MATRIX REPRESENTATIONS OF SEMIGROUPS

by D. B. McALISTER

(Received 24 August, 1965; and in revised form, 11 January, 1966)

In a series of papers [6], [7], [8], [10], Munn has considered the problem of constructing all irreducible representations of a semigroup by matrices over a field. In [10], he showed how to construct all the irreducible representations of an arbitrary inverse semigroup from those of associated Brandt semigroups. In this paper, we generalize the method of [10] to give a construction for the irreducible representations of an arbitrary semigroup from those of certain associated semigroups.

For many types of semigroups, including regular semigroups, periodic semigroups and 0-simple semigroups with non-zero idempotents, the associated semigroups are completely 0-simple. In this case, by means of Clifford's result [1] on the representations of a completely 0-simple semigroup, we can give an explicit method of construction for all irreducible representations.

I should like to express my sincere gratitude to Dr W. D. Munn, who read the first rough draft of these results and who encouraged me to prepare them for publication.

1. $\mathcal{M}$-semigroups. In general, a semigroup need have neither a zero nor an identity. However, given any semigroup S, we may embed S in a semigroup S^0 which has a zero and which is constructed from S in the following way. If S already has a zero and contains at least two members, then $S=S^0$; otherwise S^0 is the semigroup formed from S by adjoining a new symbol 0 and defining $a0=0=0a$ for each $a \in S^0=S \cup \{0\}$. The phrase "$S=S^0$" means that S is a semigroup which has a zero and at least two members.

In a similar way, we can embed a semigroup S in a semigroup S^1 that has an identity.

Because of the simple nature of the embedding of a semigroup S in the corresponding semigroup S^0, many theorems about semigroups that have no zero may be deduced from corresponding theorems for semigroups that have a zero. In particular, there will be no loss of generality if, in this paper, we consider only semigroups that have a zero.

A homomorphism θ of a semigroup $S=S^0$ onto a semigroup $\bar{S}$ is said to be 0-*restricted* if $a\theta=0\theta$ implies $a=0$; the corresponding congruence on S is also said to be 0-restricted.

PROPOSITION 1. *Let $S = S^0$ be a semigroup. Then*

$$\rho = \{(a, b) \in S \times S: \text{for all } s, t \in S^1,\ sat = 0 \text{ if and only if } sbt = 0\}$$

is a 0-*restricted congruence on S. If τ is any* 0-*restricted congruence on S, then $\tau \subseteq \rho$.*

Proof. The relation ρ is clearly an equivalence on S. Let $(a, b) \in \rho$, $x \in S$. Then, for any s, $t \in S^1$, $sat=0$ if and only if $sbt=0$. Hence, a fortiori, $saxt=0$ if and only if $sbxt=0$; thus $(ax, bx) \in \rho$. Similarly $(xa, xb) \in \rho$ and so ρ is a congruence on S.

Let $a \in S$ with $(a, 0) \in \rho$. Then, for any $s, t \in S^1$, $sat=0$; in particular, $a=0$. Hence $(a, 0) \in \rho$ implies $a=0$ so that ρ is a 0-restricted congruence on S.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

Finally, let τ be any 0-restricted congruence on S, and let $(a, b) \in \tau$. Then, by the regularity of τ with respect to multiplication, $(sat, sbt) \in \tau$ for all $s, t \in S^1$. Hence, in particular, for all $s, t \in S^1$, $sat = 0$ if and only if $sbt = 0$. This means that $(a, b) \in \rho$; hence $\tau \subseteq \rho$.

The fact that ρ is the maximum 0-restricted congruence on S may be deduced from the results of Preston [**11**] on subsets of a semigroup that are congruence classes. A proof is given here for completeness.

The congruence ρ is of importance because, in many cases, a semigroup $S = S^0$ has an image of some particular type under a 0-restricted homomorphism if and only if S/ρ is of that type. In particular, we have the following result.

PROPOSITION 2. *Let $\mathscr{X}$ be a class of semigroups that is closed under homomorphic images. Then a semigroup $S = S^0$ has an image in $\mathscr{X}$ under a 0-restricted homomorphism if and only if $S/\rho \in \mathscr{X}$.*

Proof. If $S/\rho \in \mathscr{X}$, then S has a 0-restricted homomorphic image in $\mathscr{X}$. Conversely, let τ be a 0-restricted congruence on S such that $S/\tau \in \mathscr{X}$. Then, since $\tau \subseteq \rho$, it follows, from the induced homomorphism theorem, that S/ρ is a homomorphic image of S/τ. Hence, by hypothesis, $S/\rho \in \mathscr{X}$.

Proposition 2 has several interesting corollaries. For example, let $S = S^0$ be a regular semigroup. Then, using Proposition 2, we can show that S has an image under a 0-restricted homomorphism that is an inverse semigroup if and only if, for any idempotents e, f, g, h of S, $gefh = 0$ implies $gfeh = 0$.

Munn [**10**] has shown that the following condition is important in the theory of matrix representations of a semigroup $S = S^0$.

C_1: *For any $a, x, b \in S$, if $axb = 0$, then $ax = 0$ or $xb = 0$.*

He has also shown that the next condition plays an important part in the theory, if S is an inverse semigroup.

M_2: *If M and N are nonzero ideals of S, then $M \cap N \neq \{0\}$.*

We shall see that, for arbitrary semigroups, condition

C_2: *If $a, b \in S$ and $aSb = \{0\}$, then $a = 0$ or $b = 0$,*

is more natural. The connection between C_2 and M_2 is given by the following proposition.

PROPOSITION 3. *Let $S = S^0$ be a semigroup. Then S obeys C_2 if and only if it obeys M_2 and*

C_2': *If $a \in S$ and $aSa = \{0\}$, then $a = 0$.*

Proof. Suppose first that S obeys C_2; then, clearly, S obeys C_2'. Let M and N be nonzero ideals of S, and let a, b be nonzero elements of M and N respectively. Then $aSb \subseteq M \cap N$ and, by C_2, $aSb \neq \{0\}$. Hence S obeys M_2.

Conversely, suppose that S obeys M_2 and C_2'. Given nonzero ideals M and N, let $a \in M \cap N \backslash \{0\}$. Then, by C_2', $axa \neq 0$ for some $x \in S$ so that, since $axa \in M \,.\, N$, $M \,.\, N \neq \{0\}$.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

In particular, given any nonzero elements $a, b \in S$, $S^1aS^1 . S^1bS^1 \neq \{0\}$. But

$$S^1aS^1 . S^1bS^1 = S^1aSbS^1 \cup S^1abS^1,$$

so that $aSb \neq \{0\}$ or $ab \neq 0$. If $ab \neq 0$, then, similarly, $aSab \neq \{0\}$ or $a . ab \neq 0$. Thus, in any case, $aSb \neq \{0\}$. Hence S obeys C_2.

COROLLARY. *Let $S = S^0$ be a regular semigroup. Then S obeys C_2 if and only if it obeys* M_2.

We shall make use of Proposition 2 to give a short proof that C_1 and C_2 are necessary and sufficient for a semigroup $S = S^0$ to have a 0-restricted congruence τ such that S/τ is completely 0-simple. Since a completely 0-simple semigroup is regular, it follows from Theorem 1 of [**9**] and the corollary to Proposition 3 that these conditions are necessary. Another proof that C_1 and C_2 are both necessary and sufficient for the existence of a 0-restricted congruence τ, with S/τ completely 0-simple, has been given by Lallement [**4**].

A semigroup $S = S^0$ is said to be *weakly regular* if and only if, for each nonzero member a of S, there exists $x \in S$ such that $ax = ax . ax \neq 0$.

Weakly regular semigroups have been called *E*-inversive, by Clifford and Preston [**2**], and 0-inversive, by Lallement and Petrich [**5**].

The proof of the theorem makes use of the following result which is an immediate corollary to Theorem 3 of [**5**].

PROPOSITION 4. *Let $S = S^0$ be a semigroup that obeys C_2. Then S is completely 0-simple if and only if it is weakly regular and obeys the following weak cancellation law*:

C_3: *If $a, b, x, y \in S$, then the relations $ax = bx \neq 0$ and $ya = yb \neq 0$ together imply that $a = b$.*

THEOREM 1. *Let $S = S^0$ be a semigroup that obeys C_1. Then there is a 0-restricted congruence σ on S such that S/σ obeys C_3 and such that, if τ is any 0-restricted congruence on S for which S/τ obeys C_3, then $\sigma \subseteq \tau$.*

Proof. We show first that S/ρ obeys C_3. Let $a, b, x, y \in S$ be such that none of the elements ax, bx, ya, yb is zero. Suppose, further, that $(ax, bx) \in \rho$ and $(ya, yb) \in \rho$. Then $sat = 0$, for $s, t \in S^1$, implies $sa = 0$ or $at = 0$. For, if $s, t \in S$, this is immediate from C_1 while, if, for example, $t \notin S$, then $sat = sa$. If $sa = 0$, then $s \in S$ and $sax = 0$; thus, since $(ax, bx) \in \rho$, $sbx = 0$. Hence, by C_1, since $bx \neq 0$, $sb = 0$; thus $sbt = 0$. Similarly, $at = 0$ implies $sbt = 0$ and so $(a, b) \in \rho$. Thus S/ρ obeys C_3.

Let T be the set of 0-restricted congruences τ on S such that S/τ obeys C_3; $T \neq \square$ since $\rho \in T$. Let $\sigma = \bigcap \{\tau : \tau \in T\}$. Then it is immediate that σ is a 0-restricted congruence on S. It is also straightforward to verify that S/σ obeys C_3. Thus, by its definition, σ is the smallest 0-restricted congruence τ on S such that S/τ obeys C_3.

COROLLARY. *Let $S = S^0$ be a semigroup. Then there is a 0-restricted congruence τ on S such that S/τ is completely 0-simple if and only if S obeys C_1 and C_2.*

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

Proof. We have already pointed out that conditions C_1 and C_2 are necessary. To show that the conditions are sufficient, we need only show that S/ρ is completely 0-simple.

Let $a \in S \backslash \{0\}$; then, by C_2, there exists $x \in S$ such that $axa \neq 0$. If $sat = 0$ then, as in the proof of Theorem 1, either $sa = 0$ or $at = 0$. In either case, $saxat = 0$. Conversely, if $saxat = 0$, then also $saxa = 0$ or $axat = 0$. Since $axa \neq 0$, these imply respectively that $sa = 0$ and $at = 0$; hence, in either case, $sat = 0$. Thus $(a, axa) \in \rho$ and so S/ρ is regular.

Further, since S obeys C_2 and ρ is a 0-restricted congruence, it is easy to see that S/ρ obeys C_2. By the proof of Theorem 1, S/ρ obeys C_3. Hence S/ρ obeys the conditions of Proposition 4 and so is completely 0-simple.

Let $S = S^0$ be a semigroup satisfying C_1 and let σ be the finest 0-restricted congruence τ on S such that S/τ obeys C_3 (Theorem 1). Then we shall denote S/σ by S^*.

Definition. A semigroup $S = S^0$ is called an *$\mathcal{M}$-semigroup* if it satisfies C_1 and C_2 and is such that S^* is completely 0-simple.

PROPOSITION 5. *Let $S = S^0$ be a weakly regular semigroup that obeys C_1 and C_2. Then S is an $\mathcal{M}$-semigroup.*

Proof. It is easy to verify that, if τ is any 0-restricted congruence on S, then S/τ is weakly regular. In particular, since S obeys C_1 and C_2, S^* is weakly regular and obeys C_2. Since S^* obeys C_3, it is thus immediate, from Proposition 4, that S^* is completely 0-simple. Thus S is an $\mathcal{M}$-semigroup.

COROLLARY 1. *Let $S = S^0$ be a periodic semigroup that satisfies C_1 and C_2; then S is an $\mathcal{M}$-semigroup. In particular, any finite semigroup $S = S^0$ that satisfies C_1 and C_2 is an $\mathcal{M}$-semigroup.*

Proof. Let S be a periodic semigroup that obeys C_1 and C_2. Let $a \in S \backslash \{0\}$. By C_2, there exists $x \in S$ such that $axa \neq 0$. By induction on n, it follows from C_1 that

$$(ax)^n = ax.ax. \dots .ax \neq 0$$

for any positive integer n. Hence, for some positive integer n, $(ax)^n$ is a nonzero idempotent of S. Thus, since $(ax)^n = a.(xa)^{n-1}x$, S is weakly regular. Hence the result is immediate from Proposition 5.

COROLLARY 2. *Let $S = S^0$ be a semigroup that satisfies C_1 and C_2 and that obeys the minimal conditions M_L and M_R on principal left and right ideals respectively. Then S is an $\mathcal{M}$-semigroup.*

Proof. Green [3, Theorem 4] has shown that M_L and M_R together imply the minimal condition M_J on two-sided principal ideals. Hence S has a 0-minimal principal ideal M. Since S obeys C_2, $M^2 \neq \{0\}$; thus M is 0-simple. Since S obeys M_L and M_R, M must contain a 0-minimal left ideal and a 0-minimal right ideal. Hence by [2, Corollary 2.50], M is completely 0-simple; thus it is regular.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

Let $a \in S\backslash\{0\}$ and let $x \in M\backslash\{0\}$. Then, by C_2, there exists $y \in S$ such that $ayx \neq 0$. Since $x \in M$, so does ayx and hence, since M is regular, there exists $z \in M$ such that

$$ayz = ayx . z . ayx.$$

Let $u = yxz$; then au is a nonzero idempotent of S. Hence S is weakly regular.

Another important class of $\mathcal{M}$-semigroups is the class of all 0-simple semigroups that obey C_1 and which contain nonzero idempotents. For, suppose that $S = S^0$ is such a semigroup. Then S^* is also 0-simple and contains a nonzero idempotent. Now, if e, f are nonzero idempotents of S^*, and $ef = fe \neq 0$, then

$$e . ef = ef = e . f \neq 0 \quad \text{and} \quad ef . e = fe . e = f . e \neq 0.$$

Hence, by C_3, $ef = f$; similarly, $fe = e$ so that $e = f$. Thus S^* is a 0-simple semigroup that contains a primitive idempotent. But, by [**2**, §2.7], this means that S^* is completely 0-simple. Thus S satisfies C_2 and is an $\mathcal{M}$-semigroup.

In this paper, we shall determine each irreducible 0-restricted representation Γ of an arbitrary semigroup $S = S^0$ modulo a representation of M^*, where M is a certain ideal of S, dependent on Γ, which obeys C_1 and C_2. It follows that, if M is an $\mathcal{M}$-semigroup, then the irreducible 0-restricted representations of S are known modulo those of completely 0-simple semigroups and ultimately, by Clifford's result [**1**], modulo groups.

Munn [**9**] showed that, if $S = S^0$ is an inverse semigroup that obeys C_1 and M_2, then $S^* \cong M^*$ for any nonzero ideal M of S. This does not hold in general; it need not even hold for an $\mathcal{M}$-semigroup, as the following simple example shows.

Example. Let $S = S^0$ be a completely 0-simple semigroup with no divisors of zero. Suppose further that S is not a group with zero. Let S^1 be the semigroup formed by adjoining an identity to S. Then S^1 has no divisors of zero and so S^1 obeys C_1 and C_2.

Now S is an ideal of S^1 and is completely 0-simple, hence clearly $S^* \cong S$. On the other hand S^1 has an identity, so that S^{1*} is a group with zero.

If we consider the special case of weakly regular semigroups satisfying C_1 and C_2 and in which the idempotents commute, it can be shown that S^* is a Brandt semigroup and that, in this case, there is an exact parallel with the results obtained by Munn [**9**], [**10**] for inverse semigroups. In particular, as for inverse semigroups, the finest 0-restricted congruence σ on S such that S/σ obeys C_3 has the following simple form (cf. [**9**, Theorem 2.7]): for $a, b \in S$,

$$(a, b) \in \sigma \text{ if and only if } a = 0 = b \text{ or } ax = bx \neq 0 \text{ for some } x \in S.$$

We end this section by giving a characterisation, for an arbitrary semigroup $S = S^0$ that obeys C_1, of the 0-restricted congruence σ on S whose properties were described in Theorem 1. The method of proof is similar to that used by Clifford [**11**] to describe the minimum cancellative congruence on a semigroup. As we do not need to make use of the construction, we omit the proof.

Let $S = S^0$ be a semigroup. Then, given any relation τ on S, we can construct new relations, from τ, in the following ways.

$\tau W = \{(a, b) \in S \times S$: for some $s, t \in S^1$, $(at, bt) \in \tau$ and $(sa, sb) \in \tau$, where none of sa, sb, at, bt is zero$\} \cup \{(0, 0)\}$;

$\tau C^* = \{(a,b)\in S\times S$: for some $s,t\in S^1$, $u,v\in S$, $a=sut, b=svt$ where $(u,v)\in\tau\}$;
$\tau\circ\tau = \{(a,b)\in S\times S$: for some $c\in S$, $(a,c)\in\tau$, $(c,b)\in\tau\}$;
$\tau\theta = \tau W\cup\tau C^*\cup(\tau\circ\tau)$ and $\tau\theta^n = (\tau\theta^{n-1})\theta$.

If $\mathscr{I}$ is the identity congruence on S, we write $\mathscr{I}\theta^n = \theta^n$.

THEOREM 2. *Let $S=S^0$ be a semigroup that obeys* C_1. *Let τ be any 0-restricted congruence on S. Then the least congruence ω on S, containing τ, such that S/ω obeys C_3 is $\tau\bar{\theta} = \bigcup_n \tau\theta^n$; $\tau\bar{\theta}$ is a 0-restricted congruence on S.*

In particular, if σ is the least 0-restricted congruence ω on S such that S/ω obeys C_3, then $\sigma=\bar{\theta}=\bigcup_n\theta^n$. If, further, S is an $\mathscr{M}$-semigroup, then S/σ is the maximum completely 0-simple 0-restricted homomorphic image of S.

2. Representations over a field; introduction. Let Φ be a field, and let n be a positive integer; then we denote by $(\Phi)_n$ the algebra of all $n\times n$ matrices over Φ. The $n\times n$ identity is denoted by I_n.

A *representation* Γ of a semigroup S, of degree n over a field Φ, is a homomorphism of S into the multiplicative semigroup of $(\Phi)_n$. If Γ is a representation of a semigroup $S=S^0$ of degree n over a field Φ then, by convention, we consider $\Gamma(0)$ to be the $n\times n$ zero matrix, which we shall also denote by 0. There is no loss of generality if we restrict Γ in this way; see [**10**, pp. 167–168].

If S is a semigroup, and $S\neq S^0$, then we may extend any representation Γ of S to a representation of S^0 by defining $\Gamma(0)$ to be the zero matrix. Consequently, it is sufficient to consider semigroups $S=S^0$.

Let Γ be a representation of a semigroup $S=S^0$, of degree n over a field Φ. Then we define

$V(\Gamma)=\{x\in S:\Gamma(x)=0\}$;
$r(\Gamma)=$ least positive integer s such that, for some $x\in S$, $\Gamma(x)$ has rank s;
$M=M(\Gamma)=\{x\in S:\text{rank}\,\Gamma(x)\leqq r(\Gamma)\}$, where rank $\Gamma(x)$ is the usual matrix rank of $\Gamma(x)$.

$M(\Gamma)$ and $V(\Gamma)$ are clearly ideals of S, and there is a one-to-one correspondence between the representations Γ of S that vanish on an ideal V (i.e. such that $V=V(\Gamma)$) and the 0-*restricted representations* of the Rees quotient semigroup S/V. (A representation Γ of a semigroup $S=S^0$ is said to be 0-restricted if Γ is a 0-restricted homomorphism.) It is thus sufficient to consider 0-restricted representations of semigroups; this we do.

Munn [**10**, § 1], has essentially proved the following result.

LEMMA 1. *Let Γ be a 0-restricted representation of a semigroup $S=S^0$. Then*

(i) *M is an ideal of S that obeys* C_1,

(ii) *$\Gamma(M)$ obeys* C_3.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

A representation Γ of a semigroup $S = S^0$, of degree n over a field Φ, is said to be irreducible if $\Gamma(S)$ is an irreducible matrix set, that is, if there is no fixed, nonsingular, $n \times n$ matrix C such that, for each $x \in S$,

$$C\Gamma(x)C^{-1} \quad \text{has the block form} \quad \begin{bmatrix} \Gamma_1(x) & 0 \\ A & \Gamma_2(x) \end{bmatrix},$$

where 0 denotes the zero $r \times (n-r)$ matrix, for some $1 \leqq r \leqq n$. Otherwise, Γ is reducible.

Let Γ be a representation of a semigroup $S = S^0$, of degree n over a field Φ, and let T be a subset of S. Then we denote by $[\Gamma(T)]$ the subspace of $(\Phi)_n$ generated by $\Gamma(T)$. If T is an ideal of S, then $[\Gamma(T)]$ is an ideal of the subalgebra $[\Gamma(S)]$ of $(\Phi)_n$. Further $\Gamma(T)$ is an irreducible matrix set if and only if the same is true of $[\Gamma(T)]$.

We now consider irreducible representations. The next two lemmas are classical; proofs may be found in [**2**, Chapter 5].

LEMMA 2. *An irreducible subalgebra of* $(\Phi)_n$ *is a simple algebra over* Φ.

LEMMA 3. (Schur's Lemma) *Let* $\mathscr{A}$ *be an irreducible subalgebra of* $(\Phi)_n$. *If C is a constant nonzero matrix that commutes with each member of* $\mathscr{A}$, *then C is nonsingular.*

Using Lemmas 2, 3, Munn [7] proves the following result.

LEMMA 4. *Let* Γ *be a* 0-*restricted irreducible representation of* $S = S^0$, *of degree n over a field* Φ. *Let* $\Gamma(T)$ *be an irreducible subset of* $\Gamma(S)$. *Then there exist finite sets* $e_1, \ldots, e_r \in T$, $\alpha_1, \ldots, \alpha_r \in \Phi$ *such that*

$$\sum_1^r \alpha_i \Gamma(e_i) = I_n.$$

LEMMA 5. *Let* Γ *be a* 0-*restricted irreducible representation of* $S = S^0$. *Then S obeys* C_2.

Proof. Let a, b be nonzero elements of S; then S^1aS^1, S^1bS^1 are nonzero ideals of S. If $aS^1b = \{0\}$, then $S^1aS^1 . S^1bS^1 = \{0\}$; hence $[\Gamma(S^1aS^1)] . [\Gamma(S^1bS^1)] = \{0\}$.

By Lemma 2, $[\Gamma(S)]$ is a simple algebra; hence

$$[\Gamma(S^1aS^1)] = [\Gamma(S)] = [\Gamma(S^1bS^1)].$$

Thus the hypothesis, $aS^1b = \{0\}$, implies that $[\Gamma(S)] . [\Gamma(S)] = \{0\}$. But, by Lemma 4, $I_n \in [\Gamma(S)]$, so this is impossible. Hence $aS^1b \neq \{0\}$; that is, $aSb \neq \{0\}$ or $ab \neq 0$. Suppose that $ab \neq 0$; then, as above, $aS^1ab \neq \{0\}$ and so $aSab \neq \{0\}$ or $a . ab \neq 0$. In either case $aSb \neq \{0\}$; thus S obeys C_2.

3. Representations of a 0-simple semigroup. Let $S = S^0$ be a 0-simple semigroup, and let Γ be a non-null representation of S, of degree n over a field Φ. Then, clearly, Γ is a 0-restricted representation and $M(\Gamma) = S$. Hence, by Lemma 1, S obeys C_1. By means of a proof similar to that of Lemma 5, we can show that any 0-simple semigroup obeys C_2. Hence we have the following proposition, which may be used to give a sufficient condition for the existence of non-null representations of a 0-simple semigroup; we shall consider this point in the next section.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

PROPOSITION 6. *Let $S = S^0$ be a 0-simple semigroup. Then S obeys C_2. Thus S has a completely 0-simple homomorphic image if and only if it obeys C_1.*

THEOREM 3. *Let $S = S^0$ be a 0-simple semigroup, and let S obey C_1. Let S^* denote the maximum non-null homomorphic image of S which obeys C_3; S^* is clearly 0-simple. Let Γ be a non-null representation of S, of degree n over a field Φ. Then Γ induces a non-null representation Γ^* of S^*, of degree n over Φ, according to the rule: for each $\bar{x} \in S^*$,*

$$\Gamma^*(\bar{x}) = \Gamma(x), \tag{1}$$

where $x \to \bar{x}$ is the natural homomorphism of S onto S^*.

Conversely, if Γ^* is a non-null representation of S^*, of degree n over Φ, then the mapping Γ of S onto $\Gamma^*(S^*)$, defined by, for each $x \in S$,

$$\Gamma(x) = \Gamma^*(\bar{x}),$$

is a non-null representation of S.

Proof. Since $S = M(\Gamma) = M$, $\Gamma(S) = \Gamma(M)$; hence, by Lemma 1, $\Gamma(S)$ obeys C_3. Thus $\Gamma(S)$ is a homomorphic image of S^*, and it follows, from the induced homomorphism theorem, that the mapping Γ^* of S^* onto $\Gamma(S)$, defined by (1), is a representation of S^*, of degree n over Φ.

The converse is immediate, since the composition of homomorphisms is a homomorphism.

COROLLARY 1. *Let $S = S^0$ be a 0-simple $\mathcal{M}$-semigroup. Then the non-null representations of S are those of its maximum completely 0-simple homomorphic image S^*.*

COROLLARY 2. *Let $S = S^0$ be a 0-simple semigroup with identity. Then S has a non-null representation if and only if it has no divisors of zero. In this case, the non-null representations of S are those of its maximum group-with-zero homomorphic image S^*.*

Proof. Suppose that Γ is a non-null representation of S. Then S obeys C_1, and is an $\mathcal{M}$-semigroup. Thus S^* is a completely 0-simple semigroup with identity; that is, S^* is a group-with-zero. Hence S has no divisors of zero. The remainder of the result is now immediate from Corollary 1.

Clifford [**1**] has given a construction for all non-null representations of a completely 0-simple semigroup. Taken with Corollary 1 and Theorem 3, this provides a construction for all representations of a 0-simple $\mathcal{M}$-semigroup. It should be noted however that not every 0-simple semigroup is an $\mathcal{M}$-semigroup. For example, let S be the multiplicative semigroup of all 2×2 matrices over the reals, of the form

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix},$$

where a and b are positive real numbers; then S is a simple cancellative semigroup [**2**, Chapter 5, §5, Example 7(b)]. Thus S^0 is a 0-simple semigroup that obeys C_1 and C_3. But S^0 has no nonzero idempotents and so is not completely 0-simple.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

Theorem 3 shows that, for any 0-simple semigroup $S = S^0$, there is a one-to-one correspondence between the representations of S and those of S^*. It is an easy matter to prove that this correspondence preserves equivalence, decomposition and reduction of representations. For the definitions of equivalence and decomposition of representations, see, for example, [**2**, Chapter 5].

4. Irreducible representations of an arbitrary semigroup. The main result of this section gives a method of construction for all 0-restricted irreducible representations of an arbitrary semigroup $S = S^0$, from those of certain associated semigroups. By Lemma 5, if such a representation exists, then S satisfies C_2 and, by Lemma 6 below, so also does any nonzero ideal of S. Further, if S has the property that each nonzero ideal of S that satisfies C_1 is an $\mathscr{M}$-semigroup, then each of these associated semigroups is completely 0-simple. In this case, we have an explicit construction for the irreducible 0-restricted representations of S.

THEOREM 4. *Let $S = S^0$ be a semigroup which obeys C_2. Let Γ be a 0-restricted irreducible representation of S, of degree n over a field Φ. Then Γ induces a 0-restricted irreducible representation Γ^* of M^*, where $M = M(\Gamma)$, and there are finite sets of elements $e_1, \ldots, e_r \in M$, $\alpha_1, \ldots, \alpha_r \in \Phi$ such that, for each $x \in S$,*

$$\Gamma(x) = \sum_1^r \alpha_i \Gamma^*(\overline{e_i x}), \tag{2}$$

where $x \to \bar{x}$ is the natural homomorphism $M \to M^*$.

Conversely, let M be a nonzero ideal of S that obeys C_1, and let Γ^* be a 0-restricted irreducible representation of M^*, of degree n over Φ. Then, for any finite sets $e_1, \ldots, e_r \in M$, $\alpha_1, \ldots, \alpha_r \in \Phi$ such that

$$\sum_1^r \alpha_i \Gamma^*(\bar{e}_i) = I_n, \tag{3}$$

the mapping Γ of S into $(\Phi)_n$, defined by (2), is a 0-restricted irreducible representation of S, of degree n over Φ. The representation is independent of the particular choice of elements e_i, α_i satisfying (3).

Let Γ_1 and Γ_2 be 0-restricted irreducible representations of $S = S^0$, defined, as above, from ideals M_1 and M_2 of S. Then Γ_1 and Γ_2 are equivalent if and only if they are equivalent on $M_1 \cap M_2$.

Proof. Let M and Γ satisfy the hypothesis of the first part of the theorem. By Lemma 1, $\Gamma(M)$ obeys C_3. Hence the mapping Γ^* defined by the rule

$$\Gamma^*(\bar{x}) = \Gamma(x),$$

for each $\bar{x} \in M^*$, where $x \to \bar{x}$ is the natural homomorphism of M onto M^*, is a 0-restricted

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

representation of M^* over Φ, of the same degree as Γ. Since M is an ideal of S, and $\Gamma(S)$ is an irreducible matrix set, it follows from Lemma 2 that

$$[\Gamma^*(M^*)] = [\Gamma(M)] = [\Gamma(S)].$$

Hence Γ^* is an irreducible representation of M^*.

From Lemma 4, since Γ^* is irreducible, there exist $\bar{e}_1, \ldots, \bar{e}_r \in M^*$ and $\alpha_1, \ldots, \alpha_r \in \Phi$ such that

$$\sum_1^r \alpha_i \Gamma^*(\bar{e}_i) = I_n.$$

Choose $e_i \in M$ such that $e_i \to \bar{e}_i$ for each $1 \leqq i \leqq r$. Then, for each $x \in S$,

$$\Gamma(x) = I_n \Gamma(x) = \left(\sum_1^r \alpha_i \Gamma^*(\bar{e}_i)\right) . \Gamma(x).$$

But $\Gamma^*(\bar{e}_i) = \Gamma(e_i)$ for each $1 \leqq i \leqq r$; hence, since M is an ideal of S,

$$\Gamma(x) = \sum_1^r \alpha_i \Gamma(e_i)\Gamma(x) = \sum_1^r \alpha_i \Gamma(e_i x) = \sum_1^r \alpha_i \Gamma^*(\overline{e_i x}).$$

This completes the proof of the first part.

The proof of the converse follows exactly as in the case of principal irreducible representations; cf. [7, Theorem 1].

Finally, it is clear that the criterion for equivalence is necessary. Suppose that Γ_1 and Γ_2 are equivalent on $M_1 \cap M_2$. By C_2, $M_1 \cap M_2$ is a nonzero ideal of S and hence

$$[\Gamma_1(M_1 \cap M_2)]$$

is a nonzero ideal of $\Gamma_1(S)$. But, by Lemma 2, this means that $[\Gamma_1(M_1 \cap M_2)] = [\Gamma_1(S)]$. Thus, by Lemma 4, we can choose $e_1, \ldots, e_r \in M_1 \cap M_2$ and $\alpha_1, \ldots, \alpha_r \in \Phi$ such that

$$\sum_1^r \alpha_i \Gamma_1(e_i) = I_n.$$

Since Γ_1 and Γ_2 are equivalent on $M_1 \cap M_2$, there exists a nonsingular matrix A such that, for each $m \in M_1 \cap M_2$,

$$\Gamma_2(m) = A\Gamma_1(m)A^{-1}.$$

Hence $\sum_1^r \alpha_i \Gamma_2(e_i) = I_n$; thus, for each $x \in S$,

$$\Gamma_2(x) = \sum_1^r \alpha_i \Gamma_2(e_i x) = \sum_1^r \alpha_i A\Gamma_1(e_i x)A^{-1} = A\Gamma_1(x)A^{-1}.$$

That is, Γ_1 and Γ_2 are equivalent.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

Note 1. It can readily be shown that, if, in the above theorem, S is a regular semigroup, a periodic semigroup, a semigroup satisfying M_L and M_R, or a 0-simple semigroup containing a nonzero idempotent, then M is an $\mathcal{M}$-semigroup (note Lemma 6); that is, M^* is completely 0-simple. In this case the 0-restricted irreducible representations of S can be determined explicitly by means of Clifford's theory of representations of a completely 0-simple semigroup [**1**].

Note 2. Let $S = S^0$ be a semigroup satisfying C_2 that has a unique minimal nonzero ideal. Then, by the last part of Theorem 4, the irreducible 0-restricted representations of S are determined, to within equivalence, by those of the unique minimal nonzero of S. That is, in the terminology of [7], they are the principal irreducible 0-restricted representations of S.

We shall end the paper by giving a sufficient condition for the existence of a 0-restricted representation of a semigroup $S = S^0$, that obeys C_2. Before giving this criterion, we shall prove some results about conditions C_1 and C_2.

LEMMA 6. *Let $S = S^0$ be a semigroup that obeys* C_2. *Let L be a nonzero ideal of S. Then L obeys* C_2.

Proof. Let m, n be nonzero members of L. Then, by C_2, there exists $x \in S$ such that $mxm \neq 0$. Again, by C_2, there exists $y \in S$ such that $mxm.y.n \neq 0$. Let $u = xmy$; since L is an ideal of S, $u \in L$. Then $mun \neq 0$ and so L obeys C_2.

LEMMA 7. *Let $S = S^0$ be a semigroup that obeys* C_2. *Then the set of all ideals of S that obey* C_1 *has a unique maximal member L.*

Proof. Let $L = \bigcup \{L_\alpha : \alpha \in A\}$ be the union of all ideals of S that obey C_1. If $L \neq \{0\}$, let $a \in L \backslash \{0\}$, and suppose that $sa \neq 0$ and $at \neq 0$ for $s, t \in S$; then $a \in L_\alpha$, for some $\alpha \in A$. Since, by Lemma 6, L_α obeys C_2, there exist $m, n \in L_\alpha$ such that $msa \neq 0$, $atn \neq 0$. Since L_α is an ideal that obeys C_1, it follows that $msatn \neq 0$; hence $sat \neq 0$. Thus L obeys C_1.

THEOREM 5. *Let $S = S^0$ be a semigroup that obeys* C_1 *and* C_2, *and let T be a nonzero ideal of S. If σ and τ denote, respectively, the maximum* 0*-restricted congruences on S and T, then*

$$S/\sigma \cong T/\tau.$$

Proof. Since S obeys C_1 and C_2, it follows, from Lemma 6, that the same is true of T. From the definitions of σ and τ, it is clear that, for $a, b \in T$, if $(a, b) \in \sigma$ then $(a, b) \in \tau$. Conversely, let $(a, b) \in \tau$ and let $sat = 0$, where $s, t \in S^1 \backslash 0$. Since T is an ideal of S and S obeys C_2, there exist $m, n \in T$ such that neither of ms, tn is zero. Then $msatn = 0$ and so, since $(a, b) \in \tau$, $msbtn = 0$. Since S obeys C_1, $msbtn = 0$ implies $msbt = 0$ or $sbtn = 0$. But neither of ms, tn is zero so that each of these equations implies $sbt = 0$. Similarly, $sbt = 0$ implies $sat = 0$; hence $(a, b) \in \sigma$. Thus $\tau = \sigma \cap (T \times T)$.

Let θ denote the natural homomorphism of S onto S/σ. Then, since $\tau = \sigma \cap (T \times T)$,

$$T/\tau \cong T\theta.$$

But, by the proof of the corollary to Theorem 1, $S\theta = S/\sigma$ is completely 0-simple. Thus, since T is a nonzero ideal of S, $T\theta = S\theta$. Hence we have the result.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

Let $S = S^0$ be a semigroup that obeys C_2, and let M be an ideal of S that obeys C_1. Then a sufficient condition for S to have a 0-restricted representation, over a field Φ, defined from M as in Theorem 4, is that M/ρ should have a 0-restricted irreducible representation over Φ. In fact, by Theorem 5, it is sufficient that L/ρ should have a 0-restricted irreducible representation over Φ. By the proof of the corollary to Theorem 1, L/ρ is completely 0-simple; hence we can use Clifford's results [1] to give necessary and sufficient conditions for L/ρ to have an irreducible 0-restricted representation.

Clifford proves the following. Let $\mathfrak{M}^0(G; I, \Lambda; P)$ be a regular Rees matrix semigroup over a group with zero G^0. Let Γ be a representation of G of degree n over a field Φ. Then Γ can be extended to a representation of $\mathfrak{M}^0(G; I, \Lambda; P)$ if and only if the $\Lambda \times I$ block matrix Ω over Φ, whose (λ, i)th block is the $n \times n$ matrix $\Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1} p_{1 i})$, has finite rank over Φ. Further, every representation of $\mathfrak{M}^0(G; I, \Lambda; P)$ is the extension of some representation of G; in particular, the irreducible representations are the extensions of irreducible representations of G.

Let $a \in L \backslash \{0\}$; if $a^2 \neq 0$, then (cf. the proof of the corollary to Theorem 1) $(a, a^3) \in \rho$ and $(a, a^6) \in \rho$, so that $(a, a^2) \in \rho$. Thus L/ρ is a completely 0-simple semigroup in which each element is either idempotent or nilpotent. Hence [2] L/ρ is isomorphic to a regular Rees matrix semigroup over a group-with-zero G^0; further, since each element of L/ρ is either idempotent or nilpotent, it can be verified by direct calculation that G has only one element.

Suppose that $L/\rho \cong \mathfrak{M}^0(\{e\}; I, \Lambda; P)$, where $\{e\}$ is a one element group. Let Φ be a field and let Ω be the $\Lambda \times I$ matrix over Φ where $\Omega_{\lambda i} = 1, 0, -1$ according as $p_{\lambda i}$ is greater than, is equal to, is less than $p_{\lambda 1} p_{1 i}$; $\{e\}^0$ is partially ordered by $e > 0$. If Ω has finite rank over Φ, then we say that S has finite rank over Φ; if $L = \{0\}$, then rank S is zero.

Since $\{e\}$ has only one member, every irreducible representation of $\{e\}$ over Φ is of degree one. Hence, by Clifford's results, mentioned above, L/ρ has an irreducible representation over Φ if and only if Ω has finite rank.

The above results are gathered together in the following proposition.

PROPOSITION 7. *Let $S = S^0$ be a semigroup that obeys C_2, and let Φ be a field. If S has a 0-restricted representation over Φ, then S has nonzero rank over Φ. Conversely, if S has finite nonzero rank over Φ, then S has a 0-restricted representation over Φ.*

Finally, we point out that, if $S = S^0$ is an inverse semigroup or a weakly regular semigroup in which the idempotents commute, it can be shown that the criterion of Proposition 7 is not only sufficient but is also necessary; cf. [10] for the inverse case. In this case it takes the form: S has a 0-restricted representation if and only if L/ρ is finite with at least two members.

REFERENCES

1. A. H. Clifford, Matrix representations of completely 0-simple semigroups, *American J. Math.* **64** (1942), 327–342.

2. A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, Vol. 1, American Math. Soc. Surveys, **7** (1961).

3. J. A. Green, On the structure of semigroups, *Ann. of Math.* **54** (1951), 163–172.

4. G. Lallement, *Sur les homomorphismes d'un demigroupe sur un demigroupe completement 0-simple*, Seminaire Dubreil-Pisot, 1963–64, no. 14.

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press

5. G. Lallement and M. Petrich, Some results concerning completely 0-simple semigroups, *Bull. Amer. Math. Soc.* **70** (1964), 777–778.

6. W. D. Munn, Matrix representations of semigroups, *Proc. Cambridge Phil. Soc.* **53** (1957), 145–152.

7. W. D. Munn, Irreducible matrix representations of semigroups, *Quart. J. Math.* (*Oxford Ser.*) (2) **11** (1960), 295–309.

8. W. D. Munn, A class of irreducible matrix representations of an arbitrary inverse semigroup, *Proc. Glasgow Math. Assoc.* **5** (1961), 41–48.

9. W. D. Munn, Brandt congruences on inverse semigroups, *Proc. London Math. Soc.* (3) **14** (1964), 154–164.

10. W. D. Munn, Matrix representations of inverse semigroups, *Proc. London Math. Soc.* (3) **14** (1964), 165–181.

11. G. B. Preston, *Congruences on semigroups* (Ed. J. M. Howie), N.S.F. Algebra Institute. Pennsylvania State University, Summer 1963.

Department of Pure Mathematics
Queen's University
Belfast

https://doi.org/10.1017/S0017089500000033 Published online by Cambridge University Press