu)\) and then to the pair \((\delta v, wu)\), and noting that \(bv - hu \leq \delta v - wu\), we have

\[ bv + hu \leq \delta v + wu. \]

We thus have

\[ w \delta \leq \delta v + wu. \]

Adding \(uv\) to both sides of the inequality gives

\[ (w - v)(\delta - u) \leq uv. \]

Equality is attained here when \(XY = b = \delta\) and \(h = w\).

If, on the other hand, the side of length \(v\) of \(C\) lies on \(XY\), then by the same argument we obtain \((w - u)(\delta - v) \leq uv\). In this case we write

\[ (w - v)(\delta - u) = (w - u)(\delta - v) + (w - \delta)(v - u). \]

Since \(u \leq v\) and \(w < \delta\) for triangles, we have

\[ (w - v)(\delta - u) < (w - u)(\delta - v) \leq uv. \]

Hence for circumscribed triangles \(K\), \((w - v)(\delta - u) \leq uv\) with equality when and only when the side of \(C\) of length \(u\) lies on the edge of \(K\) with length \(\delta\).

From Lemma 1, we deduce that if \(K\) is a maximal set, then \((w - v)(\delta - u) \geq uv\).
LEMMA 2. Let \(ABCD\) be a fundamental rectangular cell of \(\Lambda_R(u,v)\) labelled in an anticlockwise direction. Let \(\triangle\) be a triangle determined by the lines \(a, b\) and \(c\) with points \(A, B\) and \(C\) interior to the edges of \(\triangle\) and point \(D\) exterior to \(\triangle\). Further, let line \(c\) containing an edge of \(\triangle\) intercept the closed line segment \(AD\). Then \((w(\triangle) - v)(d(\triangle) - u) < uv\).

PROOF: Let \(b.c = P, a.c = Q\) and \(a.b = R\). By a suitable rotation of the plane together with a reflection of the set \(A\) in the mediator of the segment \(AB\), if necessary, we may assume that \(m_b > m_c \geq 0\) (see Figure 5).

Suppose first that \(\angle Q \leq \pi/2\). Let \(c\) make an acute angle \(\theta \neq 0\) with the line \(CD\). Let \(V\) be a point on \(QR\) with \(BV\) parallel to \(PQ\). Then \(BV < AB\) and \(BV\) is distant \(BC \cos \theta < BC\) from \(PQ\). We rotate \(\triangle\) about \(B\) until \(PQ\) is parallel to \(CD\). Let the rotated triangle be \(\triangle'\). Clearly \(\triangle'\) contains no lattice point in its interior and \(B\) is the only lattice point on the boundary of \(\triangle'\). Hence \(\triangle'\) may be enlarged to a triangle \(\triangle^*\) inscribing the rectangle \(ABCD\). Using Lemma 1,

\[
(w(\triangle) - v)(d(\triangle) - u) < uv.
\]

Now suppose that \(\angle Q > \pi/2\). We consider the following two cases:

CASE (i): \(Q\) lies in the closed rectangle \(ABCD\). We show that

\[
(w(\triangle) - v)(d(\triangle) - u) < uv.
\]

We first inscribe a rectangle \(R_\triangle\) in \(\triangle\) with side lengths \(u' < u\) and \(v' = v\) as follows: Let \(b'\) be a line parallel to \(b\) and distant \(v\) from \(b\). Since \(w > v\), \(b'\) intersects \(\triangle\) in a line segment \(M'N'\) of length \(s > 0\) (see Figure 6).

Let \(M\) and \(N\) be the feet of the perpendiculars from \(M'\) and \(N'\) to the line \(b\) and let \(R_\triangle\) be the rectangle with vertices \(M, N, N'\) and \(M'\). We shall show that \(s < u\). Let \(b'\) intersect the lines \(CD\) and \(AD\) in the points \(Z\) and \(Y\) respectively. Clearly \(s < YZ\).
We now consider the following two subcases:

(a) If $AB$ has length $u$ and $BC$ has length $v$, we take the coordinates of $B$, $Z$ and $Y$ to be $(u,0)$, $(x,v)$ and $(0,y)$ respectively. Hence

$$\text{Area of } \triangle BZY = \frac{1}{2}v. ZY = \frac{1}{2} \begin{vmatrix} u & 0 & 1 \\ x & v & 1 \\ 0 & y & 1 \end{vmatrix},$$

that is,

$$ZY = u + (x - u)\frac{y}{v},$$

Figure 6.

Now since $x < u$, we have $ZY < u$. We now rotate $R_\triangle$ so that the edge of $R_\triangle$ of length $s$ lies on the edge of $ABCD$ of length $u$ and $R_\triangle$ is contained in the closed rectangle $ABCD$. The same rotation transforms $\triangle$ to $\triangle'$ say. Clearly $\triangle'$ contains no interior lattice points and since $s < u$, at least one of $C$ and $D$ lies in the exterior of $\triangle'$. Hence $\triangle'$ may be enlarged to a triangle $\triangle^*$ inscribing the rectangle $ABCD$, and (2) applies immediately.

(b) If now $AB$ has length $v$ and $BC$ has length $u$, we inscribe a rectangle in $\triangle$ with side lengths $u' = s$ and $v' = v$ as described above. We now let the coordinates of $B$, $Z$ and $Y$ be $(v,0)$, $(x,u)$ and $(0,y)$ respectively. Noting that $x < v$, we obtain

$$ZY = u + (x - v)\frac{y}{v} < u.$$

By the rotation argument above, we again obtain (2).

**CASE (ii):** $Q$ lies exterior to the closed rectangle $ABCD$. Let $a$ make an acute angle $\varphi(\neq 0)$ with the line $AD$. Let $T$ be the point on $PQ$ with $BT$ parallel to $QR$. Now $BT < BC$ and $BT$ is distant $AB \cos \varphi < AB$ from $QR$. We rotate $\triangle$ clockwise about $B$ until $BT$ lies on the edge $BC$. Let the transformed triangle $\triangle'$ have vertices
$P'$, $Q'$ and $R'$ corresponding to points $P$, $Q$ and $R$ respectively. Then clearly $Q'R'$ is parallel to $AD$. We note also that points $A$ and $C$ are exterior to $\Delta P'Q'R'$. We can now construct a triangle $\Delta''$ with vertices $P''$, $Q''$, $R''$ such that line $P''Q''$ is parallel to $P'Q'$ and contains the point $C$, line $Q''R''$ is coincident with line $AD$ and line $R''P''$ is coincident with $R'P'$. Clearly $\Delta P''Q''R''$ is a triangle of the type described in Case (i). Hence

$$
(w(\Delta) - v)(d(\Delta) - u) = (w(\Delta') - v)(d(\Delta') - u)
$$

$$
< (w(\Delta'') - v)(d(\Delta'') - u)
$$

< $uv$.

This completes the proof of Lemma 2.

Suppose now that $K$ is contained in a triangle satisfying the conditions of Lemma 2. Since $K \subset \Delta$, $w(K) \leq w(\Delta)$ and $d(K) \leq d(\Delta)$. From Lemma 2, it follows that $K$ is not maximal.

Henceforth we shall use the shorthand notation $L2(a, b, c)$ to mean:

$K$ is contained in a triangle determined by the lines $a$, $b$, $c$ satisfying the conditions of Lemma 2. Hence $K$ is not maximal.

**Lemma 3.** Let $ABCD$ be a rectangular cell of $\Lambda_R(u, v)$ labelled anticlockwise and let $Q$ be a proper convex quadrilateral determined by lines $a$, $b$, $c$, $d$, with $A$, $B$, $C$ and $D$ interior to the edges of $Q$ on $a$, $b$, $c$ and $d$ respectively. Then amongst all convex sets containing no interior lattice points, a set $K$ contained in $Q$ cannot be maximal.

**Proof:** Since $K \subset Q$, it suffices to show that $Q$ is not maximal. Let $a.b = X$, $b.c = Y$, $c.d = Z$ and $d.a = W$ (Figure 7).
We now recall that the diameter of a polygonal set is the maximum distance between a pair of vertices of the polygon. Suppose first that $\delta$ is the length of an edge, $XY$ say, of $Q$. Without loss of generality, suppose that $W$ is the vertex of $Q$ furthest from $b$. Then $w \leq d(W,b)$. Let $\Delta$ be the triangle $XYW$. Clearly $d(\Delta) = XY$ and so $w(\Delta) = d(W,b)$ and $w \leq w(\Delta)$. Hence $w = w(\Delta) = d(W,b)$. Since $\Delta$ and $Q$ have the same width and diameter, it suffices to show that $\Delta$ is not maximal. Noting that the edge $WY$ contains no lattice points, $\Delta$ may be enlarged about the point $X$ to $\Delta' = \Delta W'XY'$ where $W'Y'$ contains the point $D$. By a simple variant of Lemma 2,

$$ (w(\Delta) - v)(d(\Delta) - v) < (w(\Delta') - v)(d(\Delta') - u) < uv. $$

Hence $\Delta$ (and so $Q$) is not maximal.

We now suppose that $\delta$ is the length of a diagonal of $Q$, $WY$ say. Let $t$ be the width of $Q$ in a direction perpendicular to $WY$ (see Figure 8). Since the (minimal) width of $Q$ occurs in a direction perpendicular to an edge of $Q$ (see for example [1]), we have $w < t$. Let $WY$ make an acute angle $\theta$ with $CD$ and let $XZ$ intersect $WY$ in the point $O$. Now the area of $Q$ is $(1/2)t\delta$. This area is also obtained by adding the areas of the quadrilaterals $ODWA$, $OBYC$ to $OCZD$, $OAXB$.

![Figure 8.](https://doi.org/10.1017/S0004972700017238)

Suppose first that $AB$ has length $u$ and $BC$ has length $v$. Then we have

$$ \frac{1}{2}t\delta = \frac{1}{2}v\delta \cos \theta + \frac{1}{2}ut \cos \theta. $$

Hence

$$ t\delta = (tu + \delta v) \cos \theta \leq tu + \delta v. $$

Adding $uv$ to both sides of the inequality and factorising, we have

$$ (t - v)(\delta - u) \leq uv. $$
Since $w < t$, we have
\[(w - v)(\delta - u) < uv.\]
Hence $Q$ is not maximal.

Now suppose that $AB$ has length $v$ and $BC$ has length $u$. Repeating the above argument, we obtain the corresponding inequality
\[(w - u)(\delta - v) < uv.\]
By (1), $(w - v)(\delta - u) < uv$. So again, $Q$ is not maximal.

3. PROOF OF THEOREM 3

We now assume that $K$ is a maximal set. We may assume that $\delta > w > v > u$. Let the radius of the largest circle inscribed in $K$ be $r$. It is shown in [2] that for any convex set $K$,
\[(w - 2r)\delta \leq 2\sqrt{3}r^2.\]
If $r \leq u/2 \leq v/2$, then
\[(w - v)(\delta - u) < (w - v)\delta \leq (w - 2r)\delta \leq 2\sqrt{3}r^2 \leq 2\sqrt{3}\frac{u}{2}v = \frac{\sqrt{3}}{2}uv < uv.
Hence $K$ is not maximal. We may therefore assume that $K$ contains a disk $D$ of radius $r > u/2$.

By translating $K$ through a suitable lattice vector, we may bring the centre of $\Omega$ to lie in $0 < y < v$. For easier reference, we list the properties of $\Omega$ as follows:

D1. $r > u/2$.
D2. The centre of $\Omega$ lies in $0 < y < v$.

Since $w > v$, $K^\circ$ intercepts one of $y = 0$ and $y = v$. Without loss of generality, we may assume that $K^\circ$ intercepts $y = 0$. Since $K^\circ$ contains no point of $\Lambda R(u, v)$, we may assume that $K^\circ$ intercepts $y = 0$ between two adjacent lattice points. By translating through a suitable lattice vector we may take these points to be $E(0,0)$ and $F(u,0)$. Let $G$ and $H$ be the points $(u,v)$ and $(0,v)$ respectively. We shall show that $K$ is a triangle with diameter $\delta$ and width $w = \delta v/(\delta - u)$ (see for example Figure 2).

From D1 and D2, $K^\circ$ must intercept one of the edges $EH$ and $FG$. Without losing generality, we may assume that $K^\circ$ intercepts $FG$. Hence $K$ lies above a line $f$ with $m_f > 0$. We now consider the following two cases:

Case 1: $K$ is bounded by $y = v$. By D1 and D2, lines $e$ and $f$ intersect in the halfplane $y < 0$ and $K$ is contained in the triangle $\Delta$ determined by the lines $e$, $f$ and $y = v$. Since $K^\circ$ intercepts $EF$, $m_e \neq 0$. If $m_e > 0$, then $H$ is exterior to $\Delta$.
and $L^2(e, f, g)$. We may now assume that $m_e < 0$ (possibly infinite). In this case, $\triangle$ circumscribes the rectangular cell $EFGH$. By Lemma 1, $K$ is maximal when $K$ is the triangle bounded by $y = v$ and the lines $e$ and $f$ with $m_e < 0$ (possibly infinite) and $m_f > 0$, and having diameter on the line $y = v$.

Case 2: $K$ crosses the line $y = v$. We again show that $K$ is not maximal. Suppose that $K$ crosses the line $y = v$ between the adjacent lattice points $X$ and $Y$ on the line $y = v$. Without losing generality, we may assume that $X$ and $Y$ are the points $(ku, v)$ and $((k + 1)u, v)$ respectively where $k \geq 0$. If $k = 0$, then $X = H$ and $Y = G$ and we have $m_g < 0$ and $m_h \neq 0$. If $m_h > 0$ and $m_e < 0$, then $K$ is contained in a proper convex quadrilateral $Q$, and by Lemma 3, $K$ is not maximal. If $m_h < 0$ then $L^2(f, g, h)$ or if $m_e > 0$ then $L^2(f, g, e)$. Finally, if $h$ has infinite slope, $K$ is contained in a triangle circumscribing the rectangle $EFGH$ with the edge $EH$ of length $v$ on $x = 0$. By Lemma 1, $K$ is not maximal.

We may therefore assume that $XY \neq GH$. The set $K$ is therefore bounded by lines $x$ and $y$ with $m_x > 0$. By D1 and D2, $e$ and $f$ intersect in the halfplane $y < 0$ and $x$ and $y$ intersect in the halfplane $y > v$. If $m_f > m_x > 0$, $K$ is contained in a triangle $\triangle$ determined by lines $e$, $f$ and $x$. Let $g_f$ denote the line containing $G$ and parallel to $f$ and let $\pi_H$ be the open half plane bounded by $g_f$ and containing the point $H$. Since $w(\triangle) > v > d(G, f)$, $e$ and $x$ intersect in a point $Q$ lying in the intersection of the half planes $y \leq v$ and $\pi_H$. It follows that $K$ is also contained in a triangle $\triangle'$ determined by lines $e$, $f$ and $g_x$ where $g_x$ is a line containing $G$ and parallel to $x$. Hence $L^2(e, f, g_x)$. If, on the other hand, $m_x > m_f > 0$, then by a similar argument, $K$ is contained in a triangle determined by the lines $x$, $y$ and $w_f$ where $w_f$ is the line containing the point $W(ku, 0)$ and parallel to $f$. Hence $L^2(y, x, w_f)$.

This completes the proof of Theorem 3.

4. Proof of Theorem 4

Let $K$ now be a set satisfying the conditions of Theorem 4. We may assume that the origin $O$ is one of the lattice points. Let $L(z_1, z_2)$ denote the other lattice point contained in $K^o$. Without loss of generality, we may assume that $z_1 \geq 0$ and $z_2 \geq 0$. By a reflection about the line $y = x$ if necessary, it suffices to consider the cases for which $z_1 \geq z_2$. Since $K^o$ contains no other lattice points, the open line segment $OL$ contains no lattice point. Hence we may assume that $z_1$ and $z_2$ are relatively prime.

If $z_1$ and $z_2$ are both odd, we consider the sublattice

$$
\Gamma' = \{(z, y) : z + y \equiv 1 \pmod{2}\}.
$$

Clearly $O \notin \Gamma'$, $L \notin \Gamma'$ and $K^o$ contains no point of $\Gamma'$. By Theorem 3, we have

$$
\left( w - \sqrt{2} \right) \left( \delta - \sqrt{2} \right) \leq 2.
$$
However,

\[(w - 2)(\delta - 1) - (w - \sqrt{2})(\delta - \sqrt{2}) = w(\sqrt{2} - 1) + \delta(\sqrt{2} - 2)\]
\[\leq \delta(\sqrt{2} - 1) + \delta(\sqrt{2} - 2)\]
\[= \delta(2\sqrt{2} - 3) < 0.\]

It follows that \((w - 2)(\delta - 1) < (w - \sqrt{2})(\delta - \sqrt{2}) \leq 2.\) Hence \(K\) is not maximal.

If say, \(z_1\) is odd and \(z_2\) is even, we consider the sublattice

\[\Gamma' = \{(x, y) : x = n, y = 2m + 1, m, n \in \mathbb{Z}\}.\]

Clearly \(O \not\in \Gamma', L \not\in \Gamma'\) and \(K^o\) contains no point of \(\Gamma'.\) By Theorem 3, we have

\[(w - 2)(\delta - 1) \leq 2.\]

Equality occurs when and only when \(K\) is a triangle with diameter \(\delta\) and width \(w = 2\delta/(\delta - 1)\) as shown in Figure 3.

References


Department of Pure Mathematics
The University of Adelaide
South Australia 5005
Australia

E-mail: pawyong@maths.adelaide.edu.au
pscott@maths.adelaide.edu.au